Randomized Quasi-Monte Carlo:
Theory, Choice of Discrepancy, and Applications

Pierre L'Ecuyer and David Munger

Informatique et Recherche Opérationnelle, Université de Montréal

Randomized Quasi-Monte Carlo:
Theory, Choice of Discrepancy, and Applications
featuring randomly-shifted lattice rules

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1. Monte Carlo (MC), quasi-MC (QMC), randomized QMC (RQMC).
2. Lattice rules and RQMC variance.
3. Weighted discrepancies and choice of weights.
4. Several examples.

Basic Monte Carlo setting

Want to estimate

$$\mu = \mu(f) = \int_{[0,1)^s} f(\mathbf{u})\, d\mathbf{u} = \mathbb{E}[f(\mathbf{U})]$$

where $f : [0,1)^s \to \mathbb{R}$ and $\mathbf{U}$ is a uniform r.v. over $[0,1)^s$.

Standard Monte Carlo:

- Generate n independent copies of $\mathbf{U}$, say $\mathbf{U}_1, \ldots, \mathbf{U}_n$;
- estimate μ by $\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} f(\mathbf{U}_i)$.

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Almost sure convergence as $n \to \infty$ (strong law of large numbers).
For confidence interval of level $1-\alpha$, can use central limit theorem:

$$\mathbb{P}\left[\mu \in \left(\hat{\mu}_n - \frac{c_\alpha S_n}{\sqrt{n}},\ \ \hat{\mu}_n + \frac{c_\alpha S_n}{\sqrt{n}}\right)\right] \approx 1-\alpha,$$

where S_n^2 is any consistent estimator of $\sigma^2 = \mathrm{Var}[f(\mathbf{U})]$.

Quasi-Monte Carlo (QMC)

Replace the random points $\mathbf{U}_i$ by a set of deterministic points $P_n = \{\mathbf{u}_0, \ldots, \mathbf{u}_{n-1}\}$ that cover $[0,1)^s$ more evenly.
This P_n is called a highly-uniform or low-discrepancy point set if some measure of discrepancy between the empirical distribution of P_n and the uniform distribution $\to 0$ faster than for independent random points.

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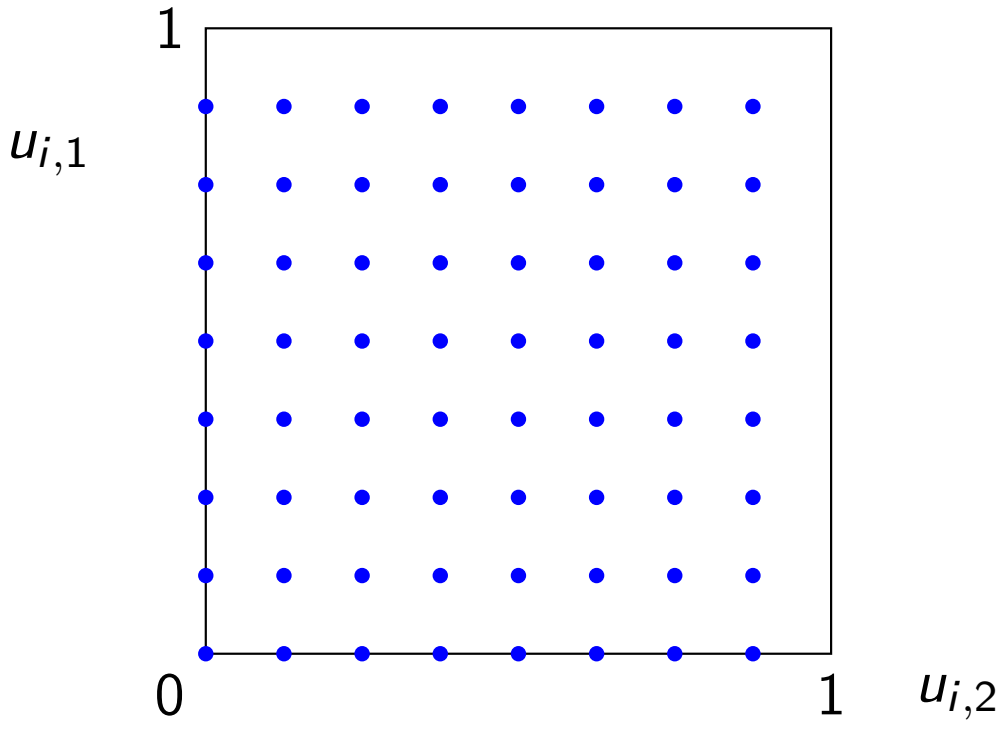
Main construction methods: lattice rules and digital nets
(Korobov, Hammersley, Halton, Sobol', Faure, Niederreiter, etc.)

Simplistic solution: rectangular grid

$P_n = \{(i_1/d, \ldots, i_s/d) \text{ such that } 0 \leq i_j < d \quad \forall j\}$ where $n = d^s$.



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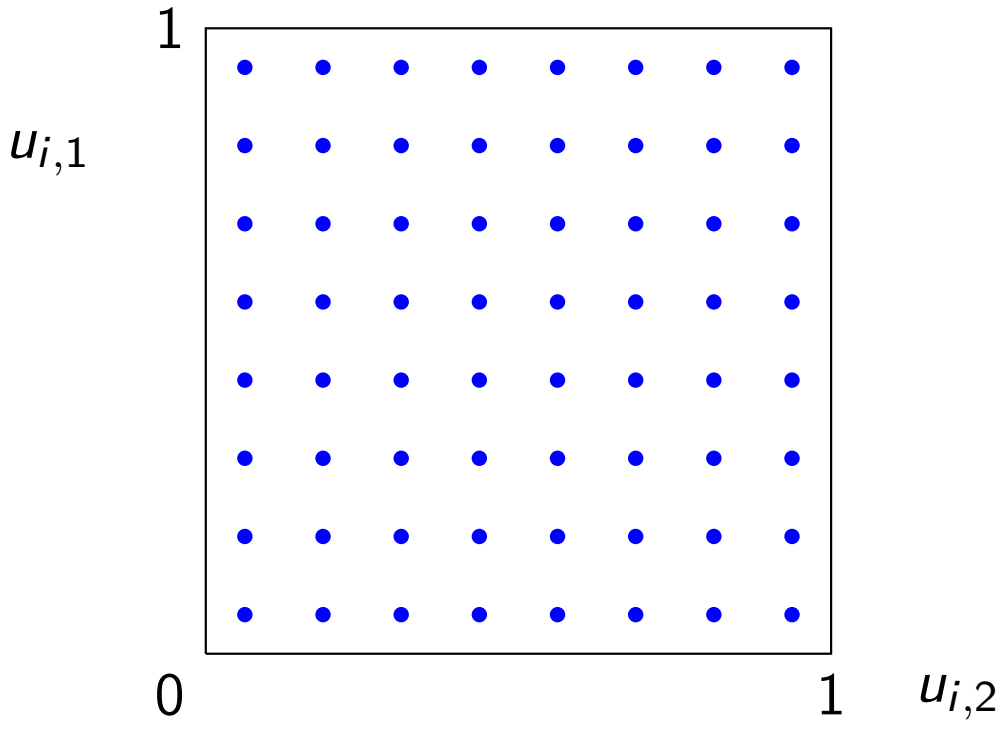

Quickly becomes impractical when s increases.
And each one-dimensional projection has only d distinct points, each two-dimensional projections has only d^2 distinct points, etc.

Example: lattice with $s = 2$, $n = 101$, $a = 12$

$$
\begin{aligned}
P_n &= \{(x/m, (ax/m) \bmod 1) : x = 0, \ldots, m-1\} \\
&= \{(x/101, (12x/101) \bmod 1) : x = 0, \ldots, 100\}
\end{aligned}
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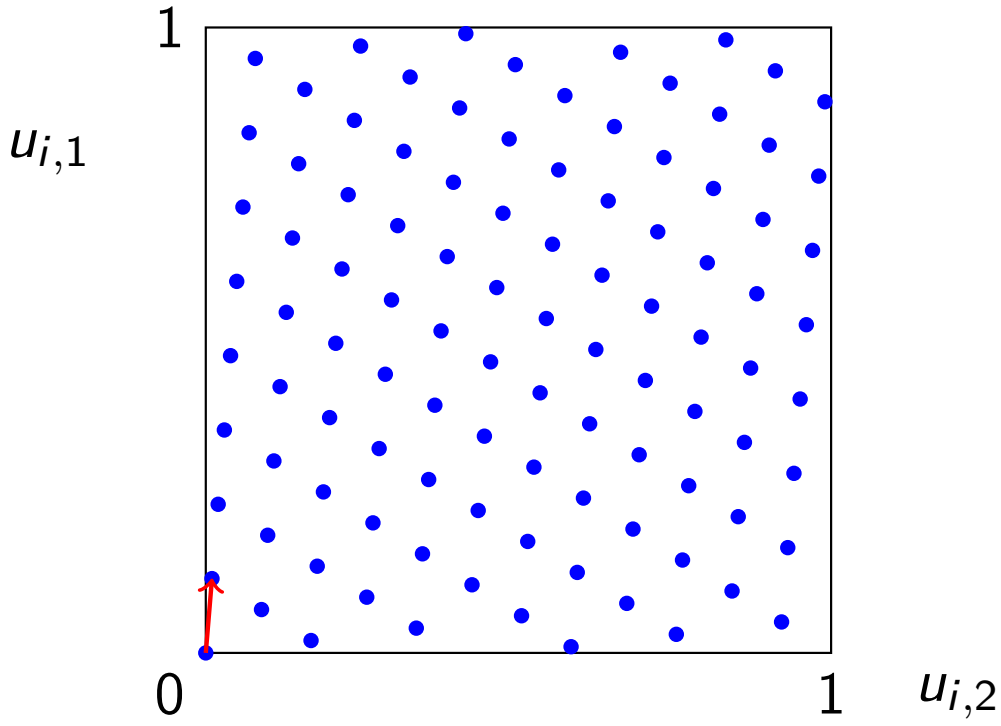

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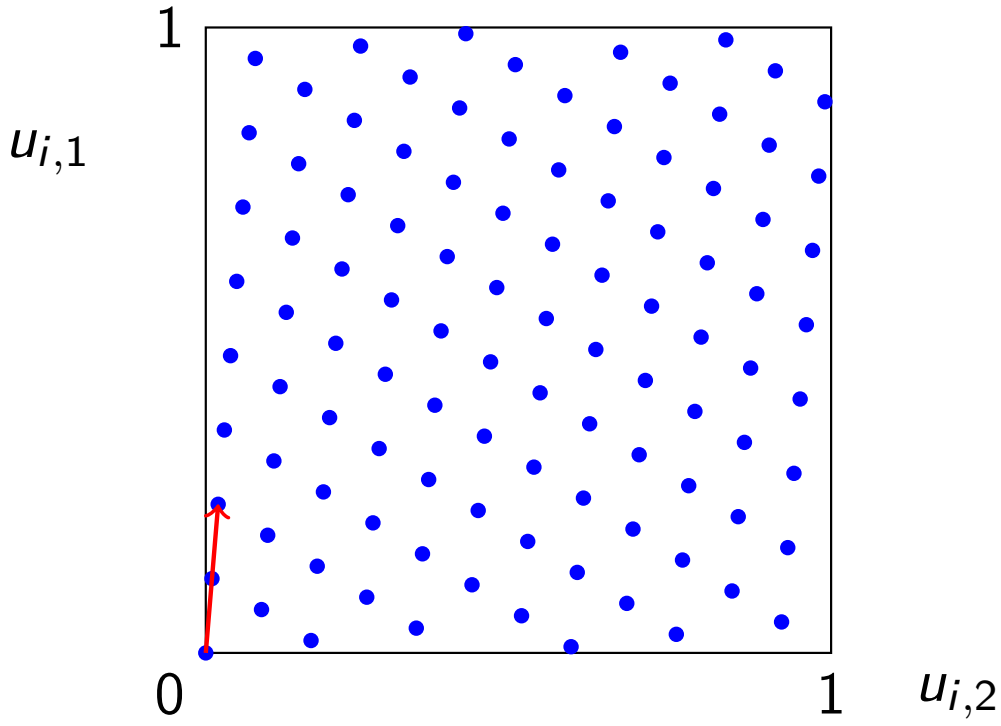

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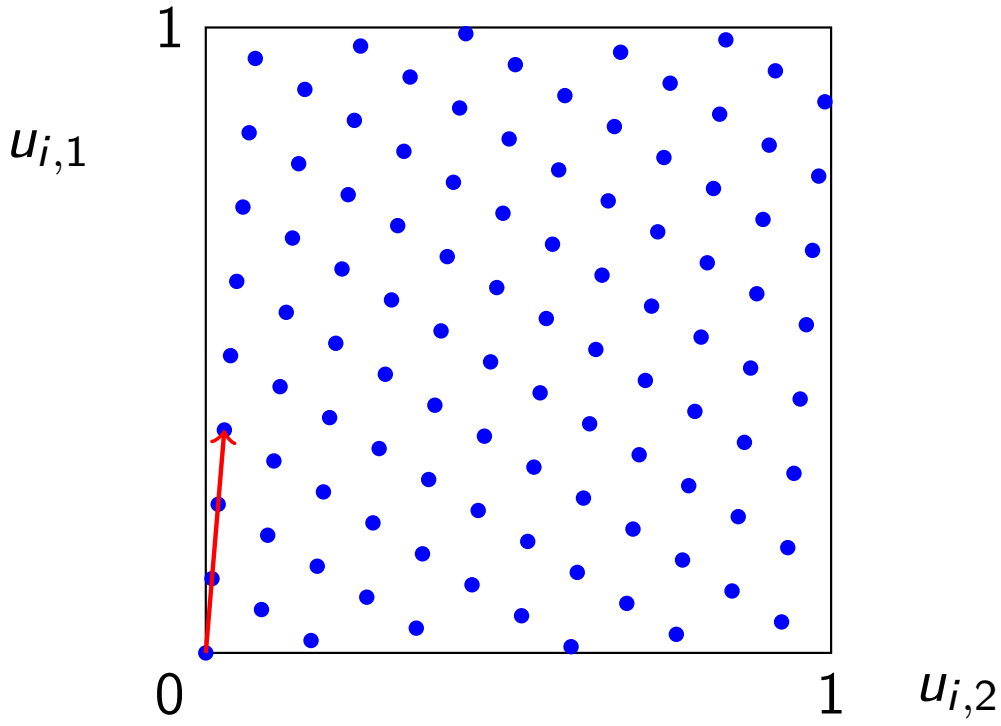

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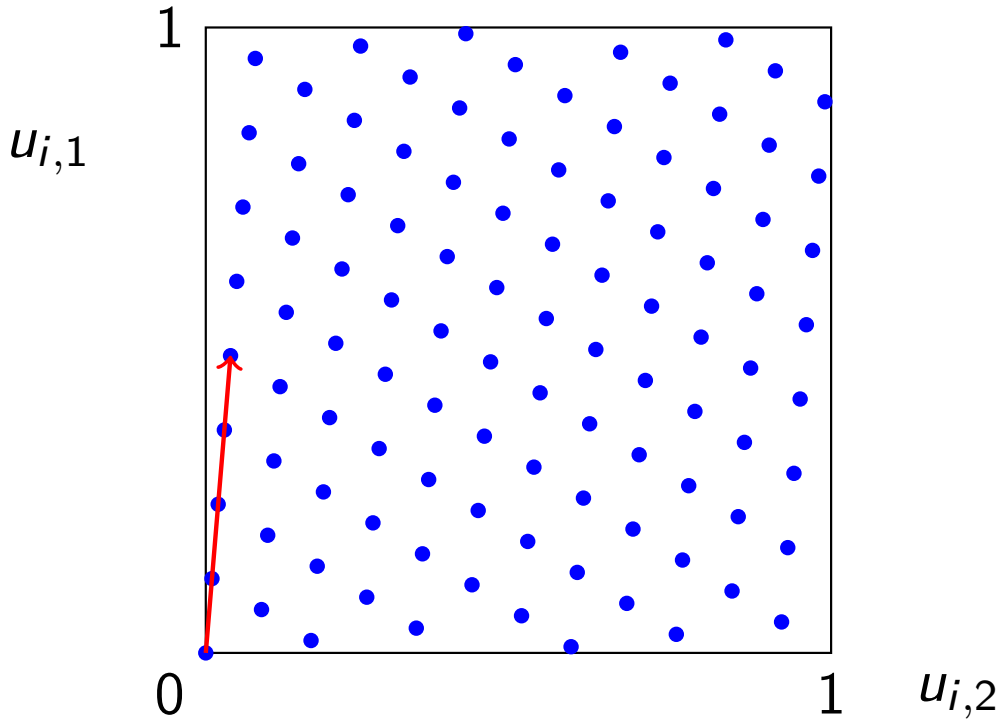

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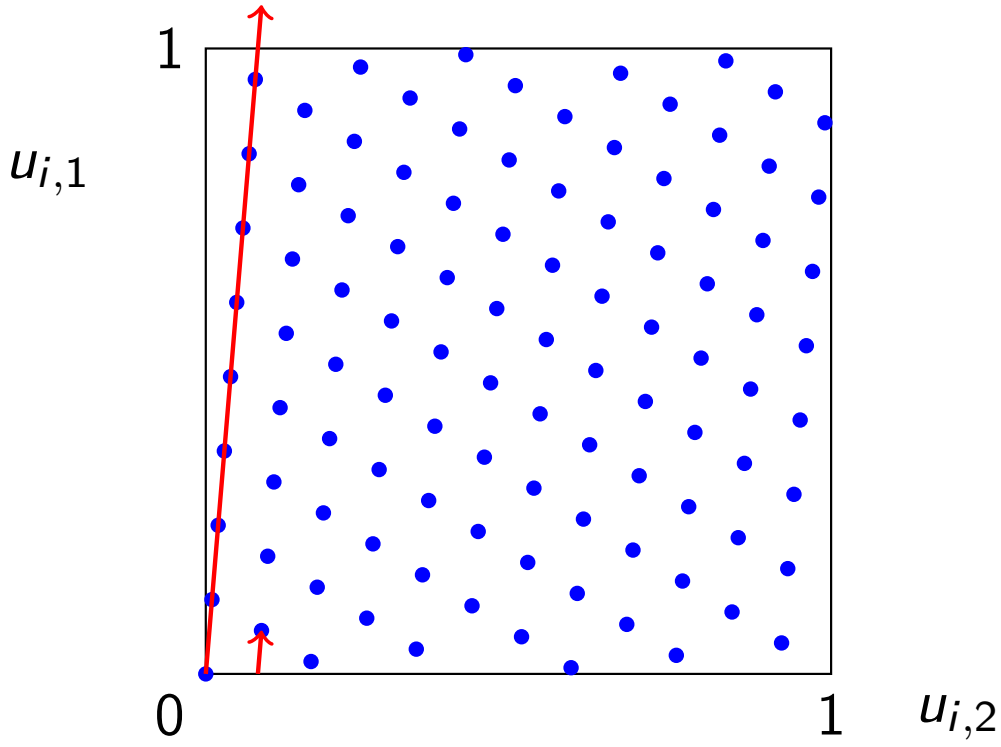

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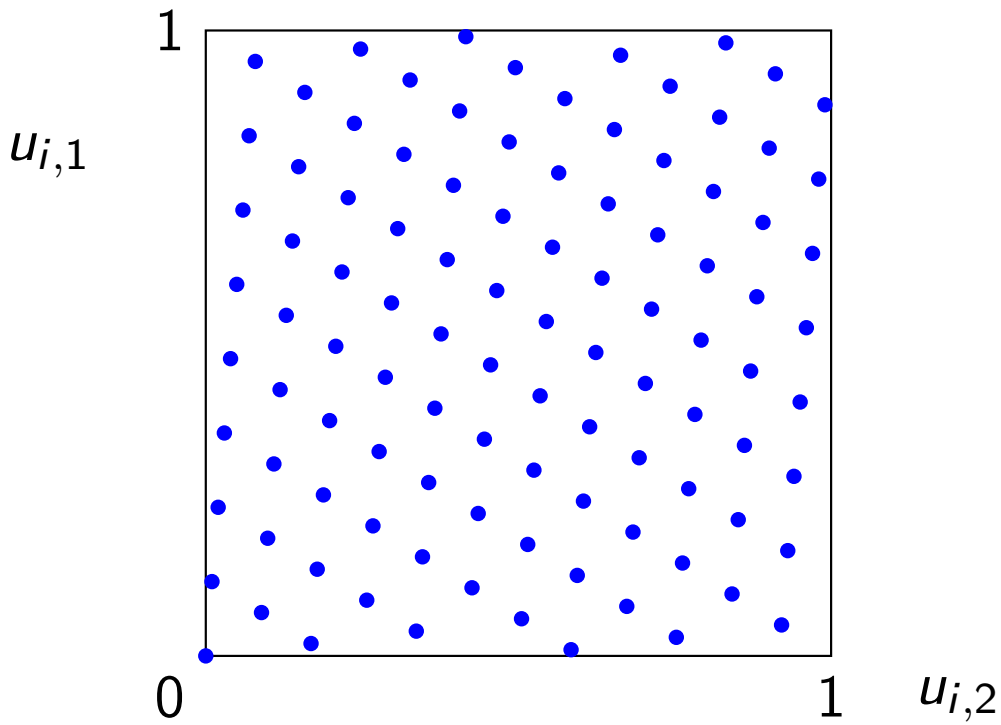

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Two problems: (1) point at $(0, 0)$ and (2) how to estimate the error?

Randomized quasi-Monte Carlo (RQMC)

An RQMC estimator of μ has the form

$$\hat{\mu}_{n,\mathrm{rqmc}} = \frac{1}{n}\sum_{i=0}^{n-1} f(\mathbf{U}_i),$$

with $P_n = \{\mathbf{U}_0, \ldots, \mathbf{U}_{n-1}\} \subset (0,1)^s$ an RQMC point set:

(i) each point $\mathbf{U}_i$ has the uniform distribution over $(0,1)^s$;

(ii) P_n as a whole is a low-discrepancy point set.

$$\mathbb{E}[\hat{\mu}_{n,\mathrm{rqmc}}] = \mu \qquad \text{(unbiased)}.$$

Can perform m independent realizations $X_1, \ldots, X_m$ of $\hat{\mu}_{n,\mathrm{rqmc}}$, then estimate μ and $\mathrm{Var}[\hat{\mu}_{n,\mathrm{rqmc}}]$ by their sample mean $\bar{X}_m$ and sample variance S_m^2 (also unbiased).

Temptation: assume that $\bar{X}_m$ has the normal distribution.

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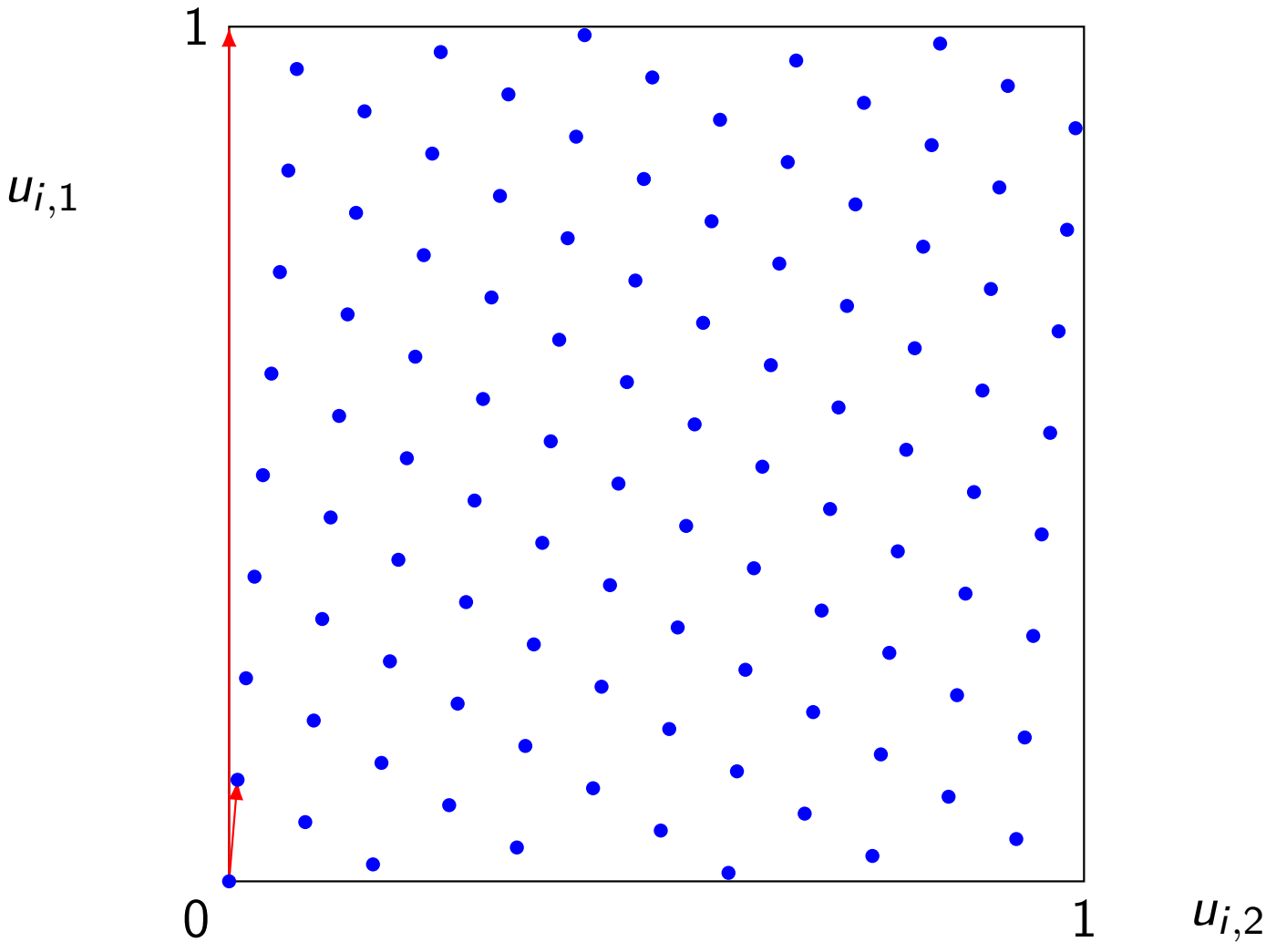

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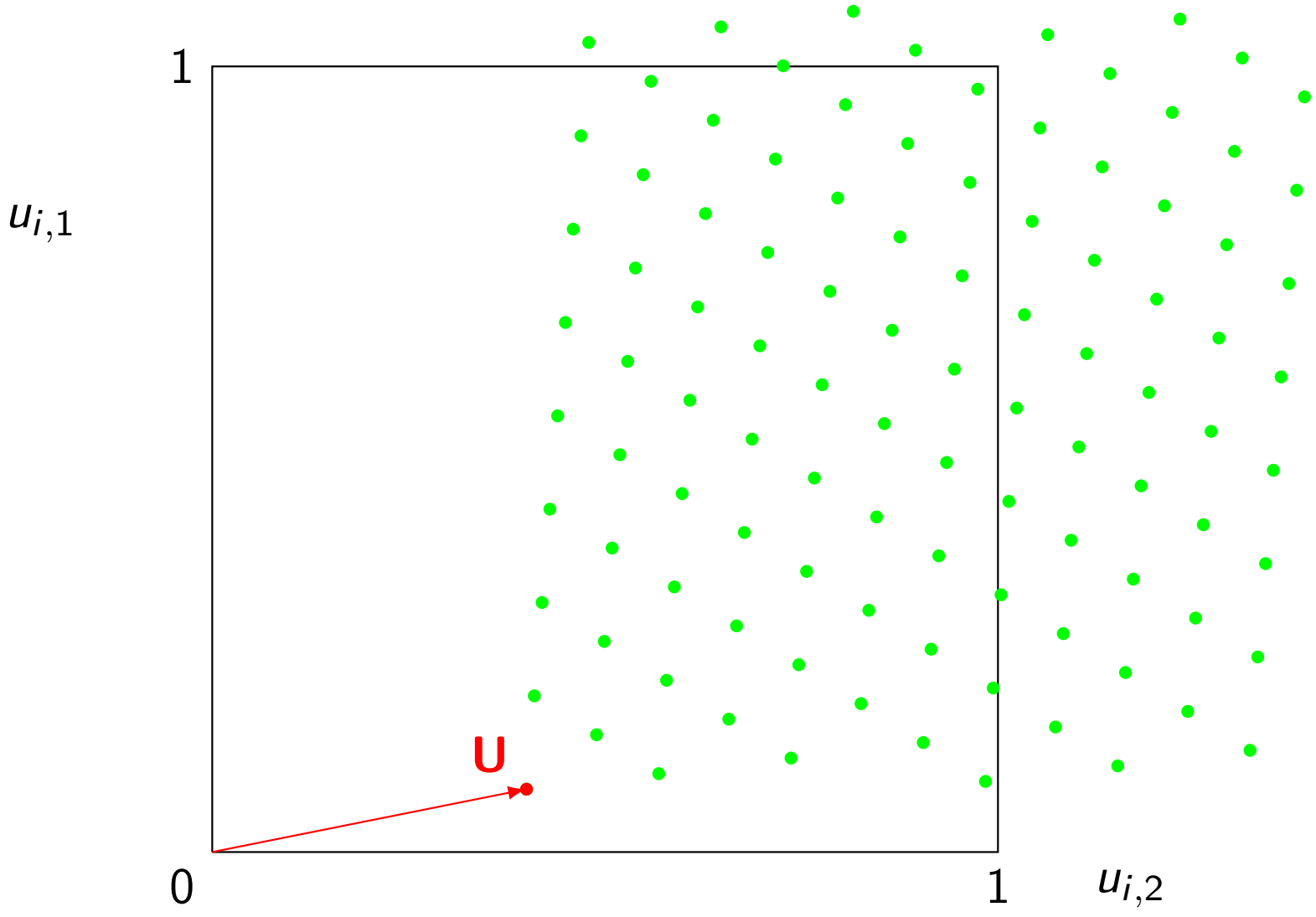

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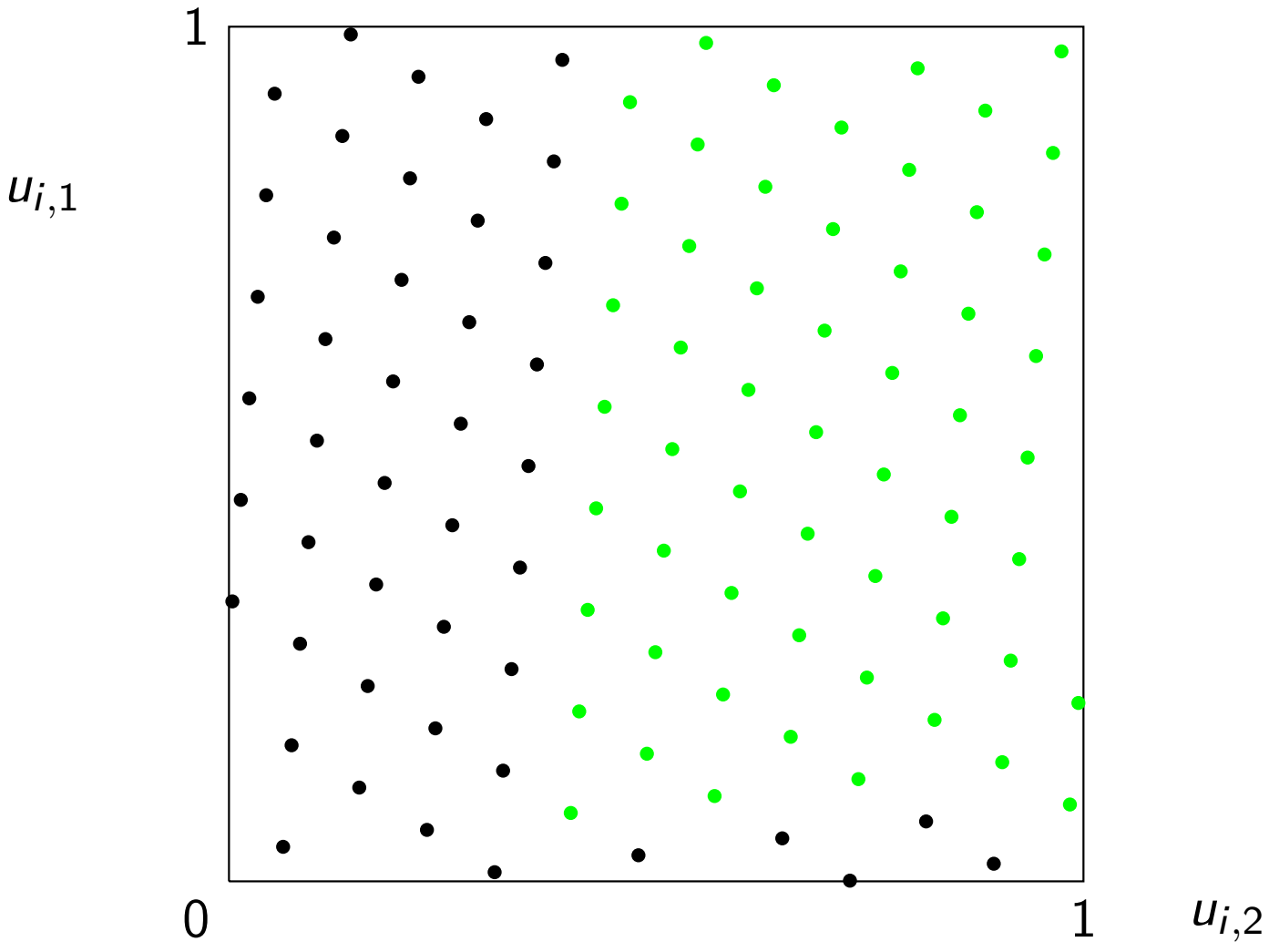

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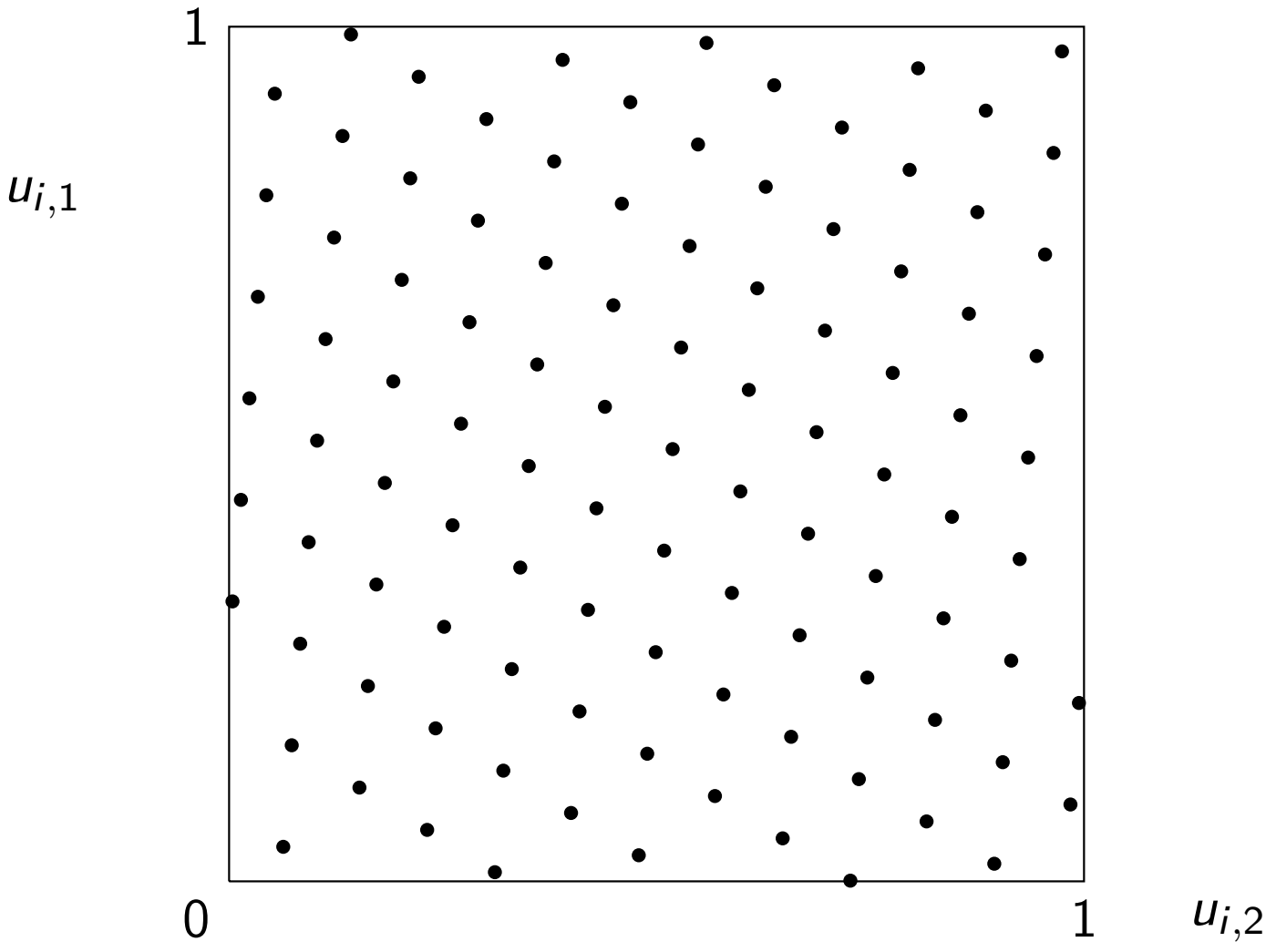

Generalized antithetic variates and RQMC

$$\begin{aligned}\mathrm{Var}[\hat{\mu}_{n,\mathrm{rqmc}}] &= \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\mathrm{Cov}[f(\mathbf{U}_i), f(\mathbf{U}_j)] \\ &= \frac{\mathrm{Var}[f(\mathbf{U}_i)]}{n} + \frac{2}{n^2}\sum_{i<j}\mathrm{Cov}[f(\mathbf{U}_i), f(\mathbf{U}_j)].\end{aligned}$$

We want to make the last sum as negative as possible.

Special cases:

- antithetic variates ($n = 2$),
- Latin hypercube sampling (LHS),
- randomized quasi-Monte Carlo (RQMC).

Lattice rules

Integration lattice:

$$L_s = \left\{ \mathbf{v} = \sum_{j=1}^{s} z_j \mathbf{v}_j \text{ such that each } z_j \in \mathbb{Z} \right\},$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_s \in \mathbb{R}^s$ are linearly independent over $\mathbb{R}$ and where L_s contains $\mathbb{Z}^s$. Lattice rule: Take $P_n = \{\mathbf{u}_0, \ldots, \mathbf{u}_{n-1}\} = L_s \cap [0,1)^s$.

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Lattice rule of rank 1: $\mathbf{u}_i = i\mathbf{v}_1 \bmod 1$ for $i = 0, \ldots, n-1$,
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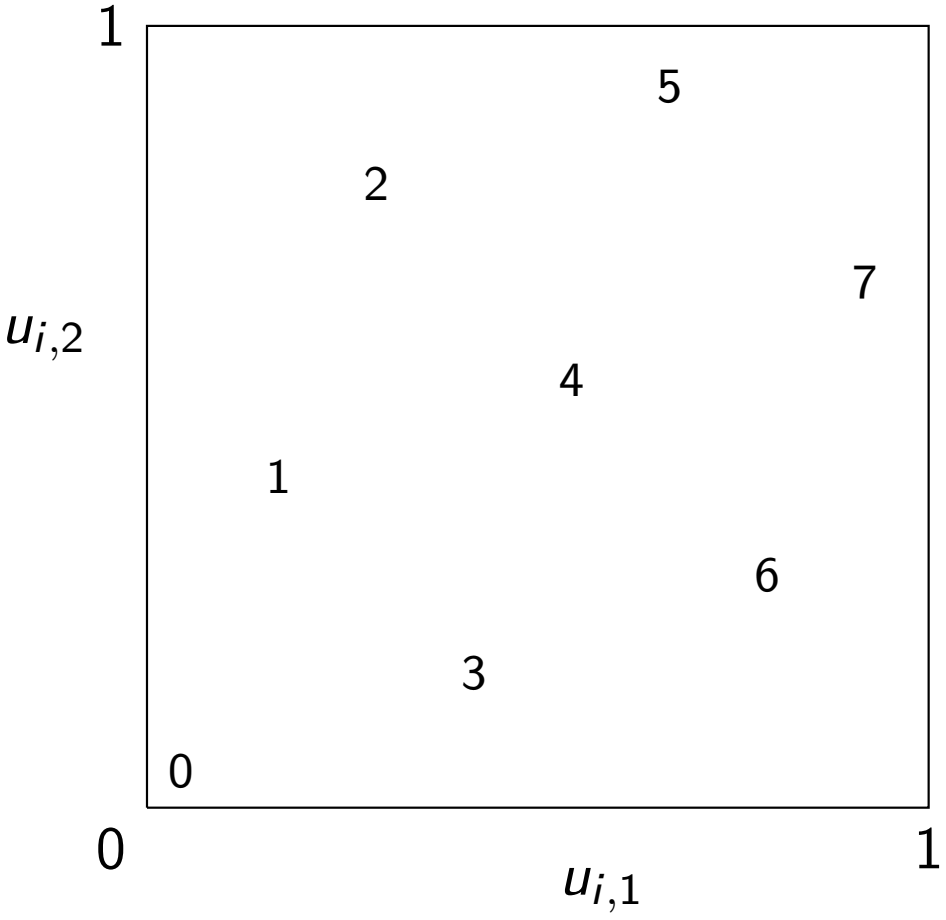
Random shift modulo 1: generate a single point $\mathbf{U}$ uniformly over $(0,1)^s$ and add it to each point of P_n, modulo 1, coordinate-wise:
$\mathbf{U}_i = (\mathbf{u}_i + \mathbf{U}) \bmod 1$. Each $\mathbf{U}_i$ is uniformly distributed over $[0,1)^s$.

A small lattice shifted by the red vector, modulo 1.



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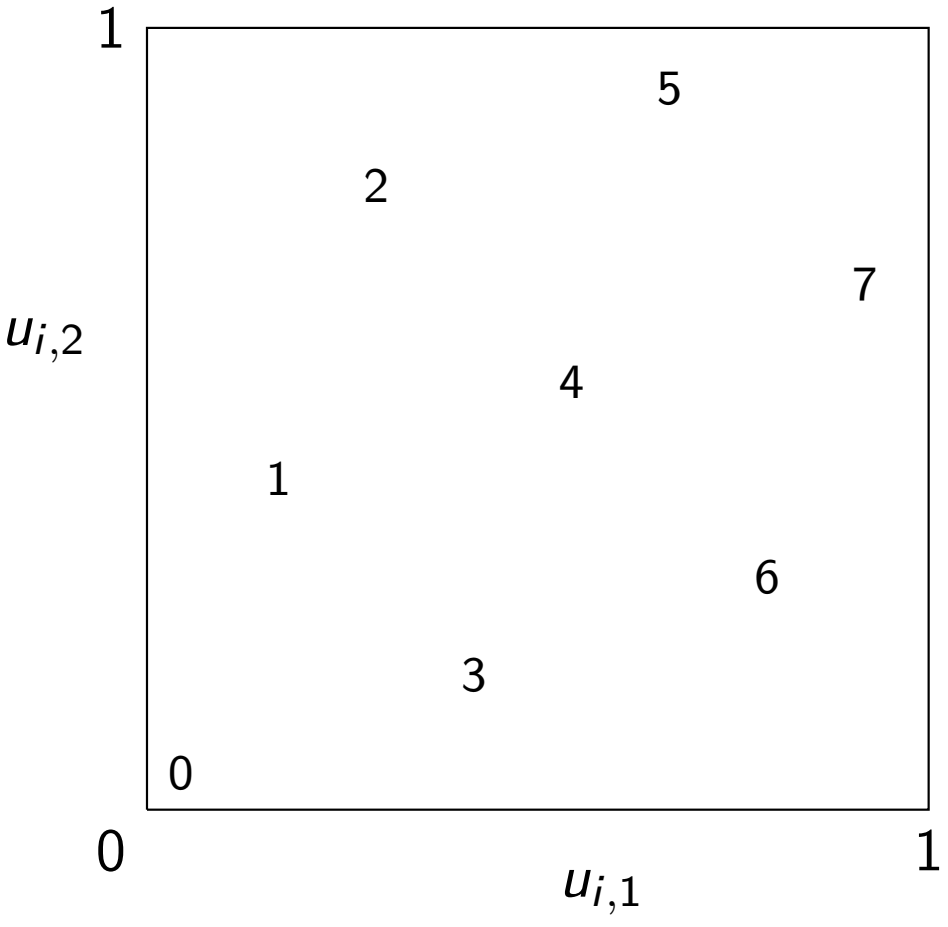
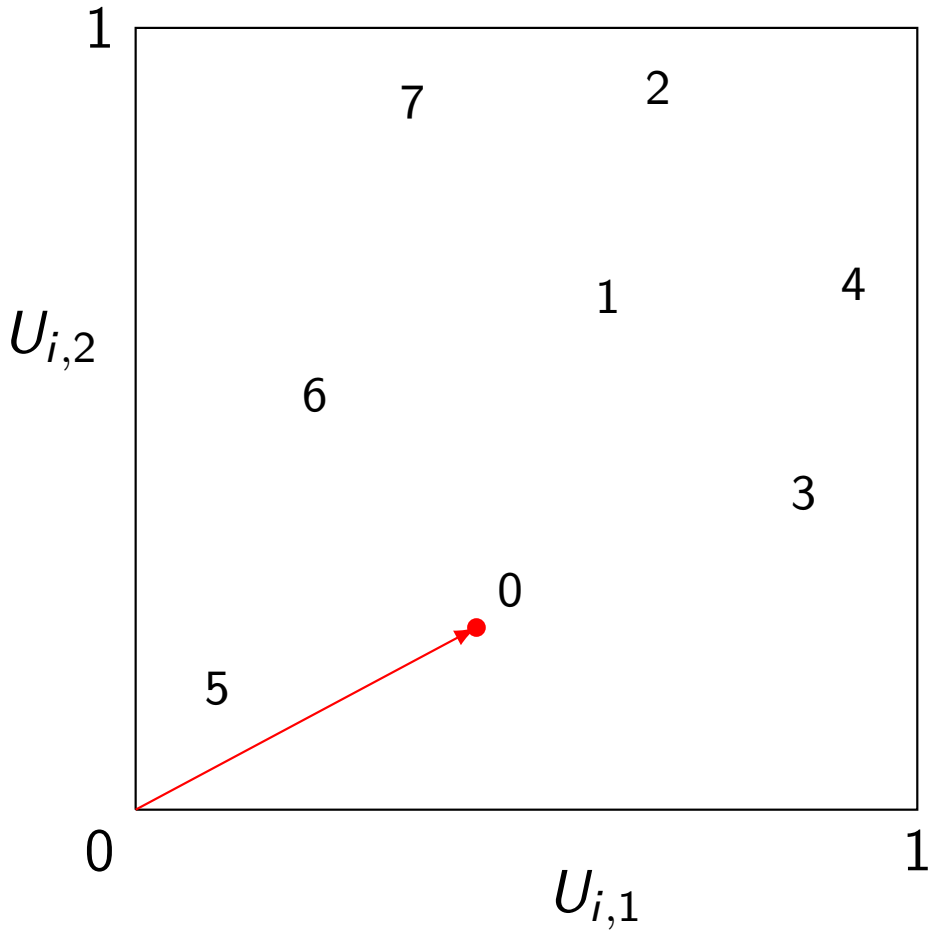
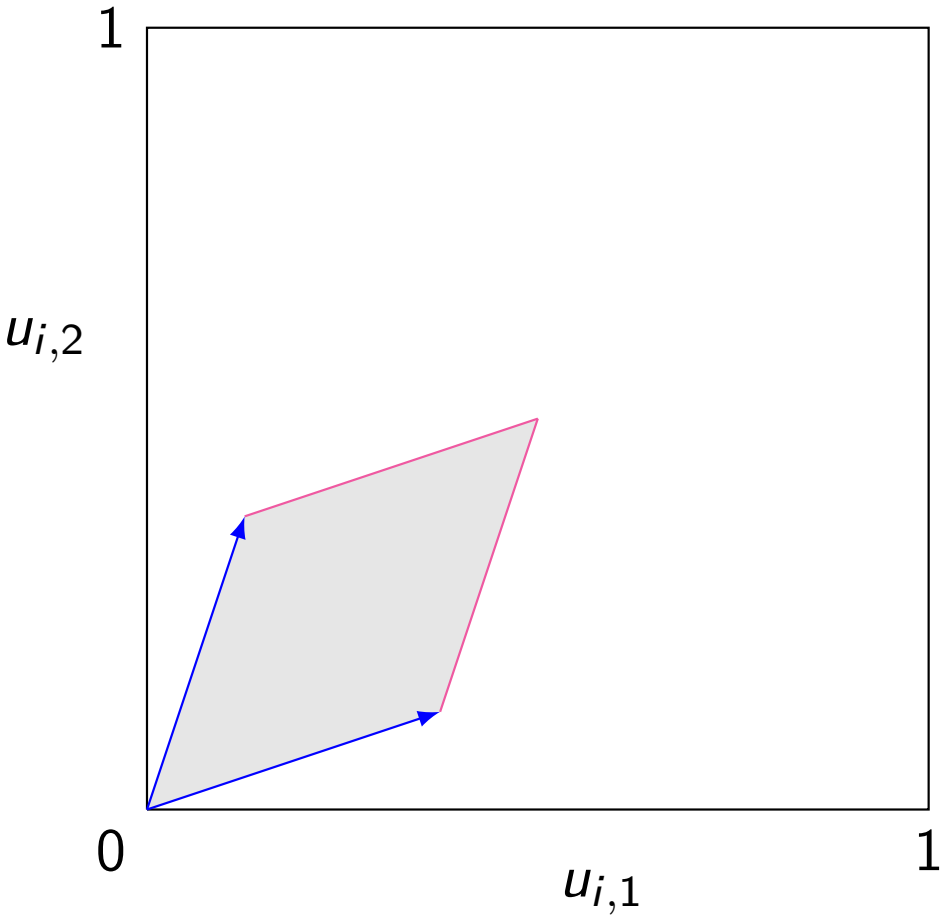




Can generate the shift uniformly in the parallelotope determined by basis vectors,



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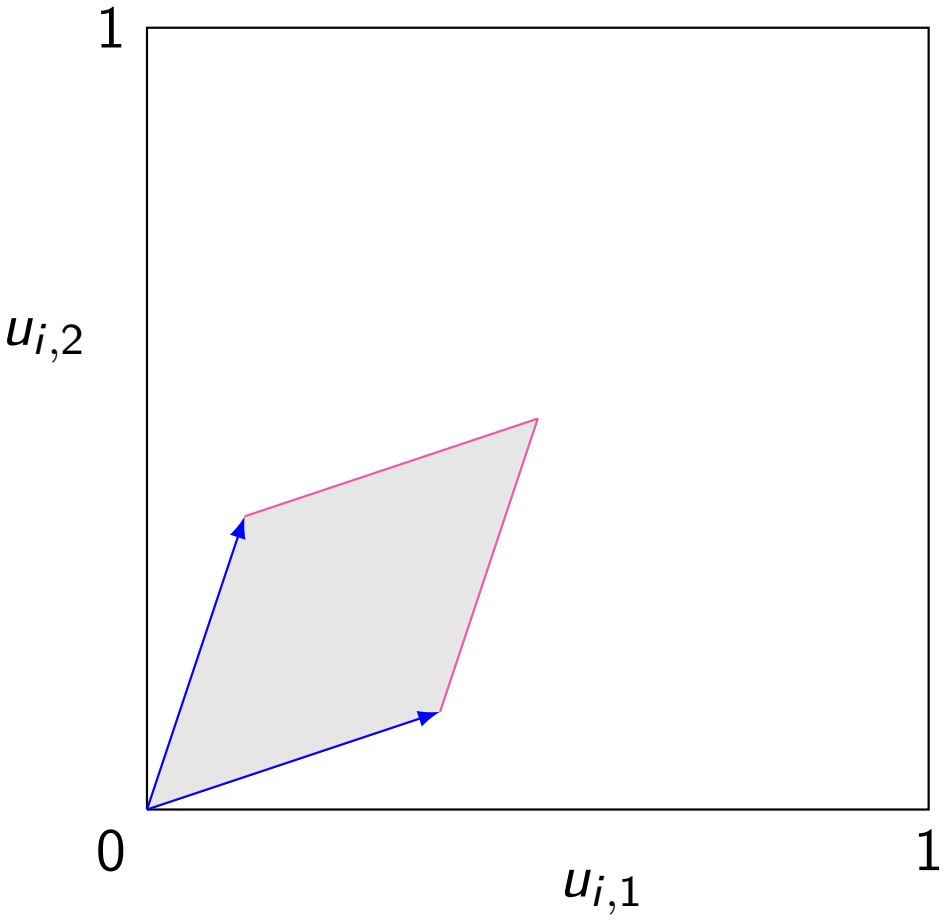
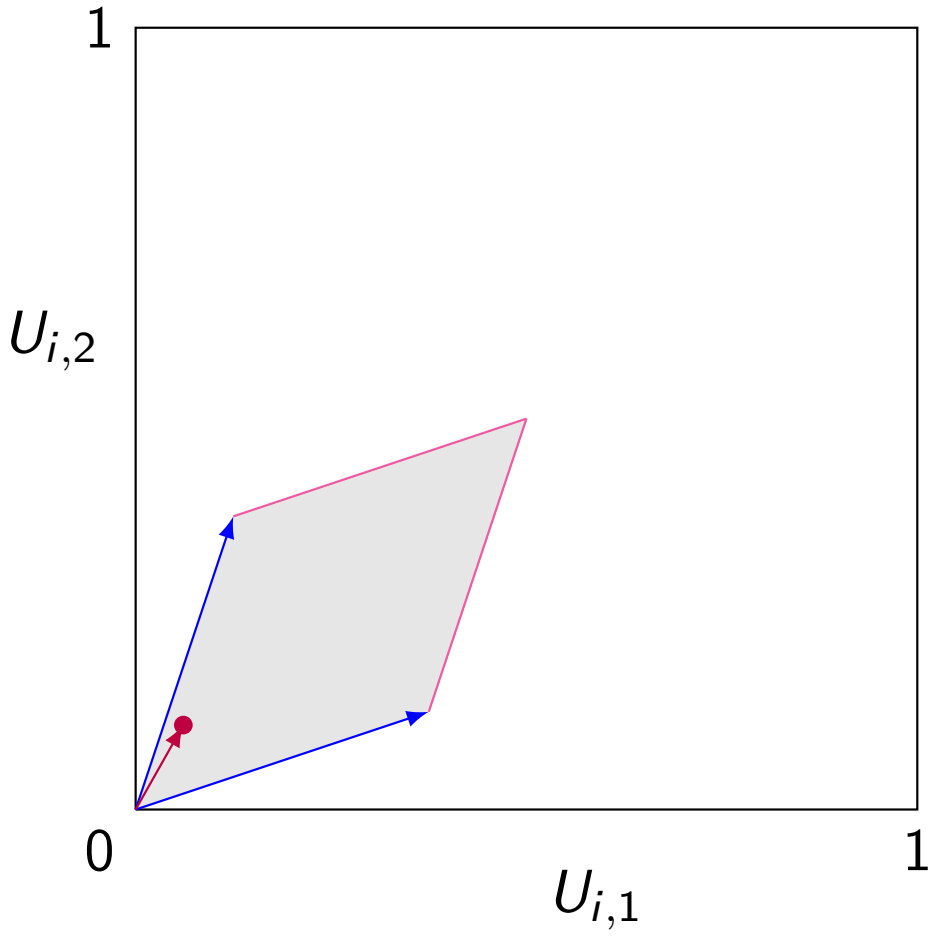
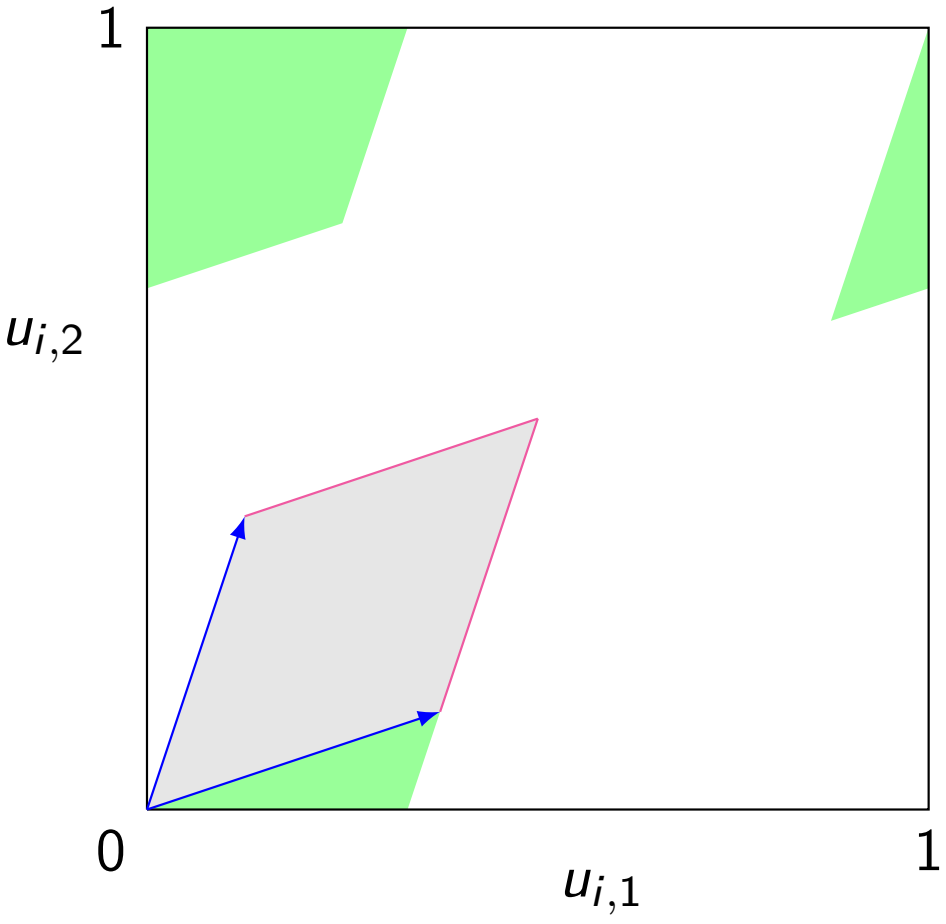
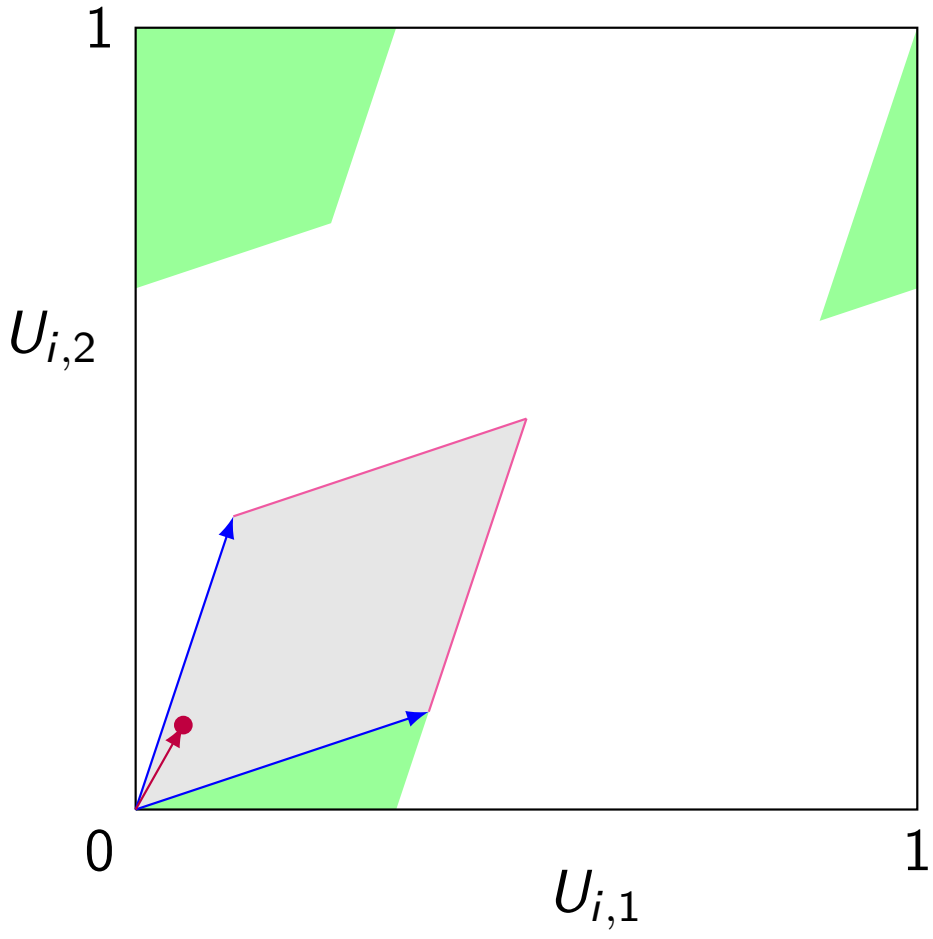


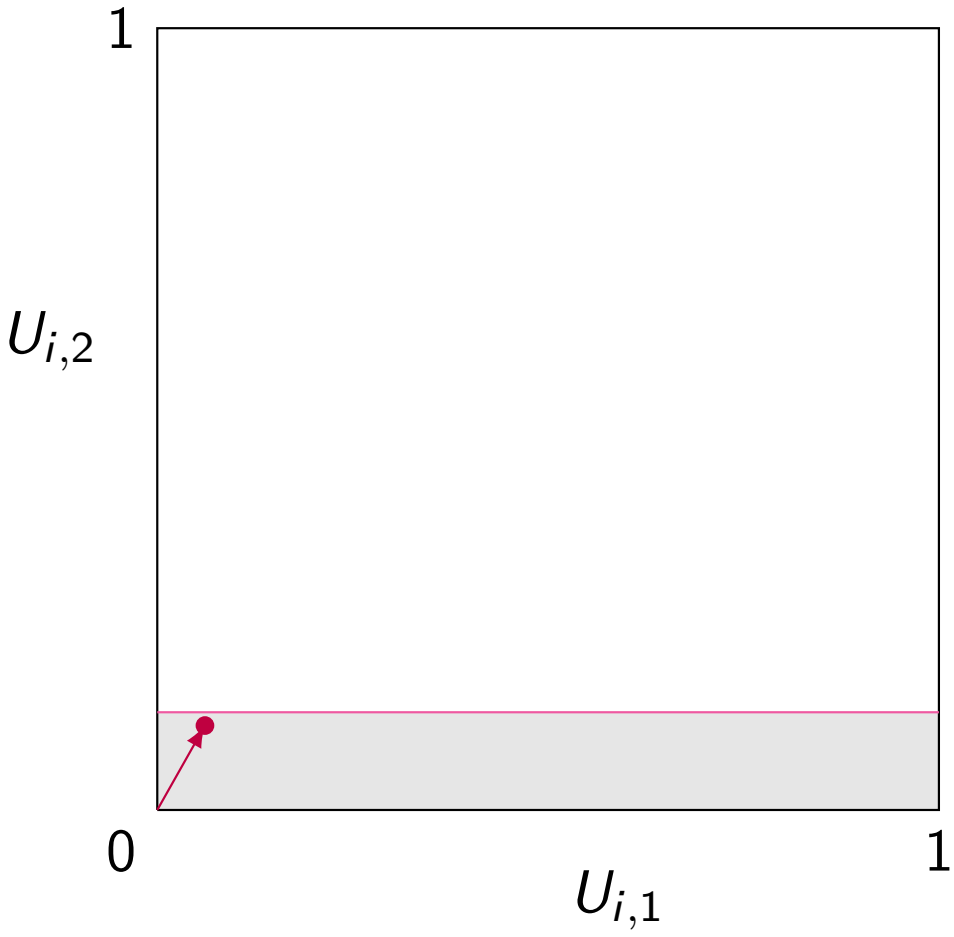


Can generate the shift uniformly in the parallelotope determined by basis vectors, or in any shifted copy of it.





Perhaps less obvious: Can generate it in any of the colored shapes below.



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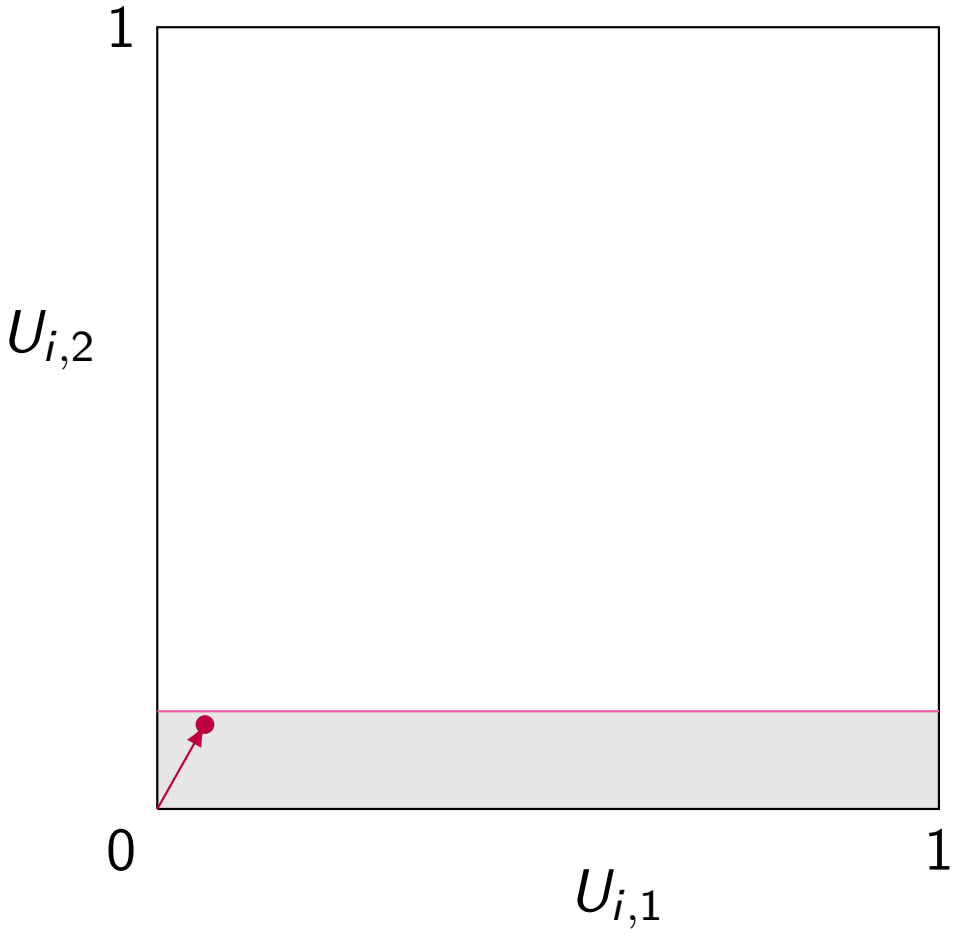

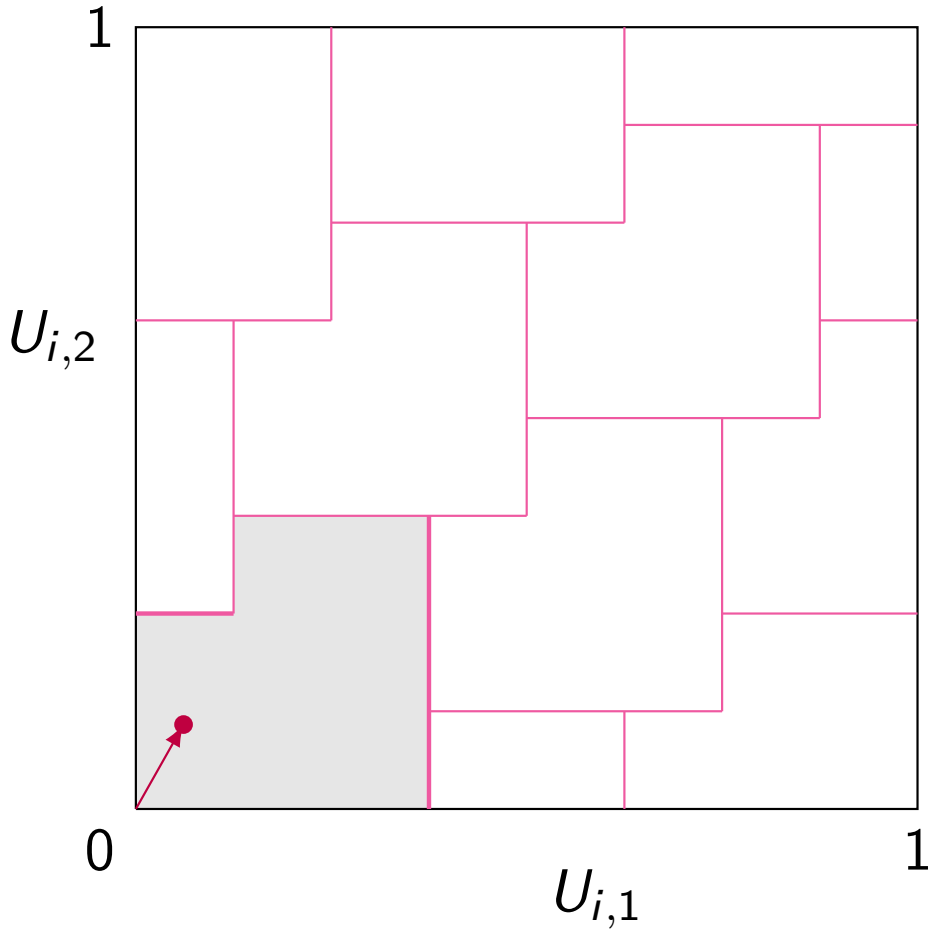

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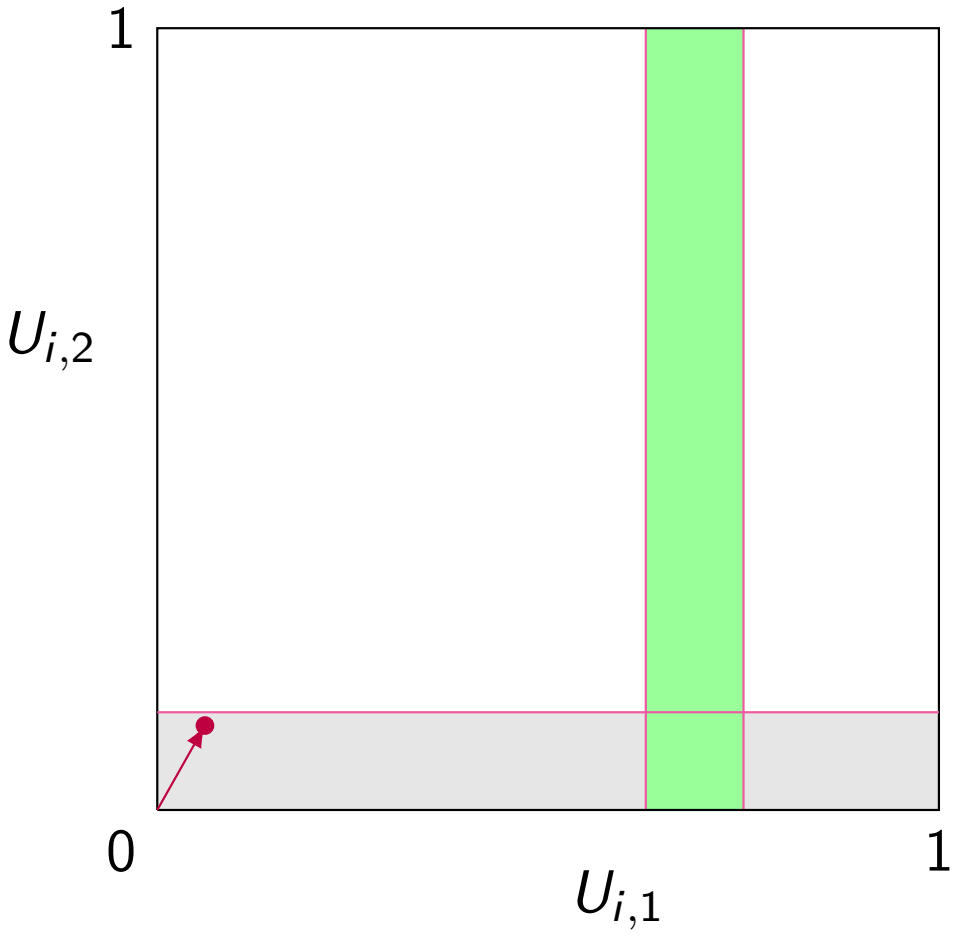
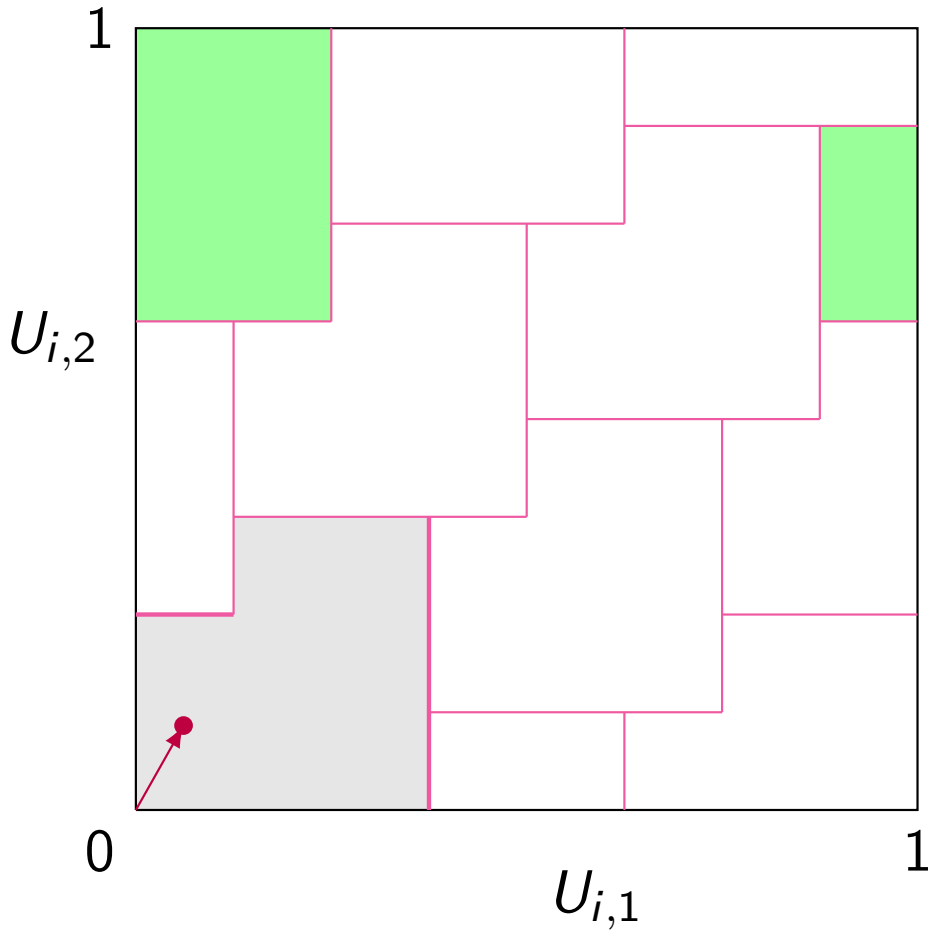




Generating the shift uniformly in one tile

Proposition. Let $R \subset [0,1)^s$ such that

$$\{R_i = (R + \mathbf{u}_i) \bmod 1,\ i = 0, \ldots, n-1\}$$

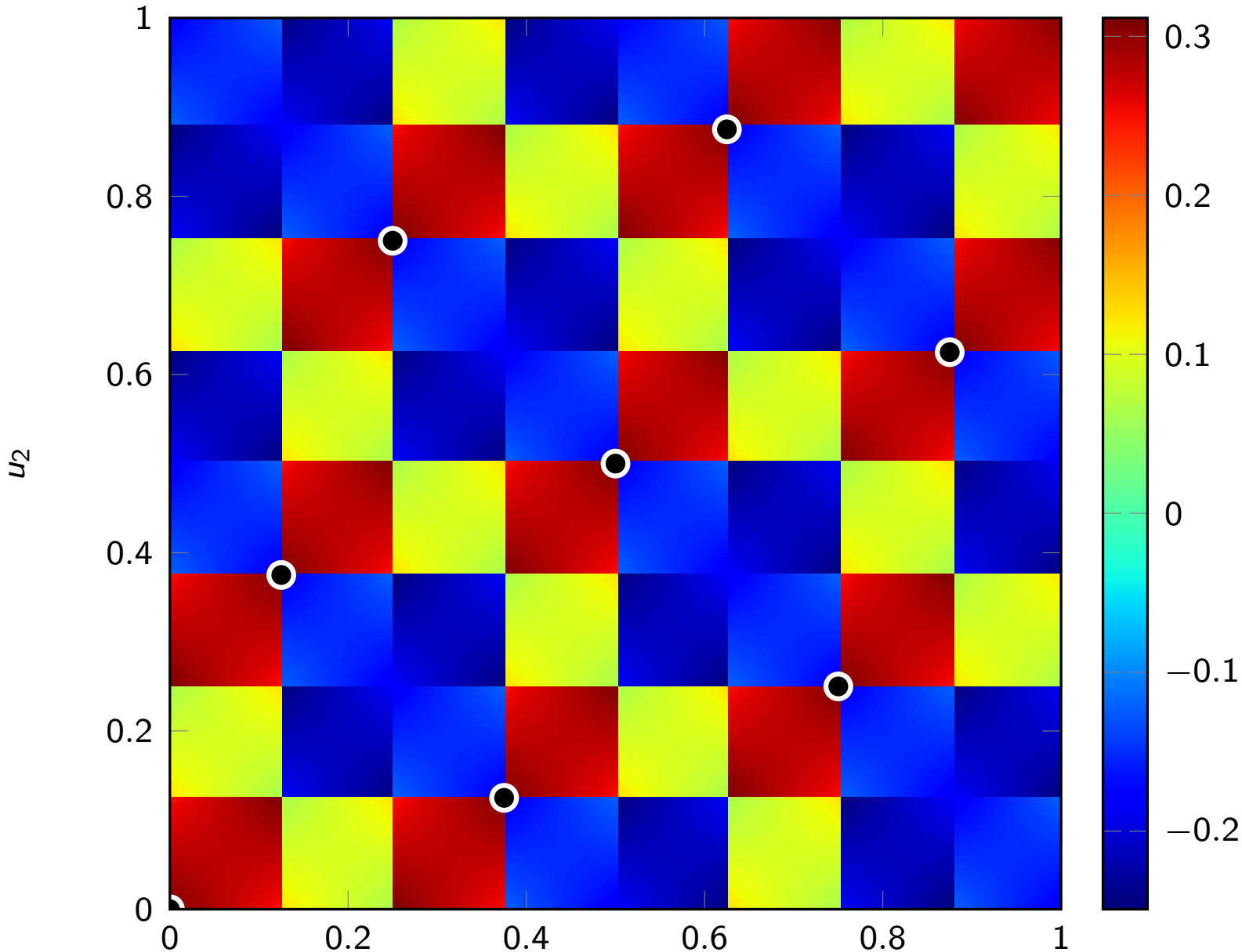
is a partition of $[0,1)^s$ in n regions of volume $1/n$.
Then, sampling the random shift $\mathbf{U}$ uniformly in any given R_i is equivalent to sampling it uniformly in $[0,1)^s$.

The error function

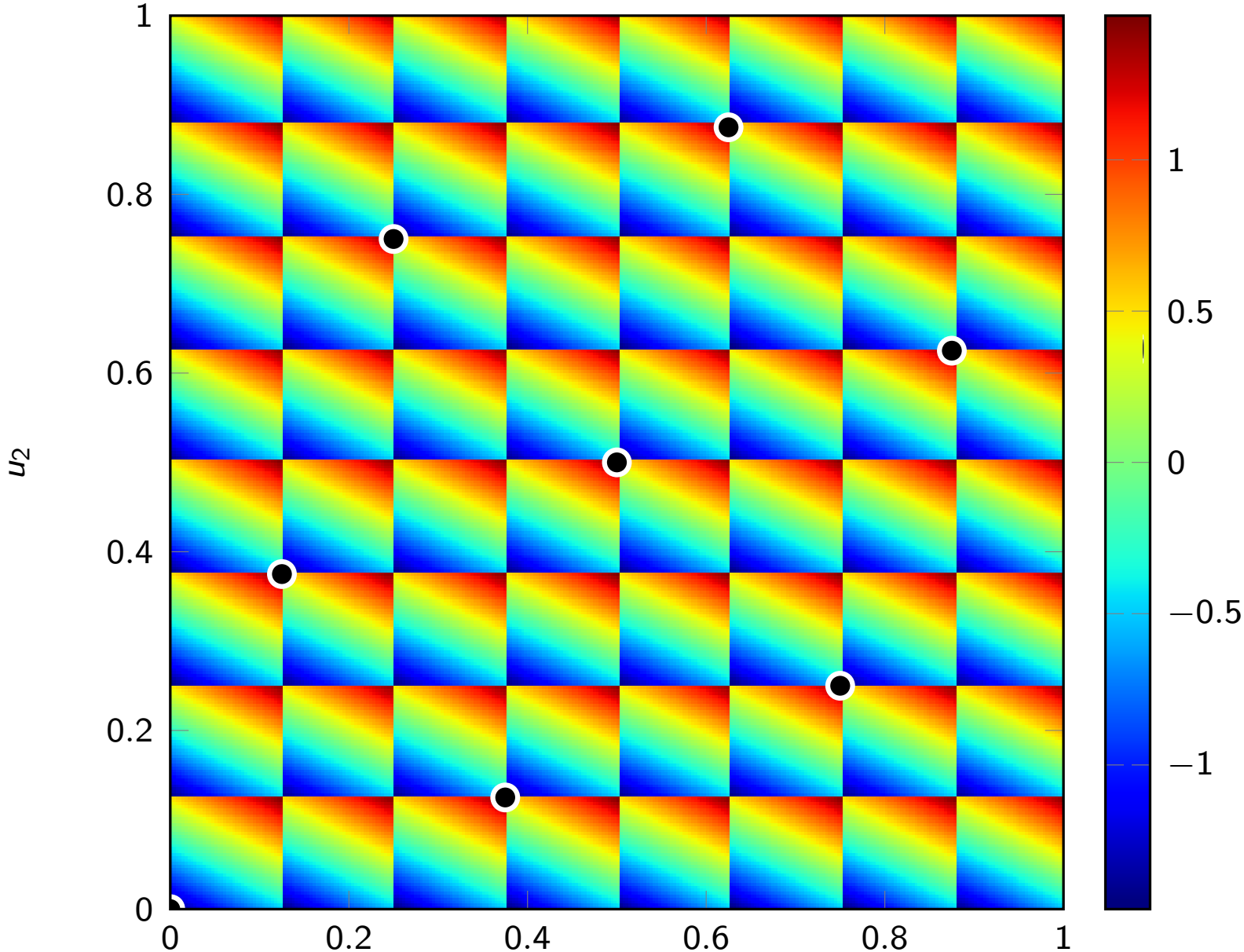
$$g_n(\mathbf{U}) = \hat{\mu}_{n,\mathrm{rqmc}} - \mu$$

over any R_i is the same as over R.

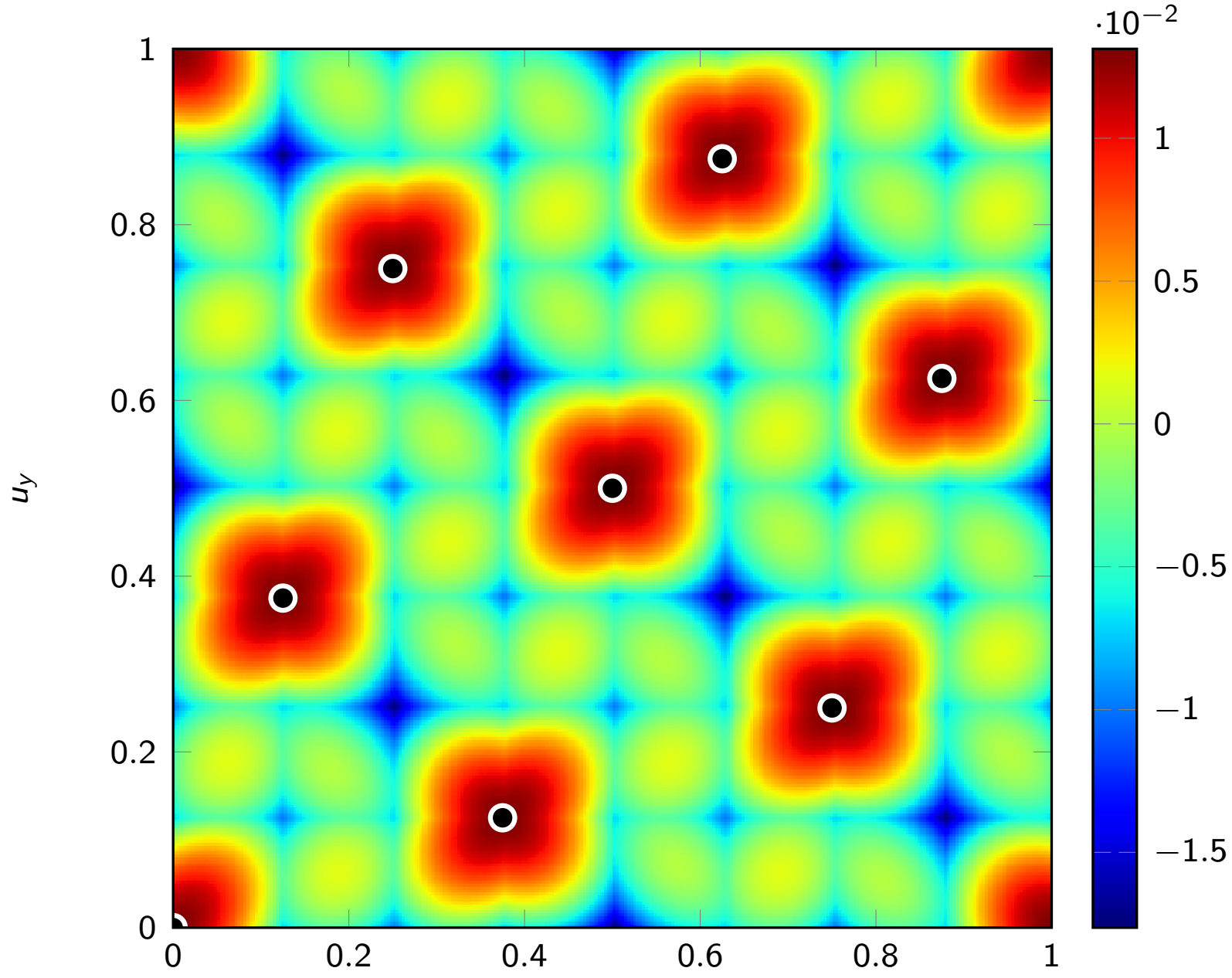
Error function $g_n(\mathbf{u})$ for $f(u_1, u_2) = (u_1 - 1/2)(u_2 - 1/2)$.



Error function $g_n(\mathbf{u})$ for $f(u_1, u_2) = (u_1 - 1/2) + (u_2 - 1/2)$.



Error function $g_n(\mathbf{u})$ for $f(u_1, u_2) = u_1 u_2 (u_1 - 1/2)(u_2 - 1/2)$.



Variance expression

Suppose f has Fourier expansion

$$f(\mathbf{u}) = \sum_{\mathbf{h}\in\mathbb{Z}^s} \hat{f}(\mathbf{h}) e^{2\pi\sqrt{-1}\mathbf{h}^t\mathbf{u}}.$$

For a randomly shifted lattice, the exact variance is (always)

$$\mathrm{Var}[\hat{\mu}_{n,\mathrm{rqmc}}] = \sum_{\mathbf{0}\neq\mathbf{h}\in L_s^*} |\hat{f}(\mathbf{h})|^2,$$

where $L_s^* = \{\mathbf{h} \in \mathbb{R}^s : \mathbf{h}^t\mathbf{v} \in \mathbb{Z} \text{ for all } \mathbf{v} \in L_s\} \subseteq \mathbb{Z}^s$ is the dual lattice.

From the viewpoint of variance reduction, an optimal lattice for f minimizes the square "discrepancy" $D^2(P_n) = \mathrm{Var}[\hat{\mu}_{n,\mathrm{rqmc}}]$.

$$\mathrm{Var}[\hat{\mu}_{n,\mathrm{rqmc}}] = \sum_{\mathbf{0}\neq\mathbf{h}\in L_s^*} |\hat{f}(\mathbf{h})|^2.$$

If f has square-integrable mixed partial derivatives up to order α, and the periodic continuation of its derivatives up to order $\alpha - 1$ is continuous across the unit cube boundaries, then

$$|\hat{f}(\mathbf{h})|^2 = \mathcal{O}((\max(1, h_1), \ldots, \max(1, h_s))^{-2\alpha}).$$

Moreover, there is a $\mathbf{v}_1 = \mathbf{v}_1(n)$ such that

$$\mathcal{P}_{2\alpha} \stackrel{\mathrm{def}}{=} \sum_{\mathbf{0}\neq\mathbf{h}\in L_s^*} (\max(1, h_1), \ldots, \max(1, h_s))^{-2\alpha} = \mathcal{O}(n^{-2\alpha+\delta}).$$

This is the variance for a worst-case f having

$$|\hat{f}(\mathbf{h})|^2 = (\max(1, h_1), \ldots, \max(1, h_s))^{-2\alpha}.$$

$$\mathrm{Var}[\hat{\mu}_{n,\mathrm{rqmc}}] = \sum_{\mathbf{0}\neq\mathbf{h}\in L_s^*} |\hat{f}(\mathbf{h})|^2.$$

If f has square-integrable mixed partial derivatives up to order α, and the periodic continuation of its derivatives up to order $\alpha - 1$ is continuous across the unit cube boundaries, then

$$|\hat{f}(\mathbf{h})|^2 = \mathcal{O}((\max(1, h_1), \ldots, \max(1, h_s))^{-2\alpha}).$$

Moreover, there is a $\mathbf{v}_1 = \mathbf{v}_1(n)$ such that

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This is the variance for a worst-case f having

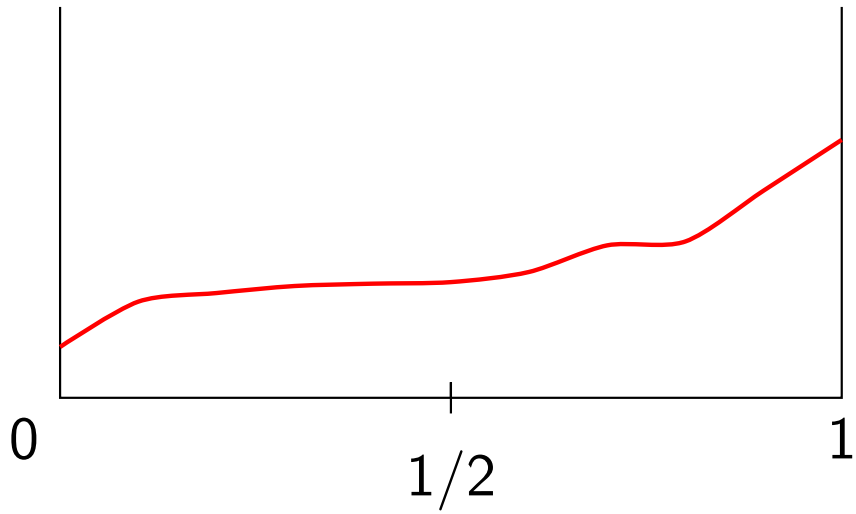
$$|\hat{f}(\mathbf{h})|^2 = (\max(1, h_1), \ldots, \max(1, h_s))^{-2\alpha}.$$

Beware of hidden factor in $\mathcal{O}$ when s is large.
This worst-case function may be far from representative in applications.

Baker's transformation

Want to make the periodic continuation continuous.

If $f(0) \neq f(1)$, define $\tilde{f}$ by $\tilde{f}(1-u) = \tilde{f}(u) = f(2u)$ for $0 \leq u \leq 1/2$.
This $\tilde{f}$ has the same integral as f and $\tilde{f}(0) = \tilde{f}(1)$.



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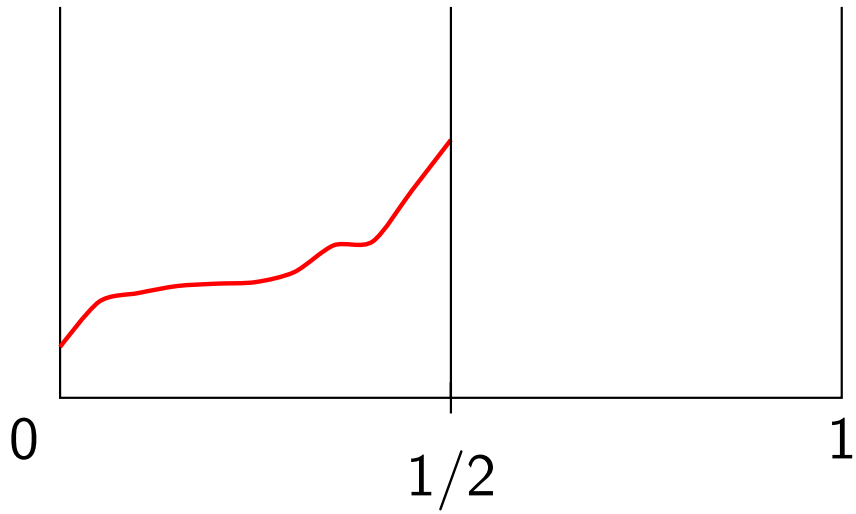

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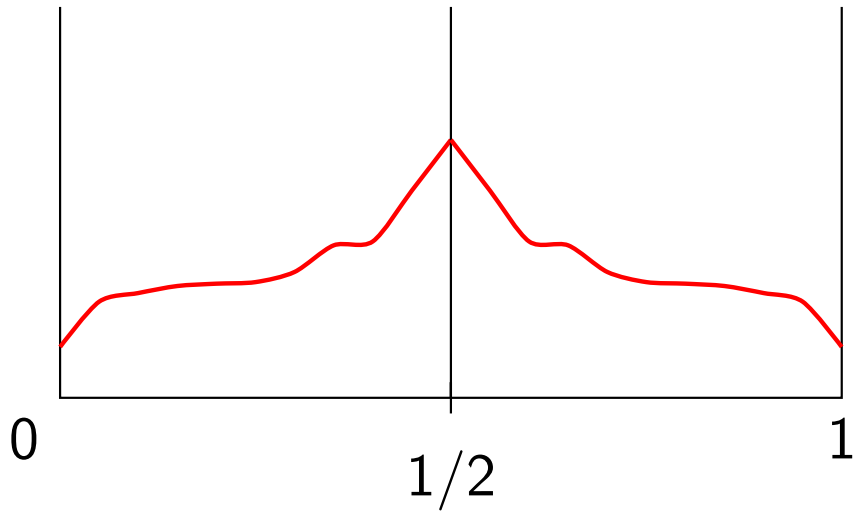

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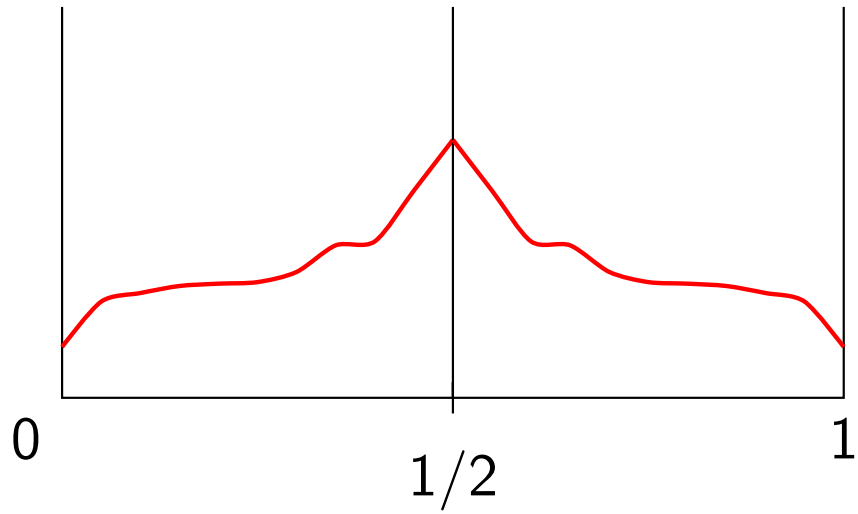
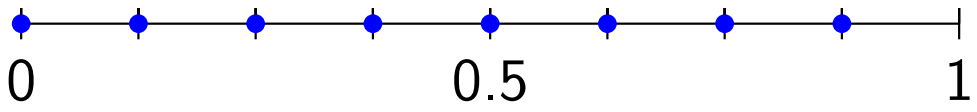


For smooth f, can reduce the variance to $O(n^{-4+\delta})$ (Hickernell 2002).
The resulting $\tilde{f}$ also symmetric with respect to $u = 1/2$.

In practice, we transform the points $\mathbf{U}_i$ instead of f.

One-dimensional case

Random shift followed by baker's transformation.
Along each coordinate, stretch everything by a factor of 2 and fold.
Same as replacing U_j by $\min[2U_j, 2(1 - U_j)]$.



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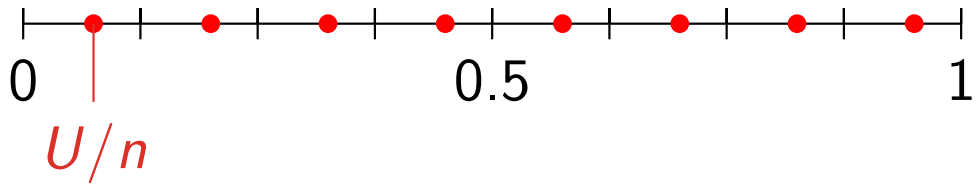

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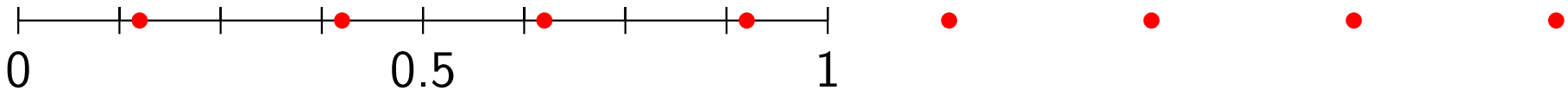

One-dimensional case

Random shift followed by baker's transformation.
Along each coordinate, stretch everything by a factor of 2 and fold.
Same as replacing U_j by $\min[2U_j, 2(1-U_j)]$.
Gives locally antithetic points in intervals of size $2/n$.

Searching for a lattice that minimizes

$$\mathrm{Var}[\hat{\mu}_{n,\mathrm{rqmc}}] = \sum_{\mathbf{0}\neq\mathbf{h}\in L_s^*} |\hat{f}(\mathbf{h})|^2$$

is unpractical, because:

- the Fourier coefficients are usually unknown,
- there are infinitely many,
- must do it for each f.

We nevertheless want to see how far we can go in that direction.

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is unpractical, because:

- the Fourier coefficients are usually unknown,
- there are infinitely many,
- must do it for each f.

We nevertheless want to see how far we can go in that direction.

We start with a simple function for which we know the Fourier expansion. Even then, the discrepancy involves an infinite number of terms!

Possible ideas: Truncate the sum to a finite subset B:

$$\sum_{\mathbf{0}\neq\mathbf{h}\in L_s^*\cap B} |\hat{f}(\mathbf{h})|^2,$$

or to the largest q square coefficients $|\hat{f}(\mathbf{h})|^2$. But hard to implement!

Dual-space exploration

The following makes sense if the $|\hat{f}(\mathbf{h})|^2$ tend to decrease with each $|h_j|$.

Start with a large set $\mathcal{L}$ of lattices (or generating vectors $\mathbf{v}_1$, for given n).

Search for vectors $\mathbf{h}$ with large weights $w(\mathbf{h}) = |\hat{f}(\mathbf{h})|^2$, via a neighborhood search starting at $\mathbf{h} = \mathbf{0}$, keeping a sorted list (as in Dijkstra's shortest path algorithm), and eliminate (successively) from $\mathcal{L}$ the lattices whose dual contains $\mathbf{h}$ for the next largest $w(\mathbf{h})$, until a single lattice remains.

Example of neighborhood $\mathcal{N}(\mathbf{h})$: only one coordinate differs, by one unit.

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Example of neighborhood $\mathcal{N}(\mathbf{h})$: only one coordinate differs, by one unit.

Component-by-component version: For $j = 1, 2, \ldots, s$, we apply the algorithm for a set $\mathcal{L}$ of j-dimensional lattices with common (fixed) $j - 1$ first coordinates, and determine the jth coordinate by visiting the j-dimensional vectors $\mathbf{h}$.

When the $|\hat{f}(\mathbf{h})|$ are unknown, we may think of estimating them as needed, dynamically.

Algorithm Dual-Space-Exploration(lattice set $\mathcal{L}$, weights w);

$\mathcal{Q} \leftarrow \mathcal{N}(\mathbf{0})$ // vectors $\mathbf{h}$ to be visited, sorted by weight $w(\mathbf{h})$;
$\mathcal{M} \leftarrow \mathcal{N}(\mathbf{0})$ // vectors $\mathbf{h}$ who already entered $\mathcal{Q}$;
while $|\mathcal{L}| > 1$ **do**
  $\mathbf{h} \leftarrow$ remove first from $\mathcal{Q}$;
  for all lattices $L_s \in \mathcal{L}$ such that $\mathbf{h} \in L_s^*$ **do**
    remove L_s from $\mathcal{L}$;
    if $|\mathcal{L}| = 1$ **then**
      return the single lattice $L_s \in \mathcal{L}$ and exit;
    end if
  end for
  for all $\mathbf{h}' \in \mathcal{N}(\mathbf{h}) \setminus \mathcal{M}$ **do**
    add $\mathbf{h}'$ to $\mathcal{M}$ and to $\mathcal{Q}$ with priority (weight) $w(\mathbf{h}')$;
  end for
end while

An example

Take the product V-shaped function

$$f(\mathbf{u}) = \prod_{j=1}^{s} \frac{|4u_j - 2| + c_j}{1 + c_j},$$

so

$$\hat{f}(\mathbf{h}) = \prod_{\{j\,:\,h_j \text{ is odd}\}} \frac{4}{(1 + c_j)\pi^2 h_j^2}.$$

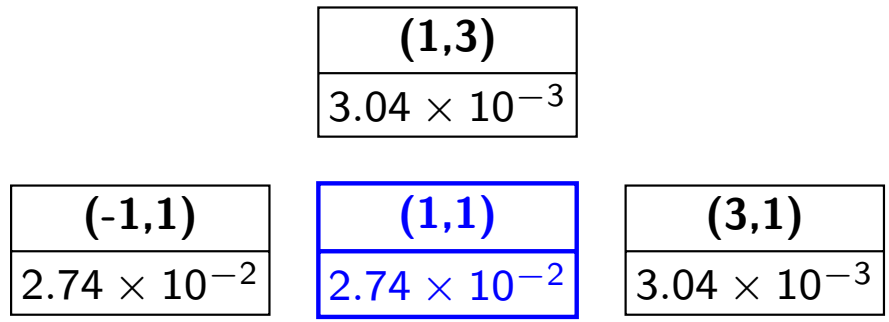
Dimensions $s = 5$ and 10.
Constants $c_j = 1, j, j^2, j^3$.

The Dual Exploration Algorithm in Action

(1,1)
2.74×10^{-2}

Colors: [in $\mathcal{Q}$] [in $\mathcal{M} \setminus \mathcal{Q}$] [visiting]

The Dual Exploration Algorithm in Action



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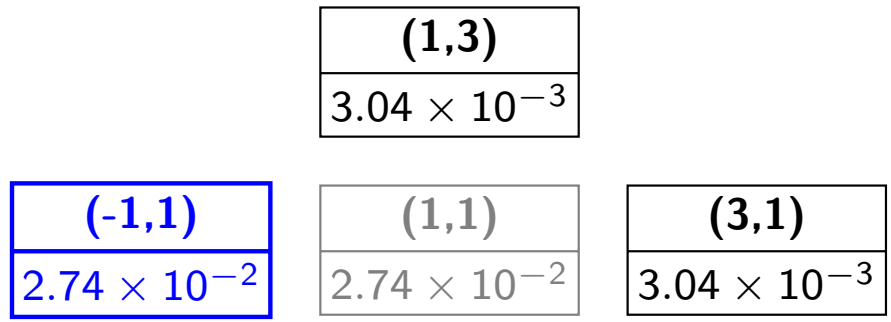

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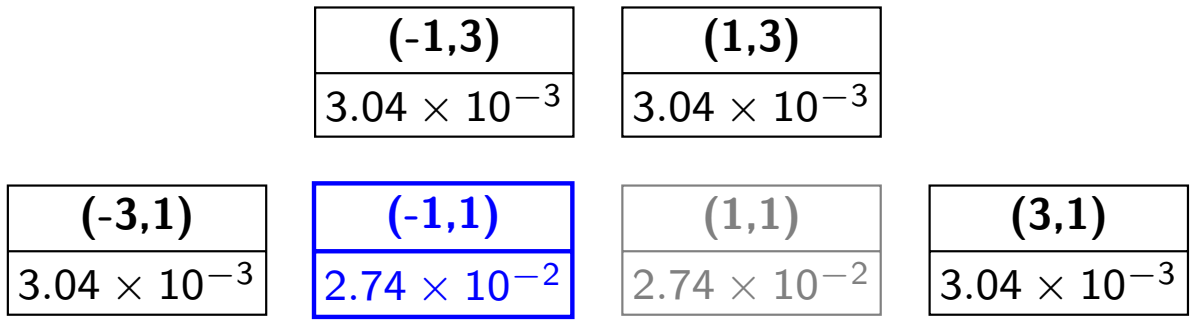

Colors: [in $\mathcal{Q}$] [in $\mathcal{M} \setminus \mathcal{Q}$] [visiting]

The Dual Exploration Algorithm in Action

	(-1,3)	**(1,3)**	
	3.04×10^{-3}	3.04×10^{-3}	
(-3,1)	(-1,1)	(1,1)	**(3,1)**
3.04×10^{-3}	2.74×10^{-2}	2.74×10^{-2}	3.04×10^{-3}

Colors: [in $\mathcal{Q}$] [in $\mathcal{M} \setminus \mathcal{Q}$] [visiting]

The Dual Exploration Algorithm in Action

	(-3,3)	**(-1,3)**	**(1,3)**		
	3.38×10^{-4}	3.04×10^{-3}	3.04×10^{-3}		
(-5,1)	**(-3,1)**	(-1,1)	(1,1)	**(3,1)**	
1.10×10^{-3}	3.04×10^{-3}	2.74×10^{-2}	2.74×10^{-2}	3.04×10^{-3}	

Colors: [in $\mathcal{Q}$] [in $\mathcal{M} \setminus \mathcal{Q}$] [visiting]

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	(-3,3)	**(-1,3)**	**(1,3)**	
	3.38×10^{-4}	3.04×10^{-3}	3.04×10^{-3}	
(-5,1)	**(-3,1)**	**(-1,1)**	**(1,1)**	**(3,1)**
1.10×10^{-3}	3.04×10^{-3}	2.74×10^{-2}	2.74×10^{-2}	3.04×10^{-3}

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		(-1,5)		
		1.10×10^{-3}		
	(-3,3)	**(-1,3)**	**(1,3)**	
	3.38×10^{-4}	3.04×10^{-3}	3.04×10^{-3}	
(-5,1)	**(-3,1)**	**(-1,1)**	**(1,1)**	**(3,1)**
1.10×10^{-3}	3.04×10^{-3}	2.74×10^{-2}	2.74×10^{-2}	3.04×10^{-3}

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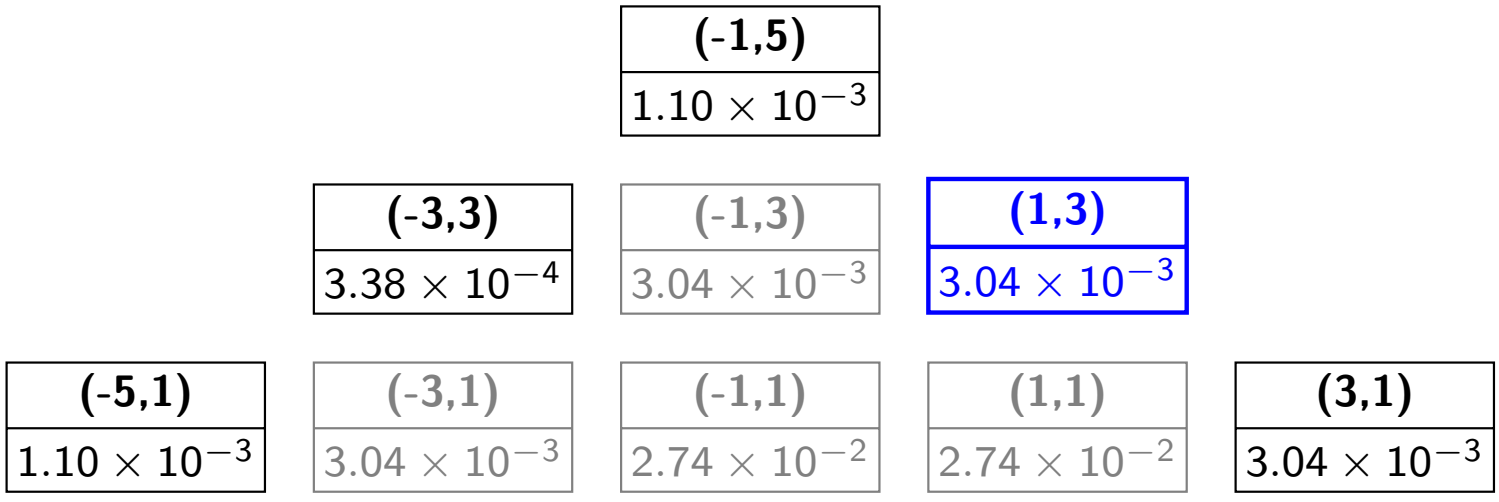


Colors: [in $\mathcal{Q}$] [in $\mathcal{M} \setminus \mathcal{Q}$] [visiting]

The Dual Exploration Algorithm in Action

		(-1,5) 1.10×10^{-3}	**(1,5)** 1.10×10^{-3}	
	(-3,3) 3.38×10^{-4}	(-1,3) 3.04×10^{-3}	**(1,3)** 3.04×10^{-3}	**(3,3)** 3.38×10^{-4}
(-5,1) 1.10×10^{-3}	(-3,1) 3.04×10^{-3}	(-1,1) 2.74×10^{-2}	(1,1) 2.74×10^{-2}	**(3,1)** 3.04×10^{-3}

Colors: [in $\mathcal{Q}$] [in $\mathcal{M} \setminus \mathcal{Q}$] [visiting]

The Dual Exploration Algorithm in Action

		(-1,5) 1.10×10^{-3}	**(1,5)** 1.10×10^{-3}	
	(-3,3) 3.38×10^{-4}	(-1,3) 3.04×10^{-3}	(1,3) 3.04×10^{-3}	**(3,3)** 3.38×10^{-4}
(-5,1) 1.10×10^{-3}	(-3,1) 3.04×10^{-3}	(-1,1) 2.74×10^{-2}	(1,1) 2.74×10^{-2}	**(3,1)** 3.04×10^{-3}

Colors: [in $\mathcal{Q}$] [in $\mathcal{M} \setminus \mathcal{Q}$] [visiting]

The Dual Exploration Algorithm in Action

		(-1,5)	**(1,5)**		
		1.10×10^{-3}	1.10×10^{-3}		
	(-3,3)	(-1,3)	(1,3)	**(3,3)**	
	3.38×10^{-4}	3.04×10^{-3}	3.04×10^{-3}	3.38×10^{-4}	
(-5,1)	(-3,1)	(-1,1)	(1,1)	**(3,1)**	**(5,1)**
1.10×10^{-3}	3.04×10^{-3}	2.74×10^{-2}	2.74×10^{-2}	3.04×10^{-3}	1.10×10^{-3}

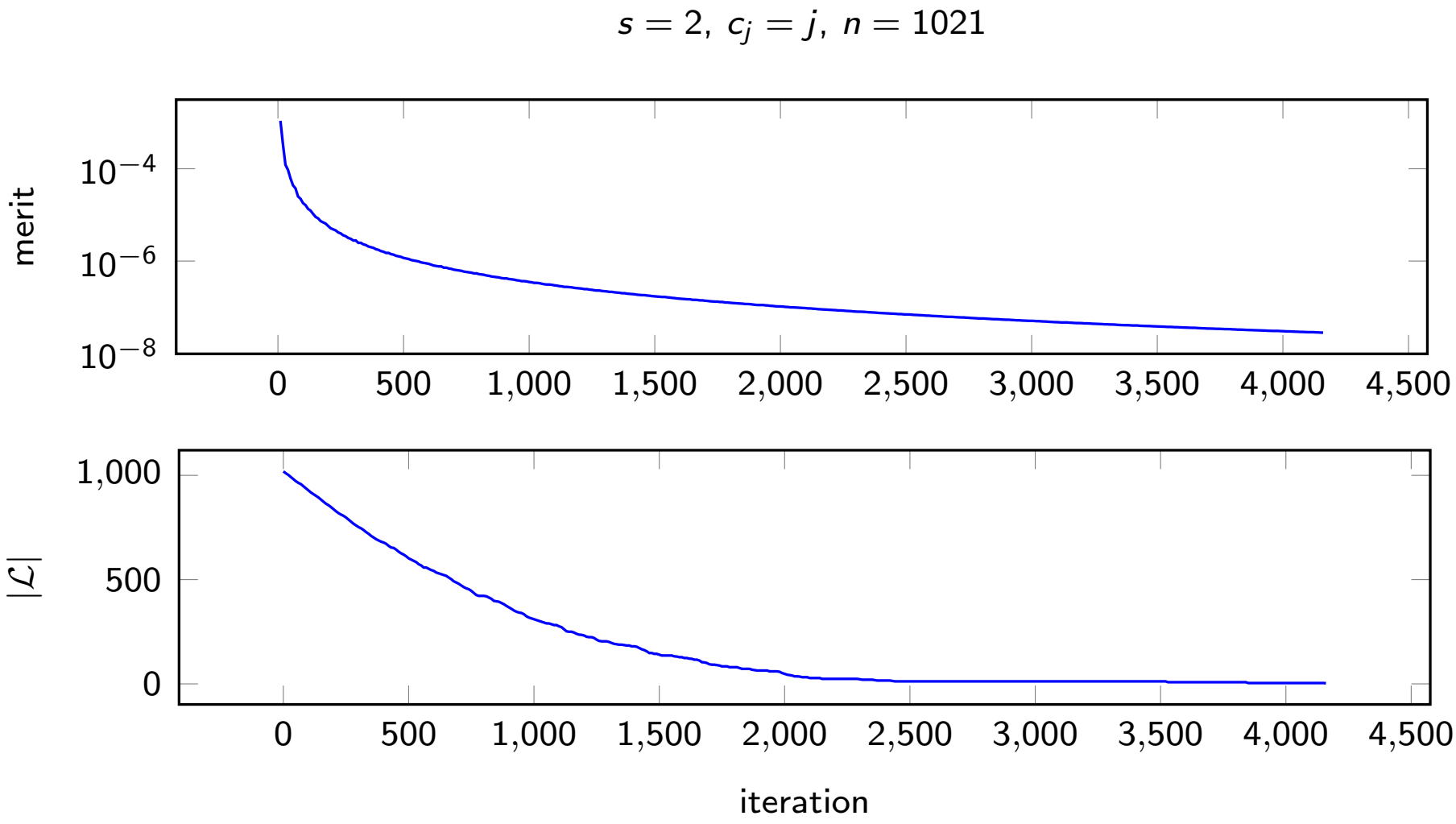
Colors: [in $\mathcal{Q}$] [in $\mathcal{M} \setminus \mathcal{Q}$] [visiting]

The Dual Exploration Algorithm in Action

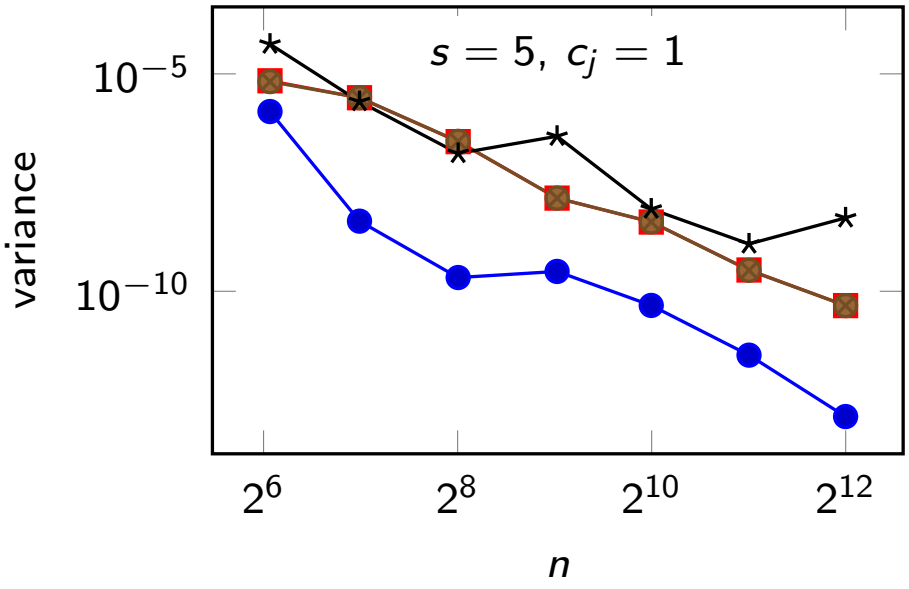
		(-1,5)	**(1,5)**			
		1.10×10^{-3}	1.10×10^{-3}			
	(-3,3)	(-1,3)	(1,3)	**(3,3)**		
	3.38×10^{-4}	3.04×10^{-3}	3.04×10^{-3}	3.38×10^{-4}		
(-5,1)	(-3,1)	(-1,1)	(1,1)	(3,1)	**(5,1)**	
1.10×10^{-3}	3.04×10^{-3}	2.74×10^{-2}	2.74×10^{-2}	3.04×10^{-3}	1.10×10^{-3}	

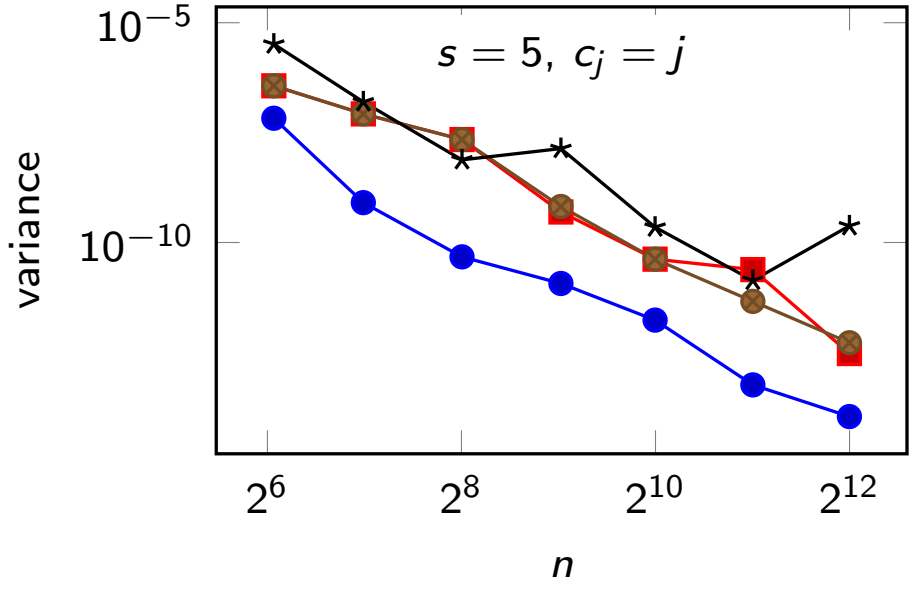
Colors: [in $\mathcal{Q}$] [in $\mathcal{M} \setminus \mathcal{Q}$] [visiting]

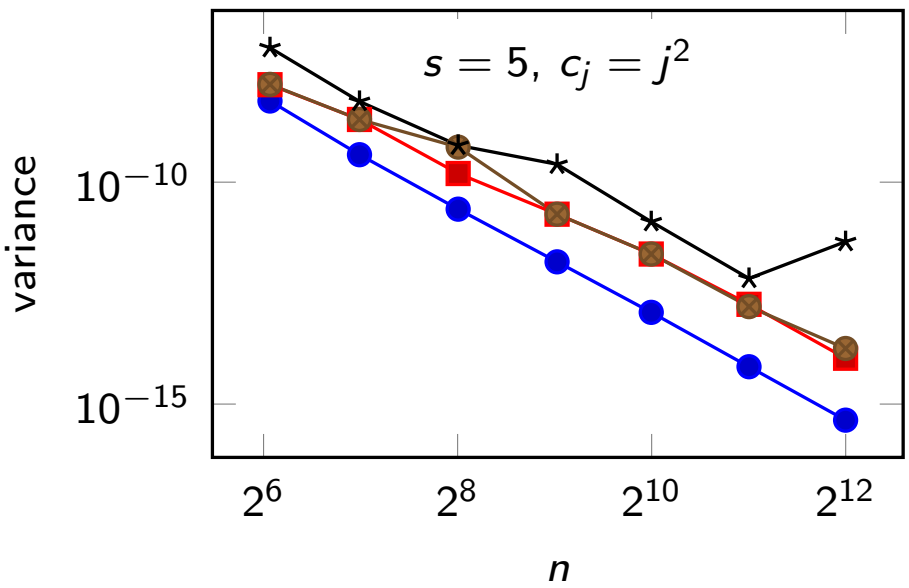
Evolution of the Dual Exploration Algorithm

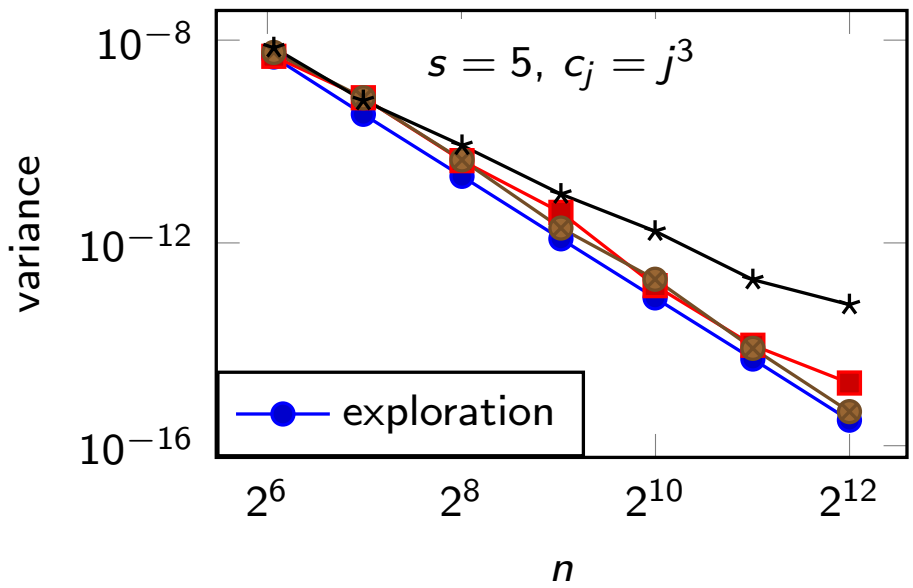


Estimated variance vs n for $s = 5$

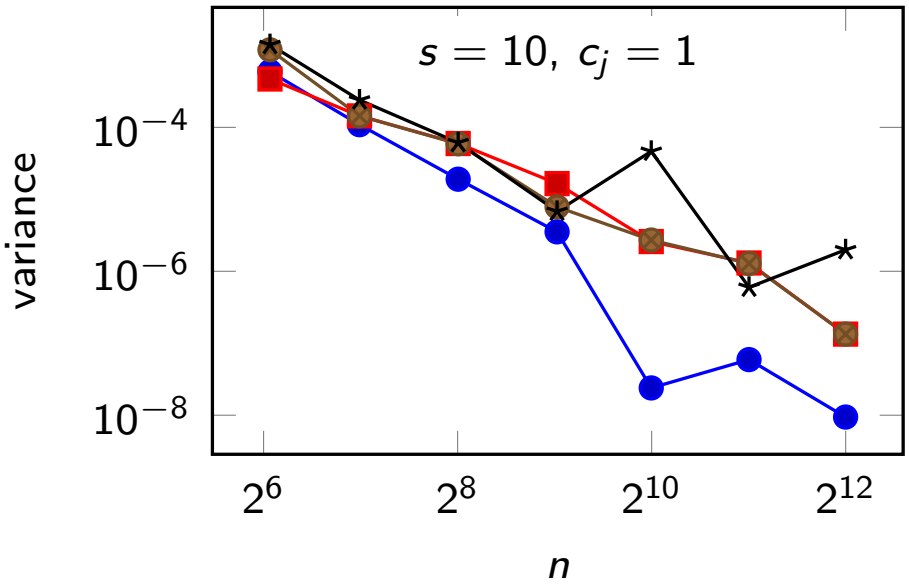


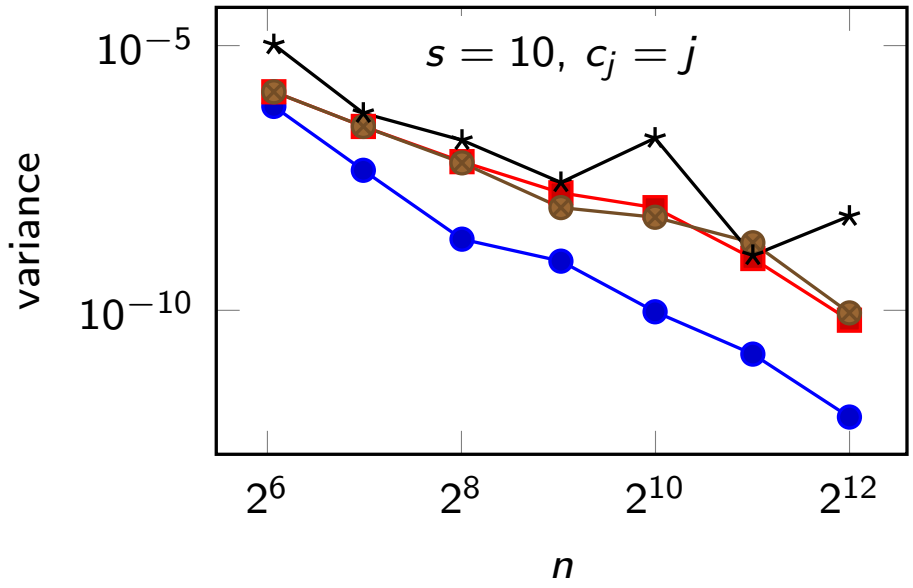


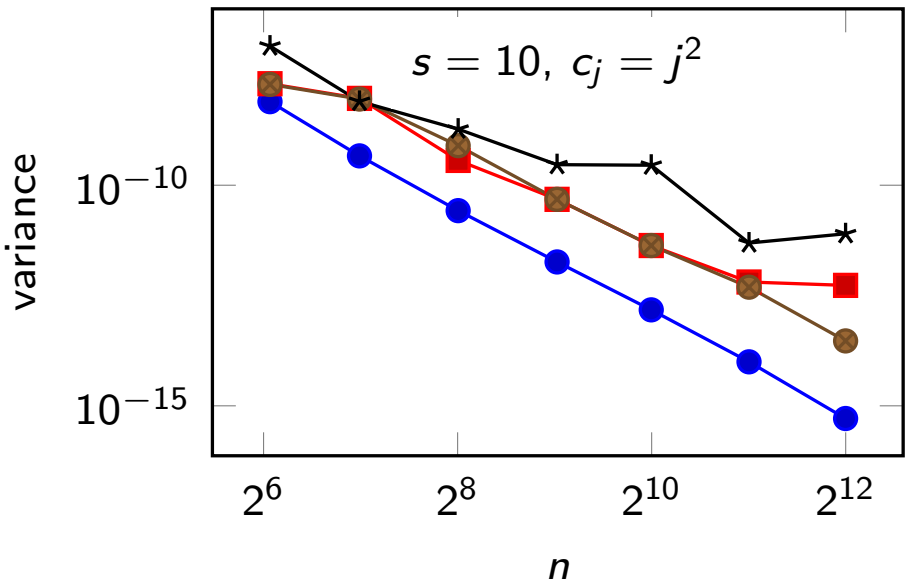


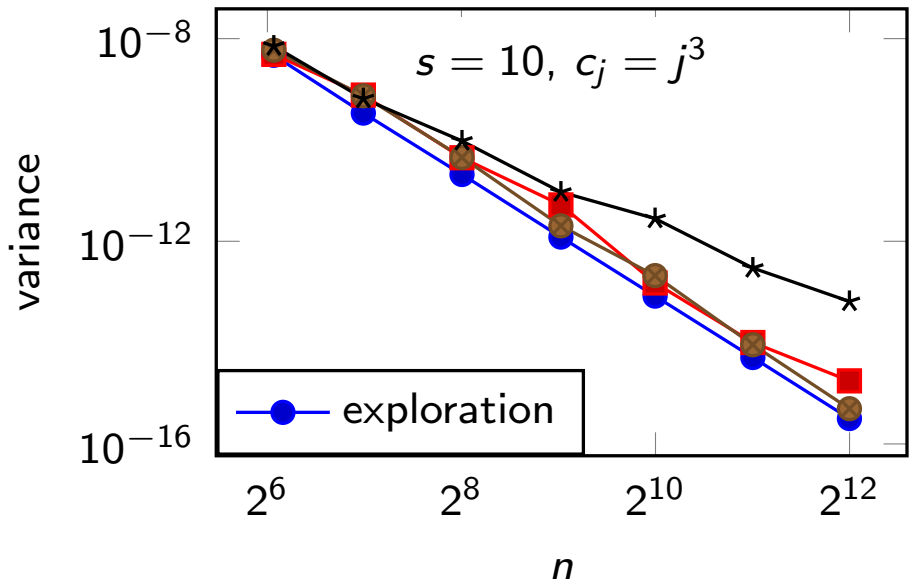


Estimated variance vs n for $s = 10$









ANOVA decomposition

The Fourier expansion has too many terms to handle. As a cruder expansion, we can write $f(\mathbf{u}) = f(u_1, \ldots, u_s)$ as:

$$f(\mathbf{u}) = \sum_{\mathfrak{u} \subseteq \{1,\ldots,s\}} f_{\mathfrak{u}}(\mathbf{u}) = \mu + \sum_{i=1}^{s} f_{\{i\}}(u_i) + \sum_{i,j=1}^{s} f_{\{i,j\}}(u_i, u_j) + \cdots$$

where

$$f_{\mathfrak{u}}(\mathbf{u}) = \int_{[0,1)^{|\bar{\mathfrak{u}}|}} f(\mathbf{u})\, \mathrm{d}\mathbf{u}_{\bar{\mathfrak{u}}} - \sum_{\mathfrak{v} \subset \mathfrak{u}} f_{\mathfrak{v}}(\mathbf{u}_{\mathfrak{v}}),$$

and the Monte Carlo variance decomposes as

$$\sigma^2 = \sum_{\mathfrak{u} \subseteq \{1,\ldots,s\}} \sigma_{\mathfrak{u}}^2, \qquad \text{where } \sigma_{\mathfrak{u}}^2 = \mathrm{Var}[f_{\mathfrak{u}}(\mathbf{U})].$$

Sensitivity indices: $S_{\mathfrak{u}} = \sigma_{\mathfrak{u}}^2/\sigma^2$. Can be estimated by MC or RQMC.

Heuristic intuition: Make sure the projections of P_n are very uniform for the important subsets $\mathfrak{u}$ (i.e., with large $S_{\mathfrak{u}}$).

Shift-invariant discrepancy

In a reproducing kernel Hilbert space (RKHS) with kernel K, and randomly-shifted points, the relevant discrepancy corresponds to the shift-invariant kernel

$$K_{\mathrm{sh}}(\mathbf{u}_i, \mathbf{u}_j) := \mathbb{E}[K(\mathbf{U}_i, \mathbf{U}_j)] = \mathbb{E}[K(\mathbf{u}_i + \mathbf{U}, \mathbf{u}_j + \mathbf{U})] = \mathbb{E}[K(\mathbf{u}_i - \mathbf{u}_j + \mathbf{U}, \mathbf{U})].$$

The mean square discrepancy can be written as

$$\mathbb{E}[D^2(P_n)] = \frac{1}{n} \sum_{\mathbf{0} \neq \mathbf{h} \in \mathbb{Z}^s} w(\mathbf{h}) \sum_{i=0}^{n-1} e^{2\pi\sqrt{-1}\mathbf{h}^{\mathrm{t}}\mathbf{u}_i} = \sum_{\mathbf{0} \neq \mathbf{h} \in L_*^s} w(\mathbf{h})$$

(for a lattice).

Key issue: choice of the weights $w(\mathbf{h})$.

Regrouping by projections: projection weights

Denote $\mathfrak{u}(\mathbf{h}) = \mathfrak{u}(h_1, \ldots, h_s)$ the set of indices j for which $h_j \neq 0$. We have

$$\mathbb{E}[D^2(P_n)] = \sum_{\mathfrak{u} \subseteq \{1,\ldots,s\}} \sum_{\mathbf{h}:\mathfrak{u}(\mathbf{h})=\mathfrak{u}} w(\mathbf{h}) \frac{1}{n} \sum_{i=0}^{n-1} e^{2\pi\sqrt{-1}\mathbf{h}^t\mathbf{u}_i} = \sum_{\mathfrak{u} \subseteq \{1,\ldots,s\}} D^2_{\mathfrak{u}}(P_n).$$

The RKHS decomposes as a direct sum and the RQMC variance has a corresponding decomposition.

$$\mathrm{Var}[\hat{\mu}_{n,\mathrm{rqmc}}] = \sum_{\mathfrak{u} \subseteq \{1,\ldots,s\}} \mathrm{Var}[\hat{\mu}_{n,\mathrm{rqmc}}(f_{\mathfrak{u}})].$$

Restriction on the weights: take $w(\mathbf{h}) = \gamma_{\mathfrak{u}(\mathbf{h})} \prod_{j \in \mathfrak{u}} h_j^{-2\alpha}$ for all $\mathbf{h}$.
Those projection weights are the so-called general weights.

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The RKHS decomposes as a direct sum and the RQMC variance has a corresponding decomposition.

$$\mathrm{Var}[\hat{\mu}_{n,\mathrm{rqmc}}] = \sum_{\mathfrak{u} \subseteq \{1,\ldots,s\}} \mathrm{Var}[\hat{\mu}_{n,\mathrm{rqmc}}(f_{\mathfrak{u}})].$$

Restriction on the weights: take $w(\mathbf{h}) = \gamma_{\mathfrak{u}(\mathbf{h})} \prod_{j \in \mathfrak{u}} h_j^{-2\alpha}$ for all $\mathbf{h}$.
Those projection weights are the so-called general weights.
Anyhow, how should we choose them?

Example: a weighted Sobolev space

Space of functions with integrable partial derivatives. RKHS with kernel

$$K(\mathbf{u},\mathbf{x}) = \sum_{u\subseteq\{1,\ldots,s\}} \gamma_u \prod_{j\in u} 2\pi^2 \left[B_2((u_j - x_j) \bmod 1)/2 + (u_j - 0.5)(x_j - 0.5)\right]$$

where $B_2(u) = u^2 - u + 1/6$. The shift-invariant kernel is

$$K_{\mathrm{sh}}(\mathbf{u},\mathbf{x}) = \sum_{u\subseteq\{1,\ldots,s\}} \gamma_u \prod_{j\in u} 2\pi^2 B_2((u_j - x_j) \bmod 1)$$

and the corresponding mean square discrepancy for a randomly-shifted lattice rule with $\mathbf{v}_1 = (v_1,\ldots,v_s) = \mathbf{z}/n$ is

$$\mathbb{E}[D^2(P_n)] = \frac{1}{n}\sum_{i=1}^{n} \sum_{u\subseteq\{1,\ldots,s\}} \gamma_u \prod_{j\in u} 2\pi^2 B_2((i\,v_j) \bmod 1).$$

From the Fourier expansion of B_2, we also have

$$\begin{aligned}\mathbb{E}[D^2(P_n)] &= \frac{1}{n}\sum_{i=1}^{n}\sum_{u\subseteq\{1,\ldots,s\}}\gamma_u\prod_{j\in u}\sum_{h_j\neq 0} h_j^{-2}e^{2\pi\sqrt{-1}ih_jv_j}\\ &= \sum_{\mathbf{0}\neq\mathbf{h}\in L_s^*}\gamma_{u(\mathbf{h})}\prod_{j\in u(\mathbf{h})}h_j^{-2} = \sum_{\mathbf{0}\neq\mathbf{h}\in L_s^*} w(\mathbf{h}).\end{aligned}$$

For those weights, we have $w(\mathbf{h}) = |\hat{f}(\mathbf{h})|^2$ for the function

$$f(\mathbf{u}) = \sum_{u\subseteq\{1,\ldots,s\}}(2\pi)^{|u|}\gamma_u^{1/2}\prod_{j\in u}(u_j - 0.5),$$

so $\mathbb{E}[D^2(P_n)]$ is the RQMC variance for this f.

On the other hand, the ANOVA variance components for this f are

$$\sigma_u^2 = (4\pi^2)^{|u|}\gamma_u\prod_{j\in u}\mathrm{Var}[U-0.5] = (4\pi^2/12)^{|u|}\gamma_u = (\pi^2/3)^{|u|}\gamma_u.$$

The optimal weights for this f are then

$$\gamma_u = (3/\pi^2)^{|u|}\sigma_u^2 \approx (0.30396)^{|u|}\sigma_u^2.$$

Using the same kernel and a different heuristic argument, Wang and Sloan (2006) come up with weights that generalize to (they do this for product weights only):

$$\gamma_{\mathfrak{u}}^2 = (45/\pi^4)^{|\mathfrak{u}|}\sigma_{\mathfrak{u}}^2,$$

that is,

$$\gamma_{\mathfrak{u}} = (\sqrt{45}/\pi^2)^{|\mathfrak{u}|}\sigma_{\mathfrak{u}} \approx (0.6797)^{|\mathfrak{u}|}\sigma_{\mathfrak{u}},$$

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With $\gamma_{\mathfrak{u}} = 1$, we obtain the classical (unweighted) $\mathcal{P}_2$.

Weighted $\mathcal{P}_{2\alpha}$:

$$\mathcal{P}_{2\alpha} = \sum_{\mathbf{0}\neq\mathbf{h}\in L_s^*} \gamma_{\mathfrak{u}(\mathbf{h})}(\max(1,h_1),\ldots,\max(1,h_s))^{-2\alpha}$$

Variance for a worst-case function whose square Fourier coefficients are

$$|\hat{f}(\mathbf{h})|^2 = \gamma_{\mathfrak{u}(\mathbf{h})}(\max(1,h_1),\ldots,\max(1,h_s))^{-2\alpha}.$$

This is the RQMC variance for the function

$$f(\mathbf{u}) = \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \sqrt{\gamma_{\mathfrak{u}}} \prod_{j\in\mathfrak{u}} \frac{(2\pi)^\alpha}{\alpha!} B_\alpha(u_j).$$

We also have for this f:

$$\sigma_{\mathfrak{u}}^2 = \gamma_{\mathfrak{u}}\left[\mathrm{Var}[B_\alpha(U)]\frac{(4\pi^2)^\alpha}{(\alpha!)^2}\right]^{|\mathfrak{u}|} = \gamma_{\mathfrak{u}}\left[|B_{2\alpha}(0)|\frac{(4\pi^2)^\alpha}{(2\alpha)!}\right]^{|\mathfrak{u}|}.$$

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For $\alpha = 1$, we should take $\gamma_u = (3/\pi^2)^{|u|}\sigma_u^2 \approx (0.30396)^{|u|}\sigma_u^2$.
For $\alpha = 2$, we should take $\gamma_u = [45/\pi^4]^{|u|}\sigma_u^2 \approx (0.46197)^{|u|}\sigma_u^2$.

The ratios weight / variance should decrease exponentially with $|u|$.

Heuristics for choosing the weights

Idea 1: take $\gamma_u \approx \sigma_u^2$ or $\gamma_u \approx S_u$ for each u. Too simplistic.

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Idea 3: In general, one can define a simple parametric model for the weights and then estimate the parameters by matching the ANOVA variances (e.g., Wang and Sloan 2006).

For example, $\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$ for some constants $\gamma_j \geq 0$ (product weights).

Fewer parameters: take $\gamma_j = a\beta^j$ for $a, \beta > 0$ (geometric).

With a weighted $\mathcal{P}_\alpha$-type criterion, we should have $\gamma_{\mathfrak{u}} = \rho^{|\mathfrak{u}|}\sigma^2_{\mathfrak{u}}$ for some $\rho > 0$.

Proposal 4: A strategy for order-dependent weights.

Assume $\gamma_{\mathfrak{u}} = \Gamma_{|\mathfrak{u}|}$. Need to select $\Gamma_1, \ldots, \Gamma_s$.

For each $\mathfrak{u}$, let $v_{\mathfrak{u}}^2$ be an estimate of the optimal $\gamma_{\mathfrak{u}}$.

Strategy: take Γ_r as the average

$$\Gamma_r = \binom{s}{r}^{-1} \sum_{\{\mathfrak{u}:|\mathfrak{u}|=r\}} v_{\mathfrak{u}}^2.$$

Here, scaling all weights by the same factor changes nothing.

Proposal 5: A strategy for product weights.

Ignore one-dimensional projections; they are the same for all lattices.

The idea is to fit the estimated "optimal weights" over all two-dimensional projections via a least-squares procedure. Then we rescale all the weights by a constant factor to match the ratio of average estimated "optimal weights" over the three-dimensional projections to that over the two-dimensional projections.

Let τ_j be the unscaled weight for projection j. We first minimize

$$R = \sum_{k=1}^{s} \sum_{j=1}^{k-1} \left(\tau_j \tau_k - v_{\{j,k\}}^2 \right)^2 .$$

Differentiating w.r.t. τ_j and equaling to 0, we obtain, for each j,

$$\tau_j \sum_{k=1,\, k \neq j}^{s} \tau_k^2 = \sum_{k=1,\, k \neq j}^{s} \tau_k v_{\{j,k\}}^2 .$$

This can be solved by an iterative fixed-point algorithm:

$$\tau_j^{(0)} = \max_{k,l=1,\ldots,s} v_{\{k,l\}}, \qquad \tau_j^{(i+1)} = \frac{\sum_{k=1,\, k\neq j}^{s} \tau_k^{(i)} v_{\{j,k\}}^2}{\sum_{k=1,\, k\neq j}^{s} \left(\tau_k^{(i)}\right)^2},$$

for $i = 1, 2, \ldots$.

We then rescale the weights via $\gamma_j = c\tau_j$ where the constant c satisfies

$$\frac{\sum_{k=1}^{s}\sum_{j=1}^{k-1} \tau_j\tau_k}{\sum_{k=1}^{s}\sum_{j=1}^{k-1}\sum_{l=1}^{j-1} \tau_j\tau_k\tau_l} = c\, \frac{\sum_{k=1}^{s}\sum_{j=1}^{k-1} v_{\{j,k\}}^2}{\sum_{k=1}^{s}\sum_{j=1}^{k-1}\sum_{l=1}^{j-1} v_{\{j,k,l\}}^2}.$$

Idea 6: Control the shortest vector in dual lattice, for each projection.

Spectral test for LCGs (Knuth, Fishman, etc.):

$$\min_{2 \le r \le t_1} \frac{\ell_{\{1,\ldots,r\}}}{\ell_r^*(n)}$$

where $\ell_{\mathfrak{u}}$ is the length of a shortest vector in $L_s^*(\mathfrak{u})$ and $\ell_r^*(n)$ is a theoretical upper bound on this length, in r dimensions.

Advantages: Computing time of $\ell_{\mathfrak{u}}$ are almost independent of n, although exponential in $|\mathfrak{u}|$. Poor lattices can be eliminated quickly: search is fast.

Idea 6: Control the shortest vector in dual lattice, for each projection.

Lemieux and L'Ecuyer (2000, etc.) maximize

$$M_{t_1,\ldots,t_d} = \min \left[\min_{2\le r\le t_1} \frac{\ell_{\{1,\ldots,r\}}}{\ell_r^*(n)}, \min_{2\le r\le d} \min_{\substack{\mathfrak{u}=\{j_1,\ldots,j_r\}\subset\{1,\ldots,s\} \\ 1=j_1<\cdots<j_r\le t_r}} \frac{\ell_{\mathfrak{u}}}{\ell_r^*(n)} \right],$$

where $\ell_{\mathfrak{u}}$ is the length of a shortest vector in $L_s^*(\mathfrak{u})$ and $\ell_r^*(n)$ is a theoretical upper bound on this length, in r dimensions.

Advantages: Computing time of $\ell_{\mathfrak{u}}$ are almost independent of n, although exponential in $|\mathfrak{u}|$. Poor lattices can be eliminated quickly: search is fast.

This can of course be generalized by adding weights to projections.

Searching for lattice parameters

Korobov lattices.

Search for $\mathbf{z} = (1, a, a^2, \ldots, \ldots)$ over all admissible integers a.

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Component by component (CBC) construction.

Let $z_1 = 1$;

For $j = 2, \ldots, s$, find $z_j \in \{1, \ldots, n-1\}$, $\gcd(z_j, n) = 1$, such that $(z_1, \ldots, z_{j-1}, z_j)$ minimizes the selected discrepancy for the first j dimensions.

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Partial randomized CBC construction.

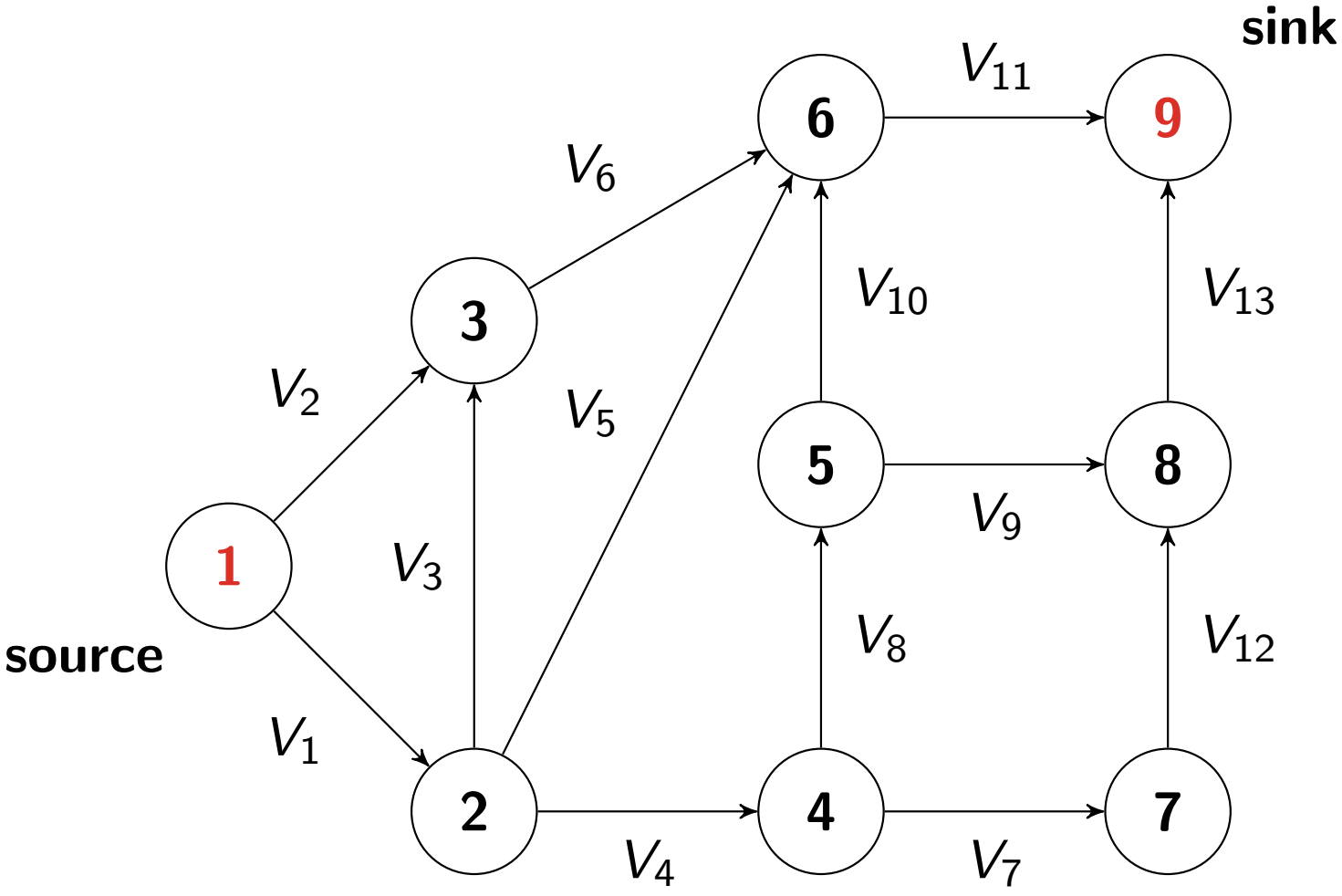
Let $z_1 = 1$;
For $j = 2, \ldots, s$, try r random $z_j \in \{1, \ldots, n-1\}$, $\gcd(z_j, n) = 1$, and retain the one for which $(z_1, \ldots, z_{j-1}, z_j)$ minimizes the selected discrepancy for the first j dimensions.

Example: stochastic activity network

[Elmaghraby 1977]. Each arc j has random length $V_j = F_j^{-1}(U_j)$.

Let $T = f(U_1, \ldots, U_{13})$ = length of longest path from node 1 to node 9.

Want to estimate $q(x) = \mathbb{P}[T > x]$ for a given constant x.



To estimate $q(x)$ by **MC**, we generate n independent realizations of T, say $T_1, \ldots, T_n$, and take $(1/n)\sum_{i=1}^{n} \mathbb{I}[T_i > x]$.

For **RQMC**, we replace the n realizations of $(U_1, \ldots, U_{13})$ by the n points of a randomly-shifted lattice.

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Illustration: $V_j \sim \text{Normal}(\mu_j, \sigma_j^2)$ for $j = 1, 2, 4, 11, 12$, and $V_j \sim \text{Exponential}(1/\mu_j)$ otherwise.

The μ_j: 13.0, 5.5, 7.0, 5.2, 16.5, 14.7, 10.3, 6.0, 4.0, 20.0, 3.2, 3.2, 16.5.

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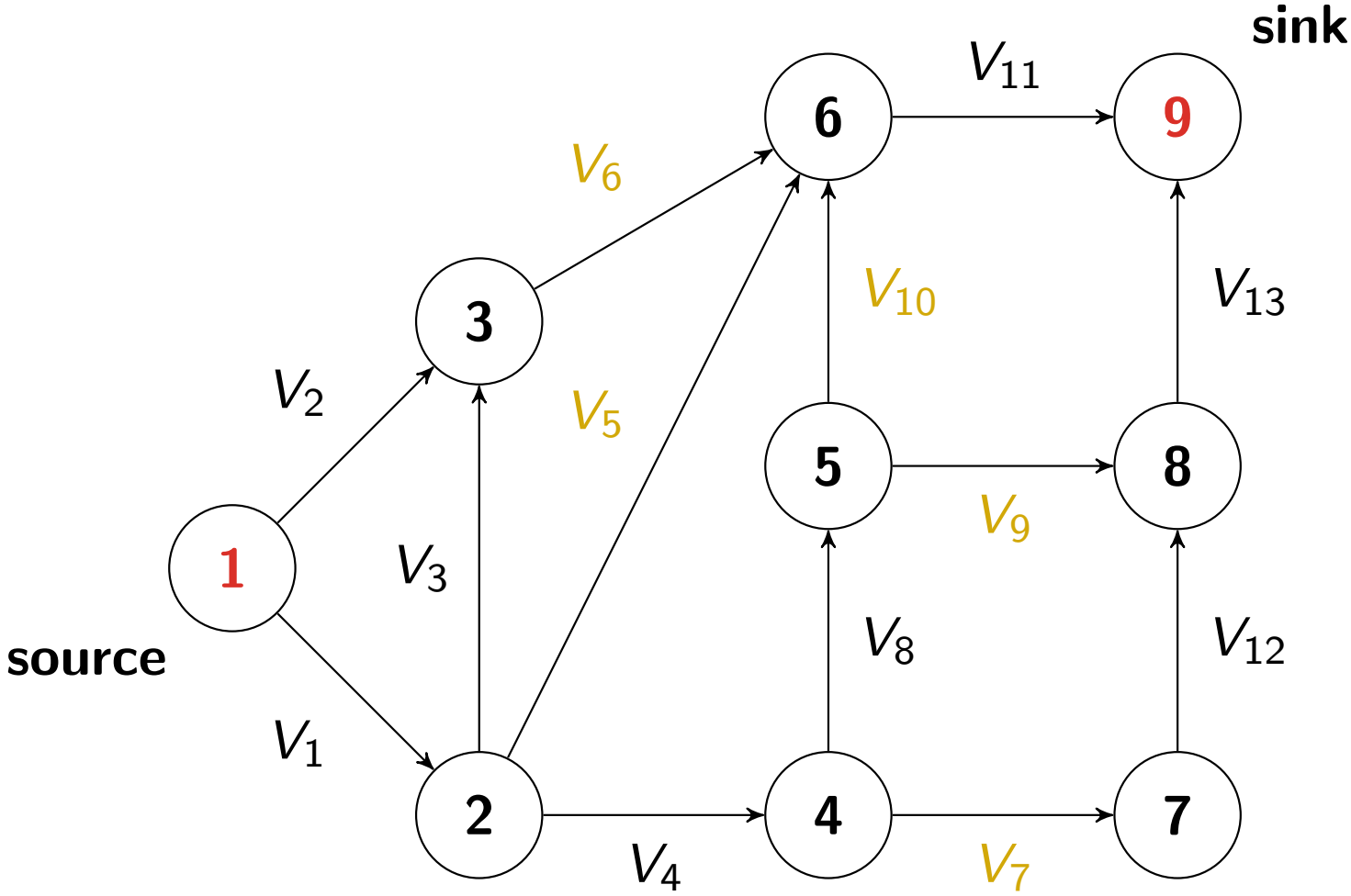
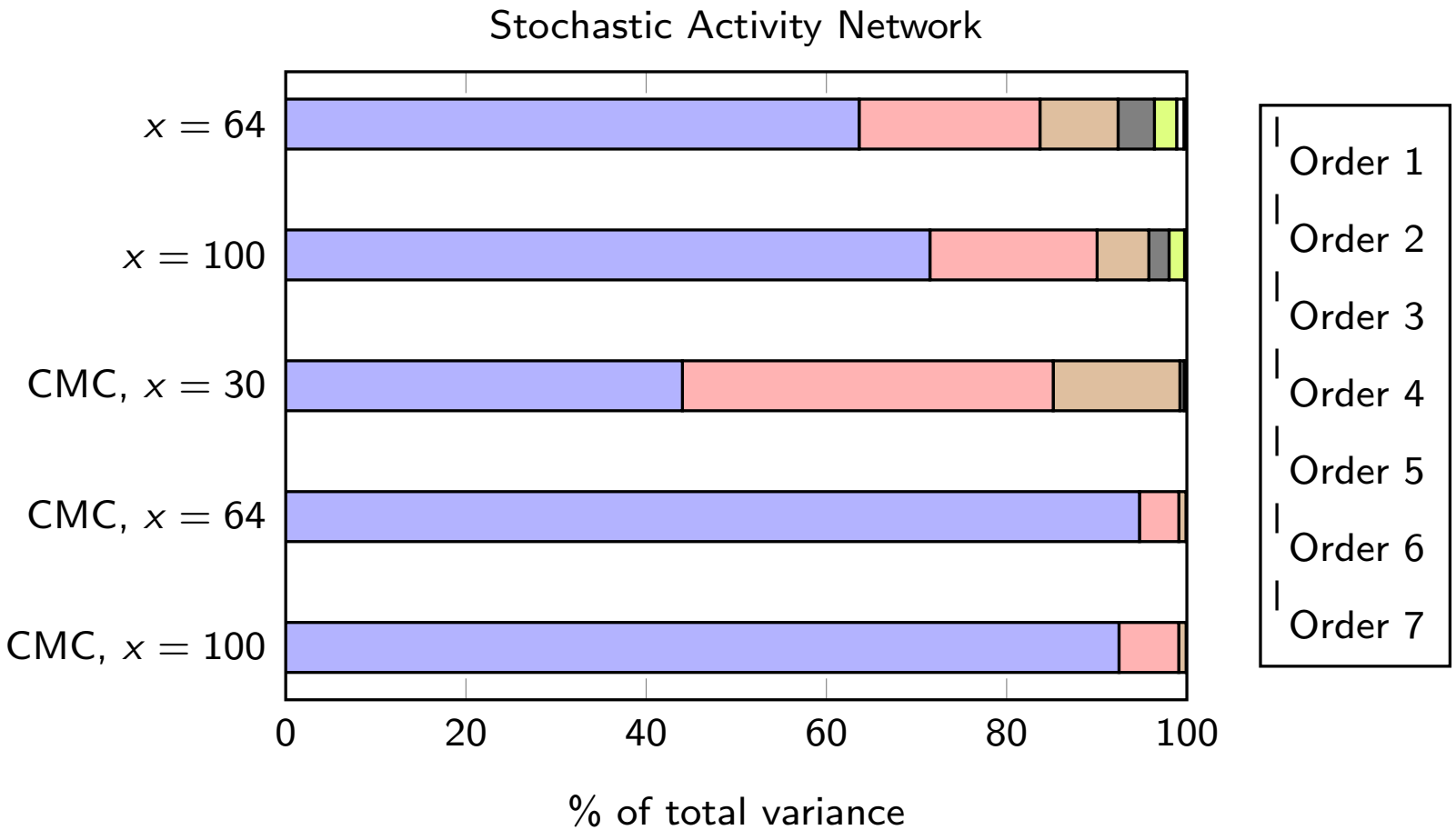
CMC estimator. Generate the V_j's only for the 8 arcs that do not belong to the cut $\mathcal{L} = \{5, 6, 7, 9, 10\}$, and replace $\mathbb{I}[T > x]$ by its conditional expectation given those V_j's, $\mathbb{P}[T > x \mid \{V_j, j \notin \mathcal{L}\}]$.
This makes the integrand continuous in the U_j's.


sink
source
1
2
3
4
5
6
7
8
9
V_1
V_2
V_3
V_4
V_5
V_6
V_7
V_8
V_9
V_{10}
V_{11}
V_{12}
V_{13}

ANOVA Variances for the Stochastic Activity Network



ANOVA decomposition

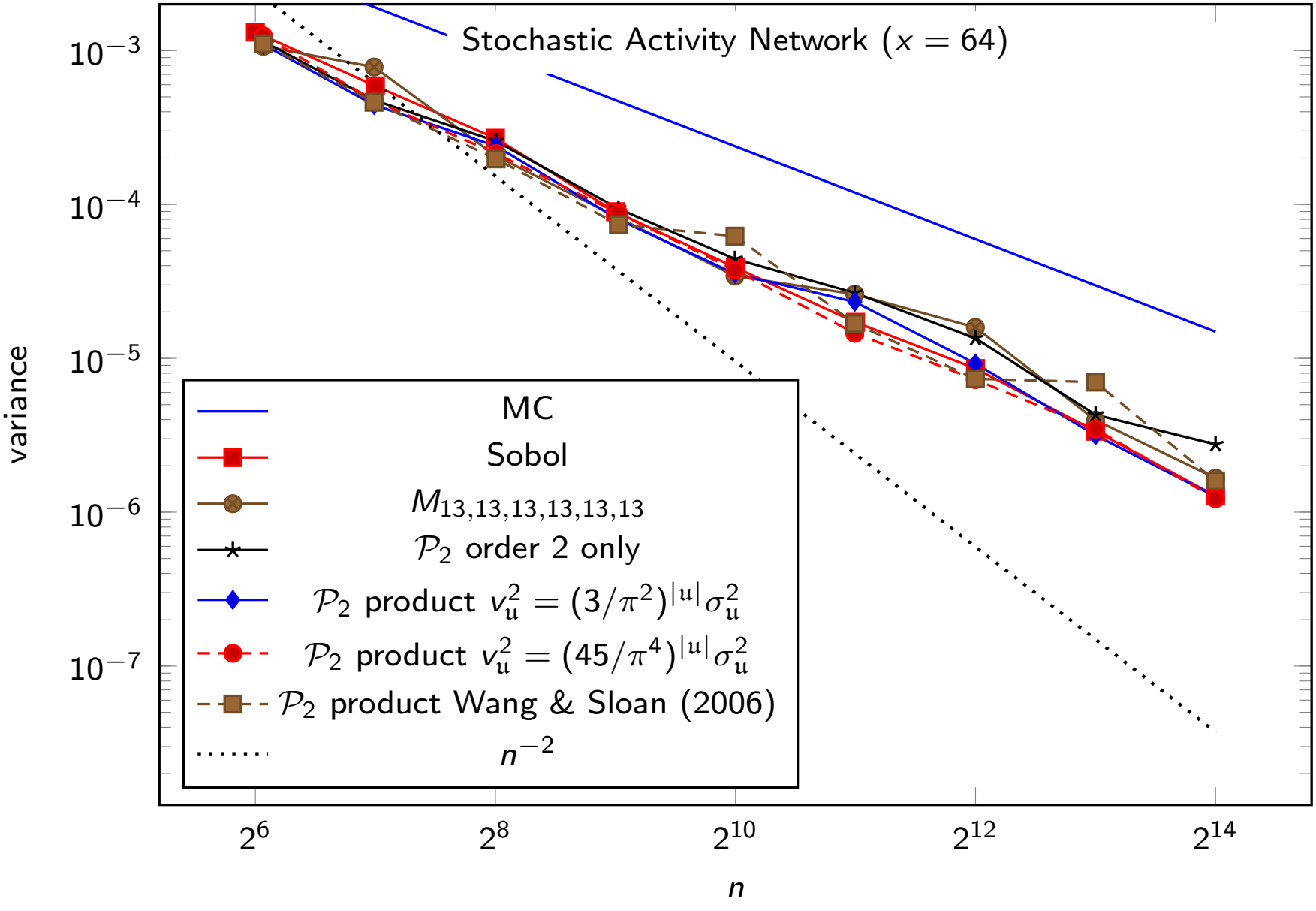
There are six paths from 1 to 9:

$$\{\{1,5,11\},\{2,6,11\},\{1,3,6,11\},\{1,4,7,12,13\},\{1,4,8,9,13\},\{1,4,8,10,11\}\}.$$

Intuition: the important projections should be only the subsets of those paths.
Fraction of the total variance that lies in these projections:

	$x = 30$	$x = 64$	$x = 100$
crude MC		80.6 %	96.3 %
conditional MC	88.8 %	99.5 %	100 %

Lattices of Rank 1 with CBC



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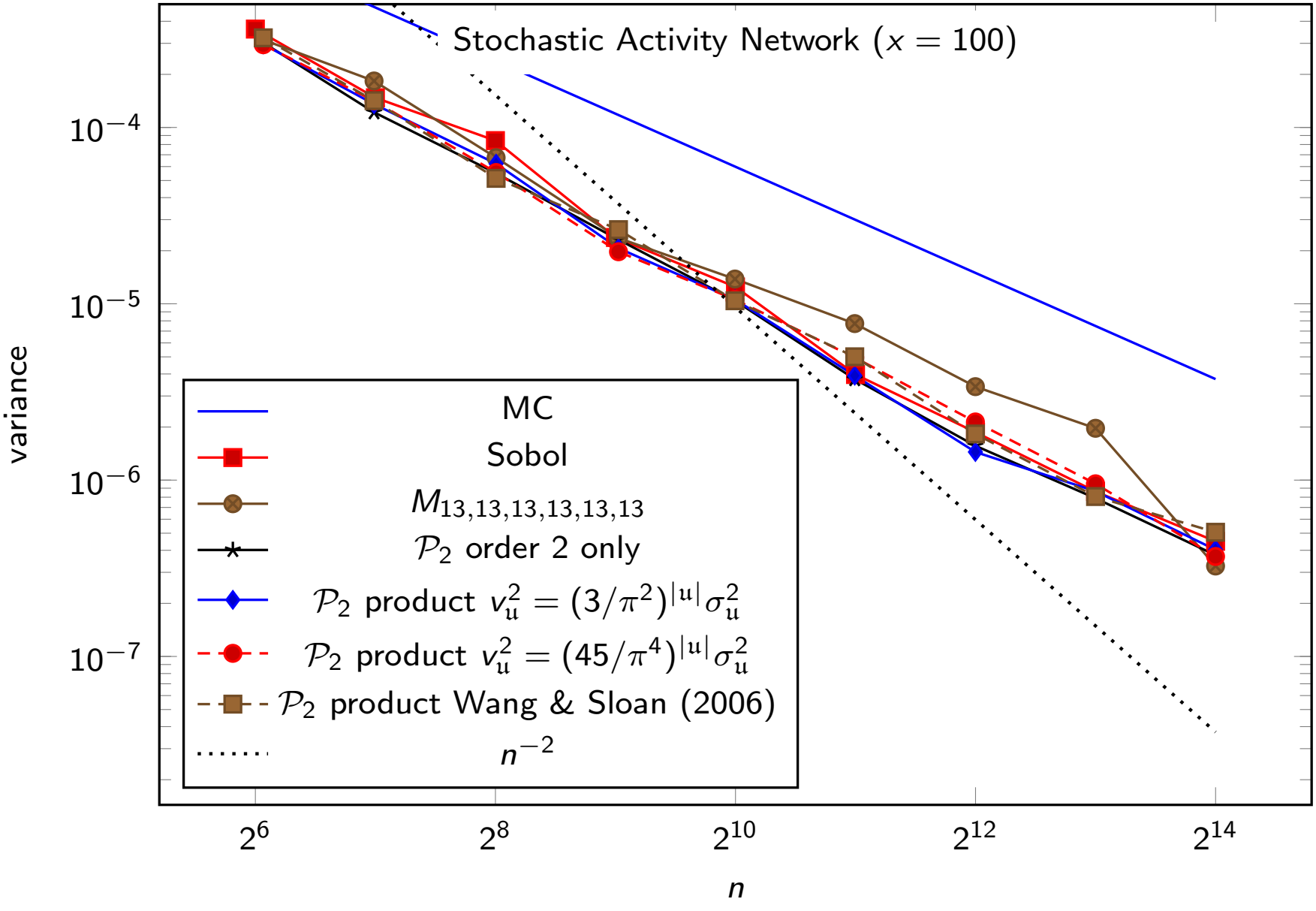

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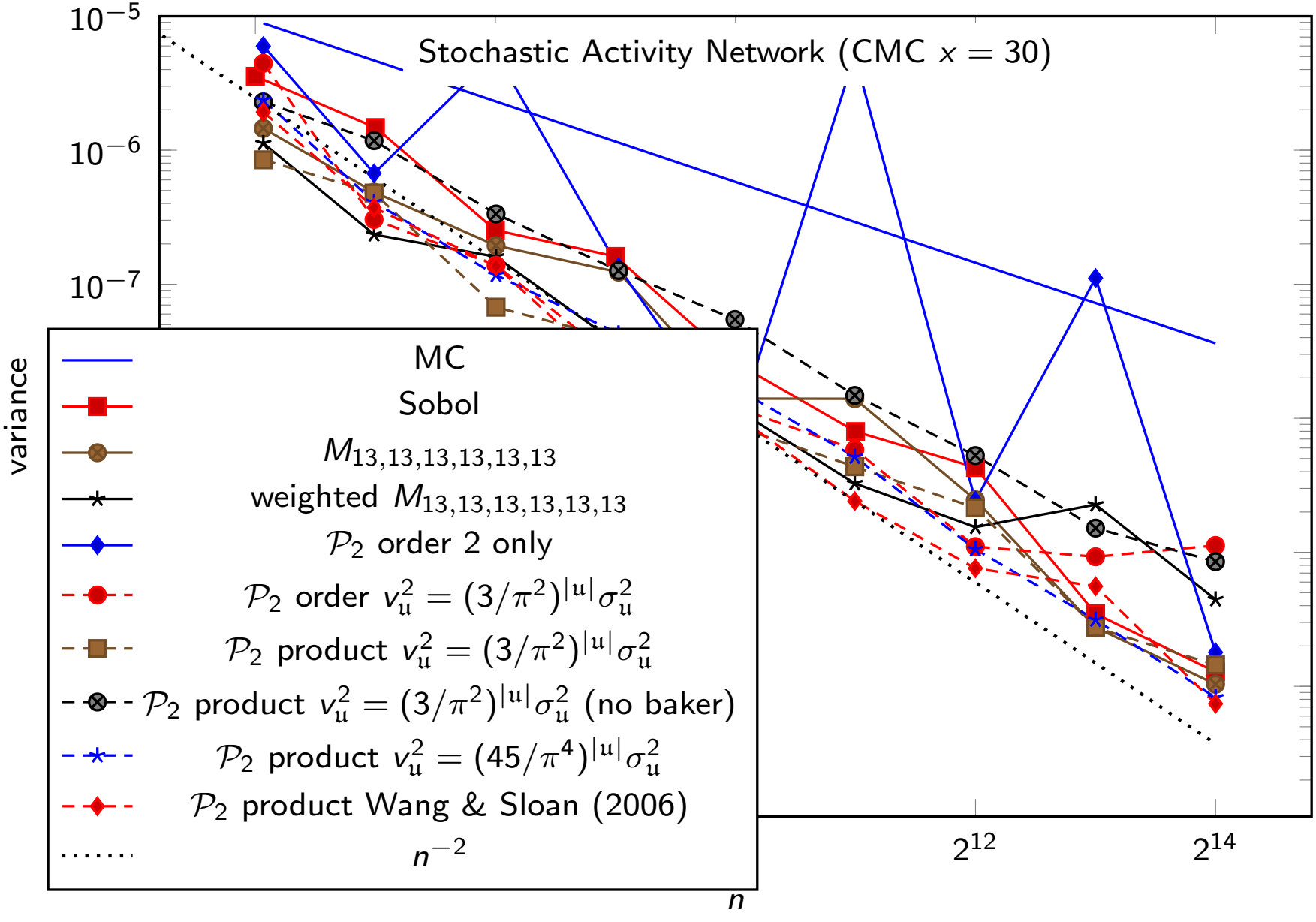

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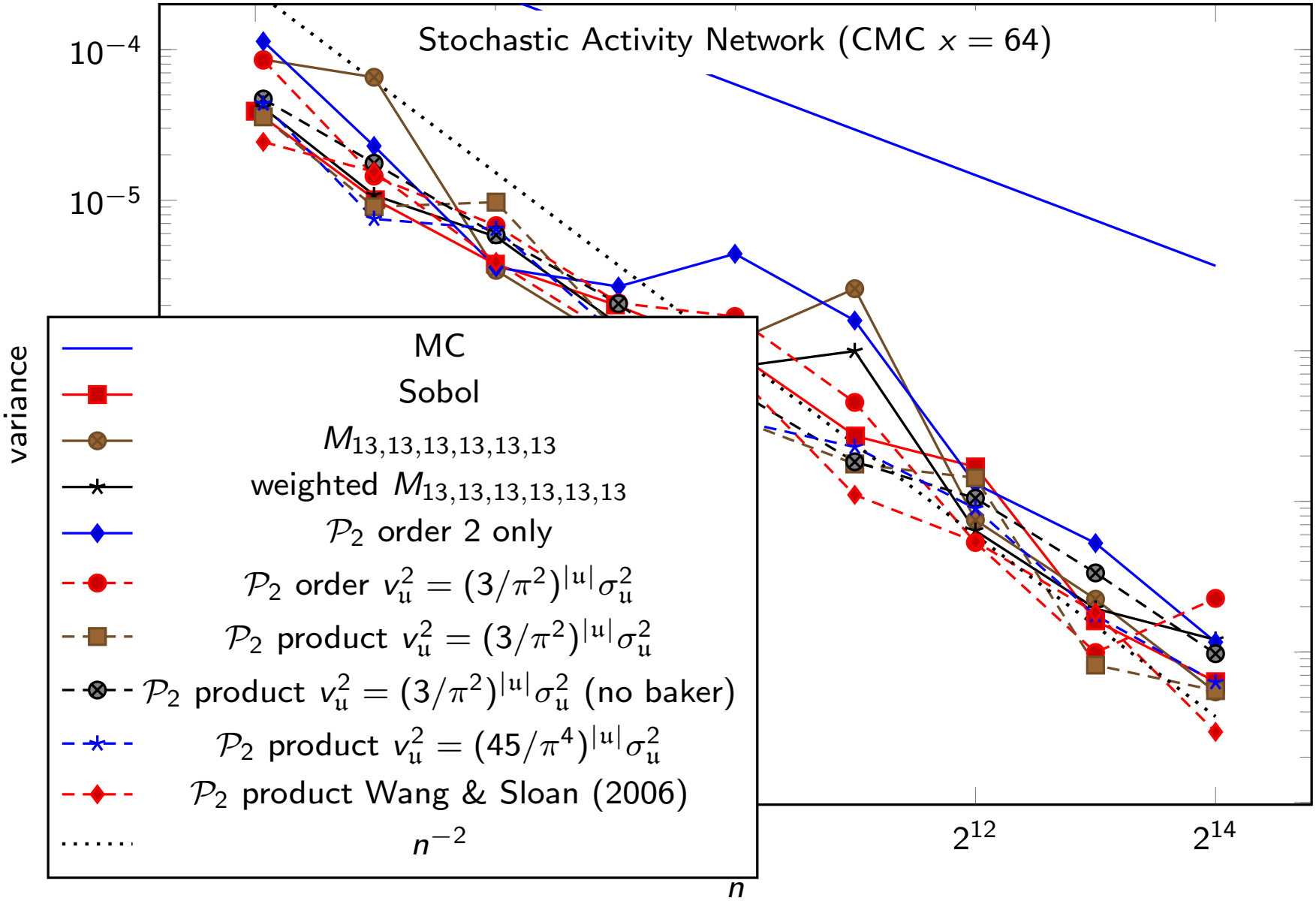

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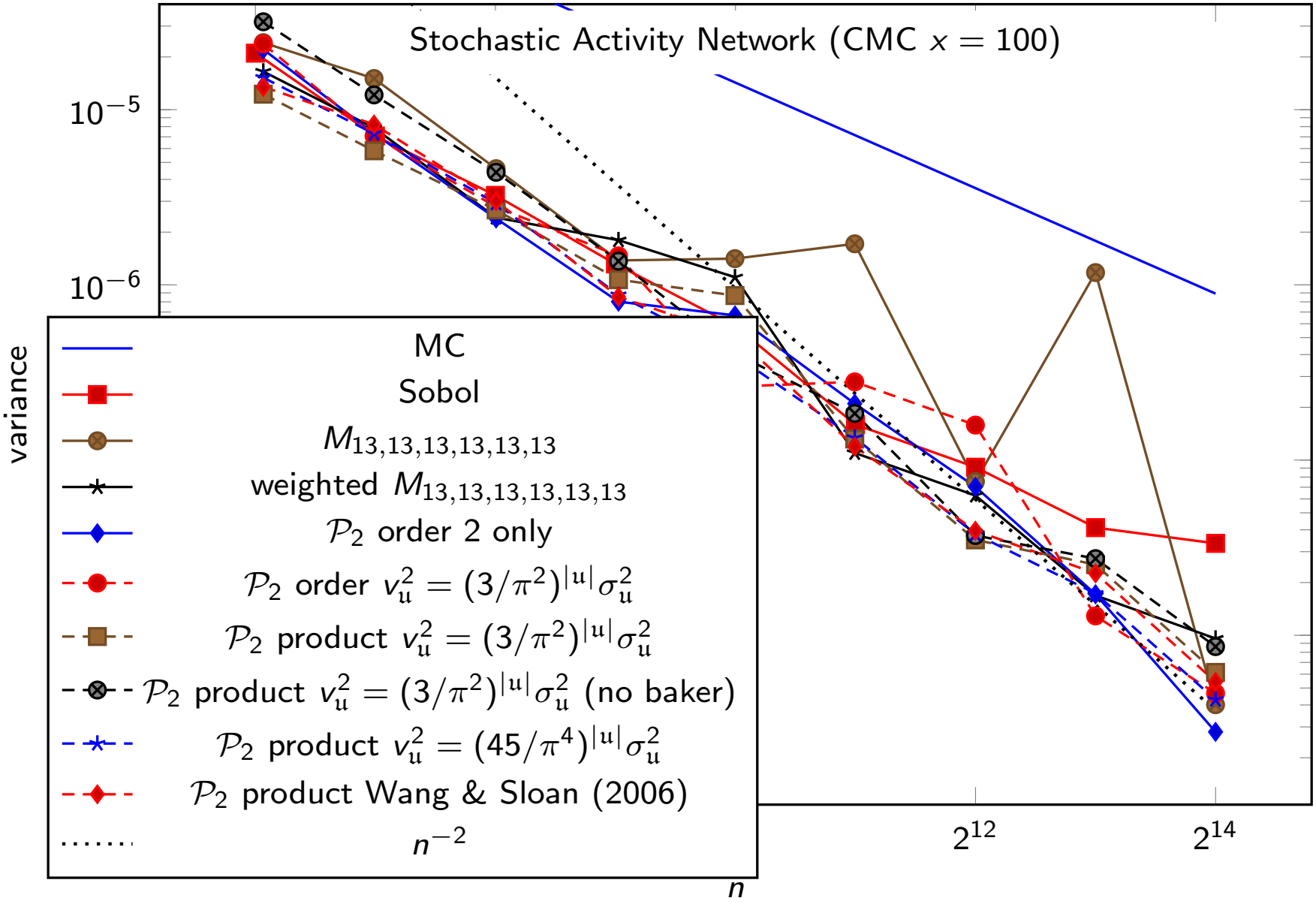
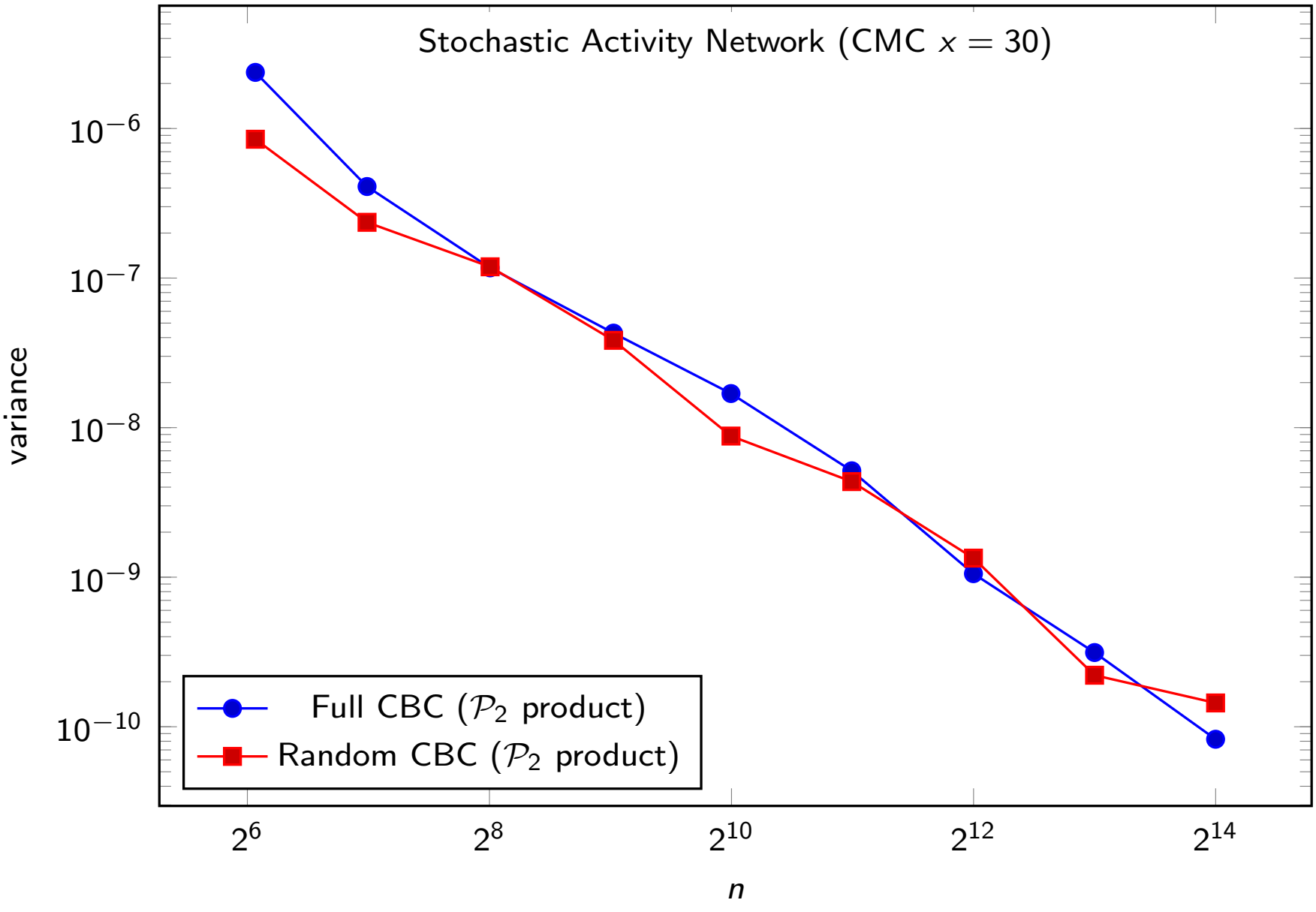


Random vs. Full CBC



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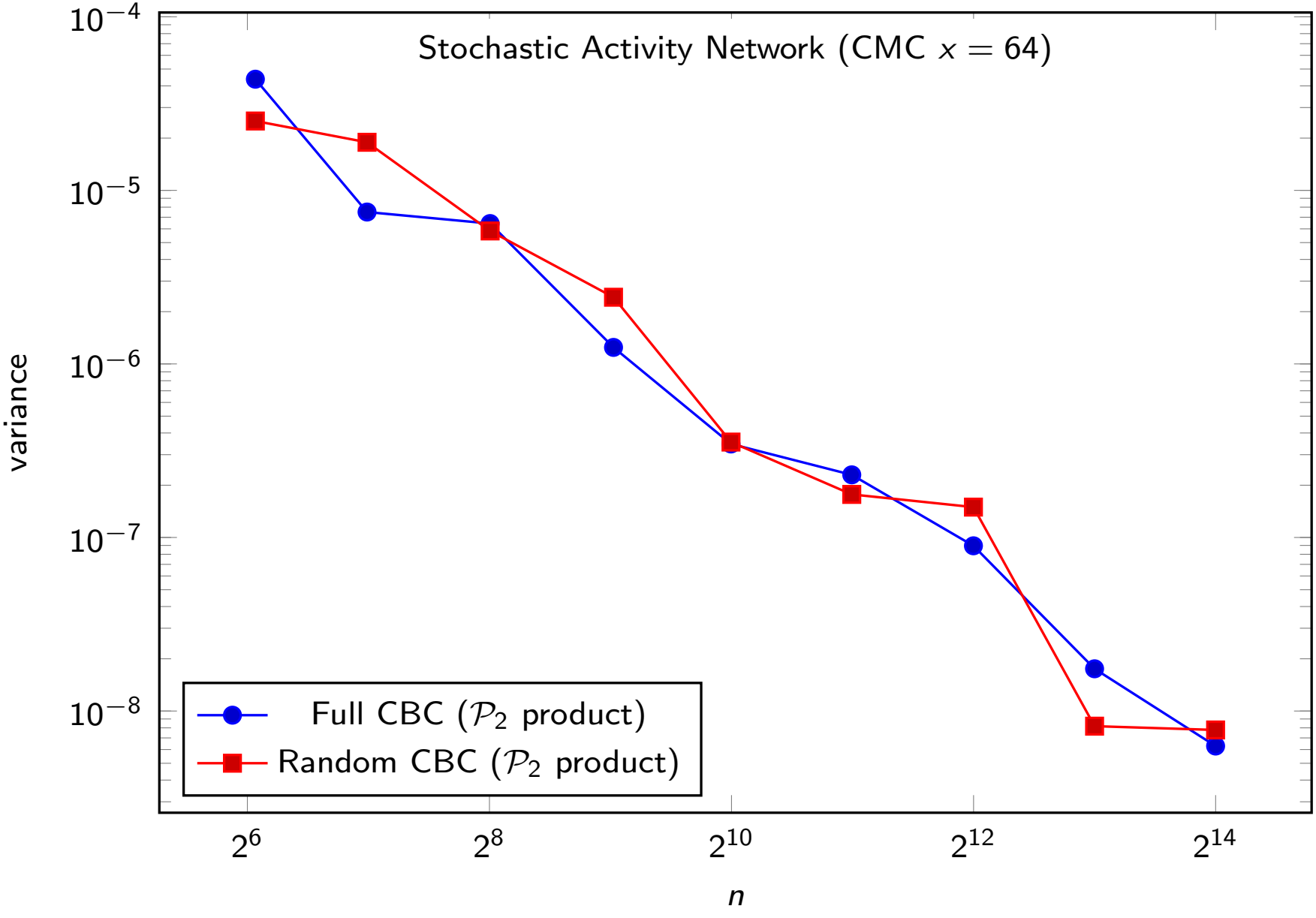

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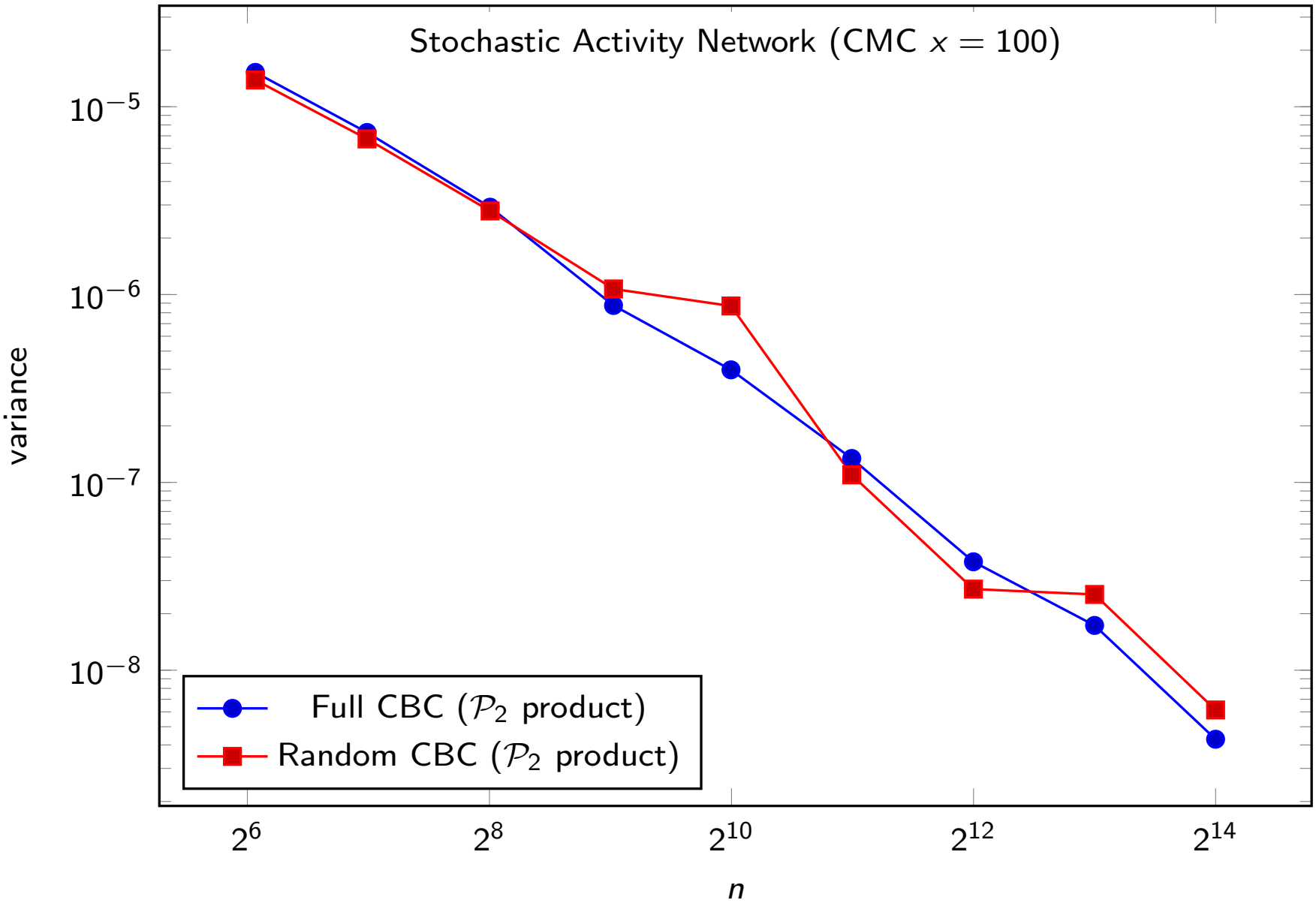
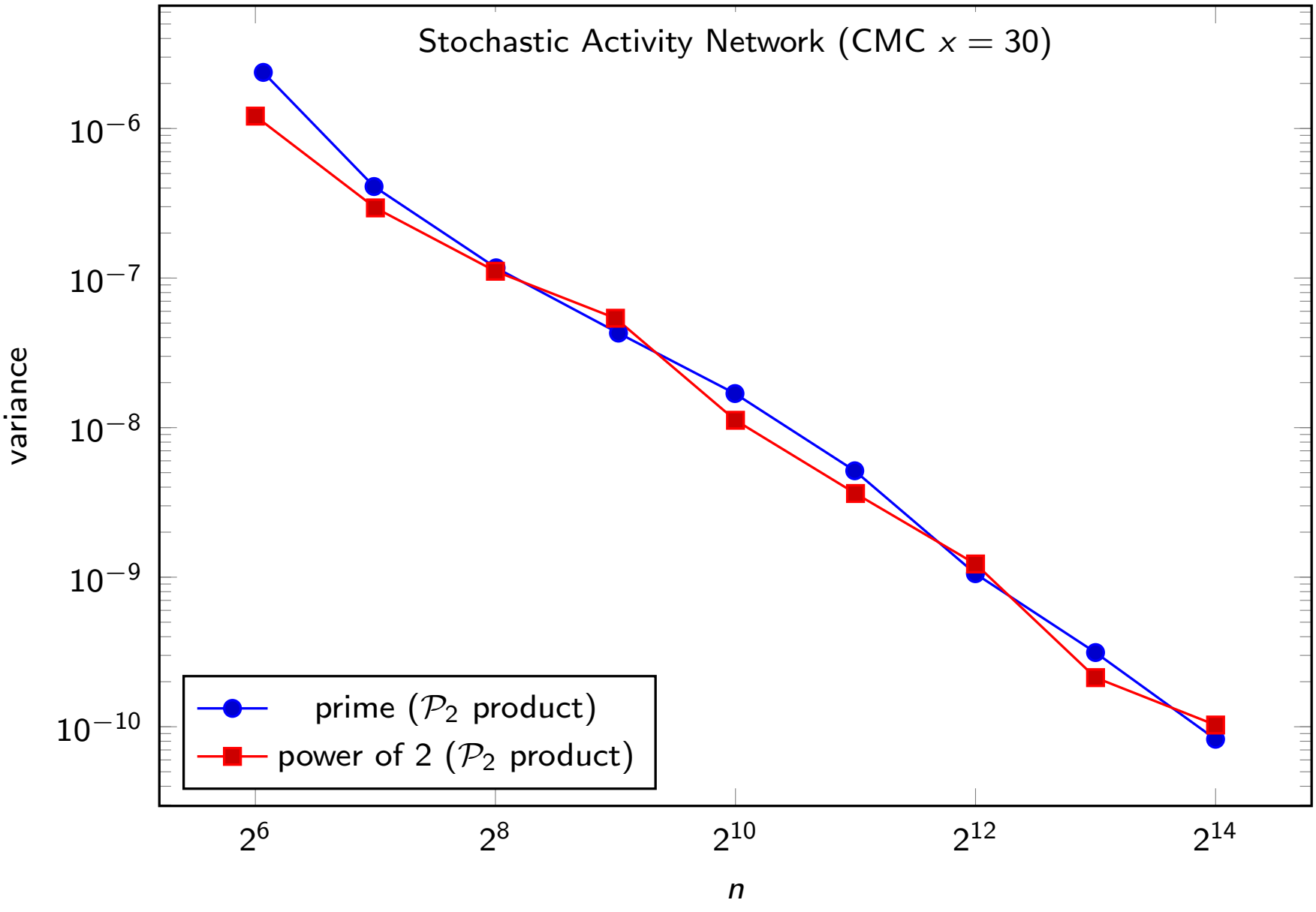


Prime vs. Power-of-2 Number of Points



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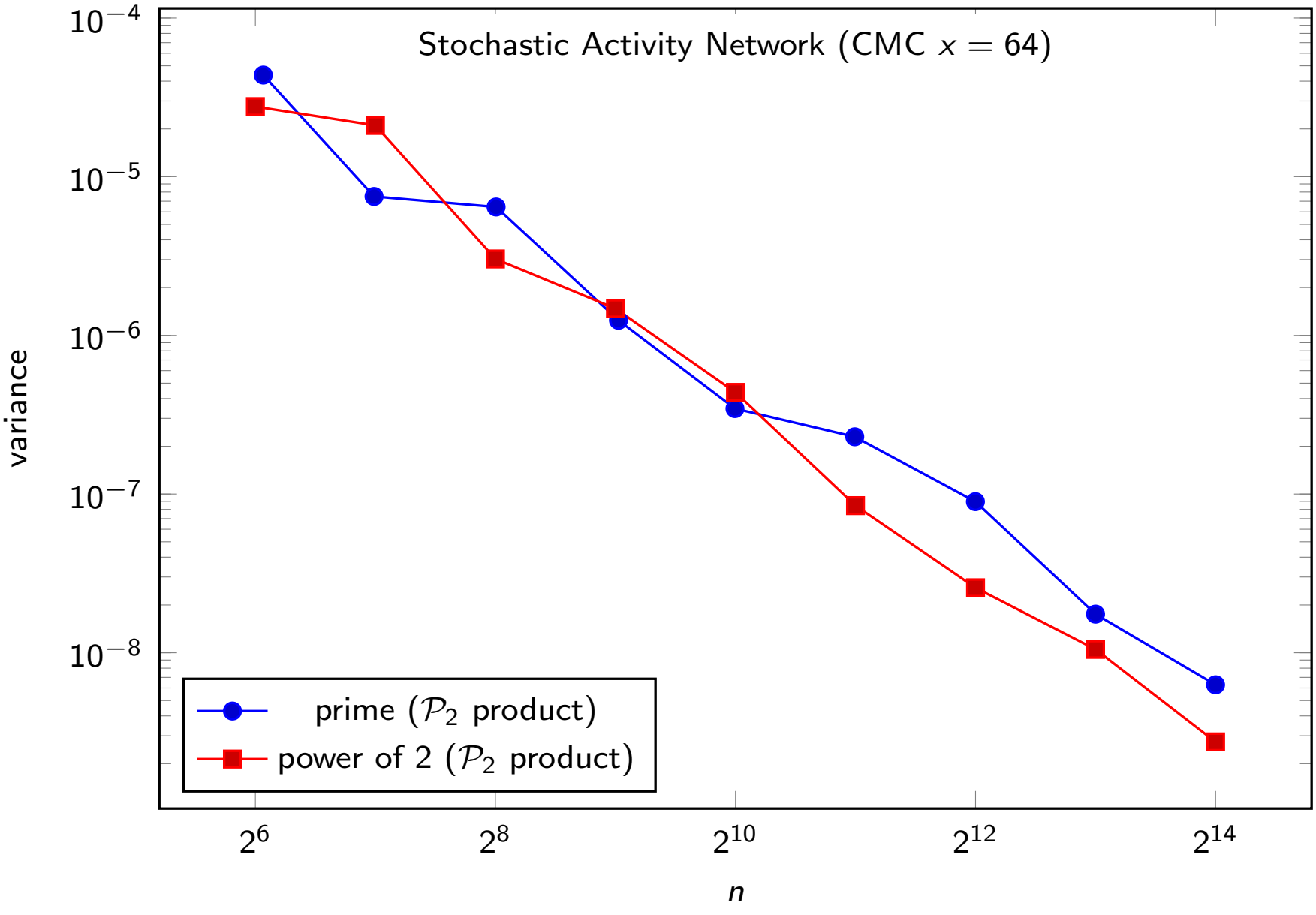

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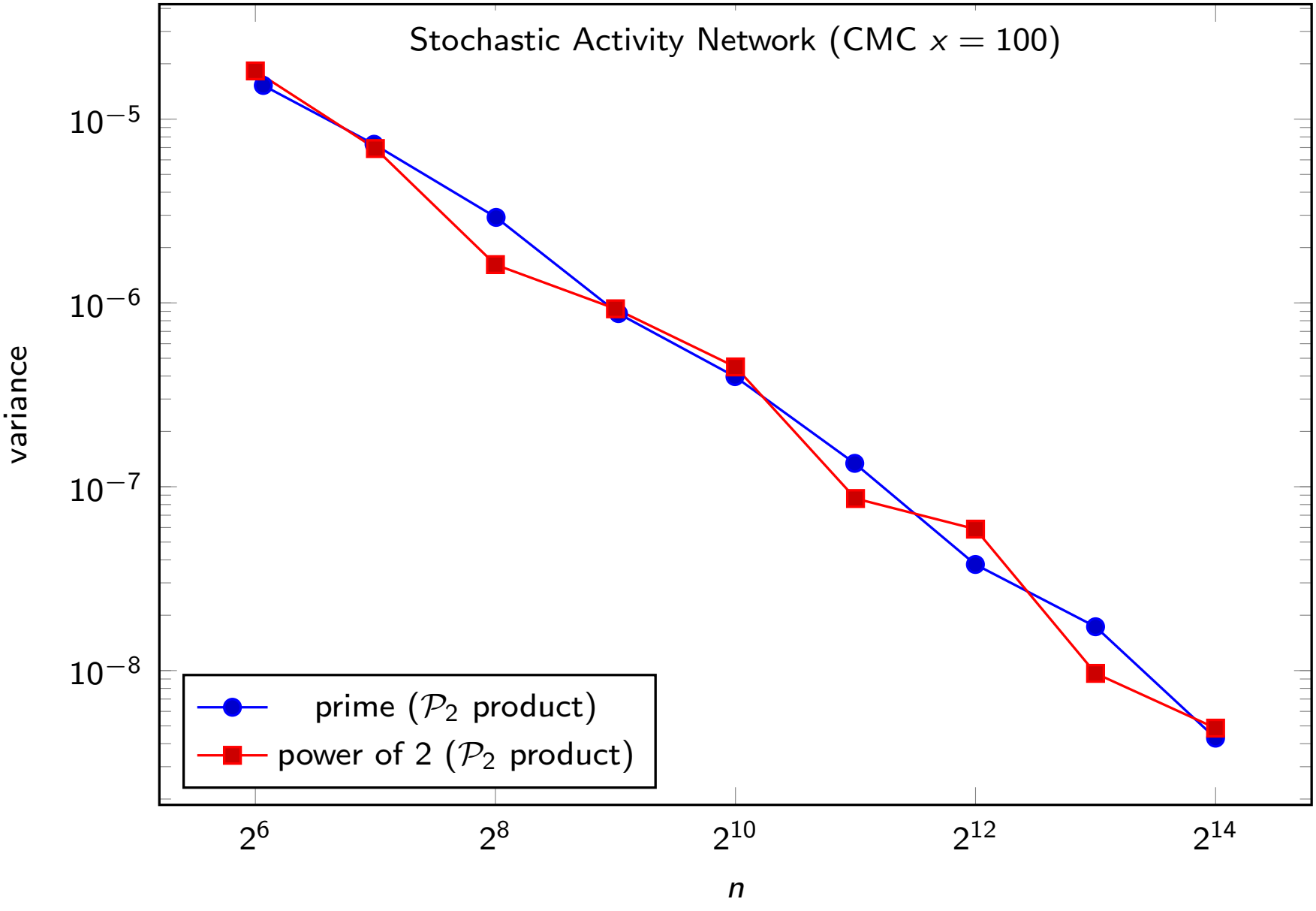
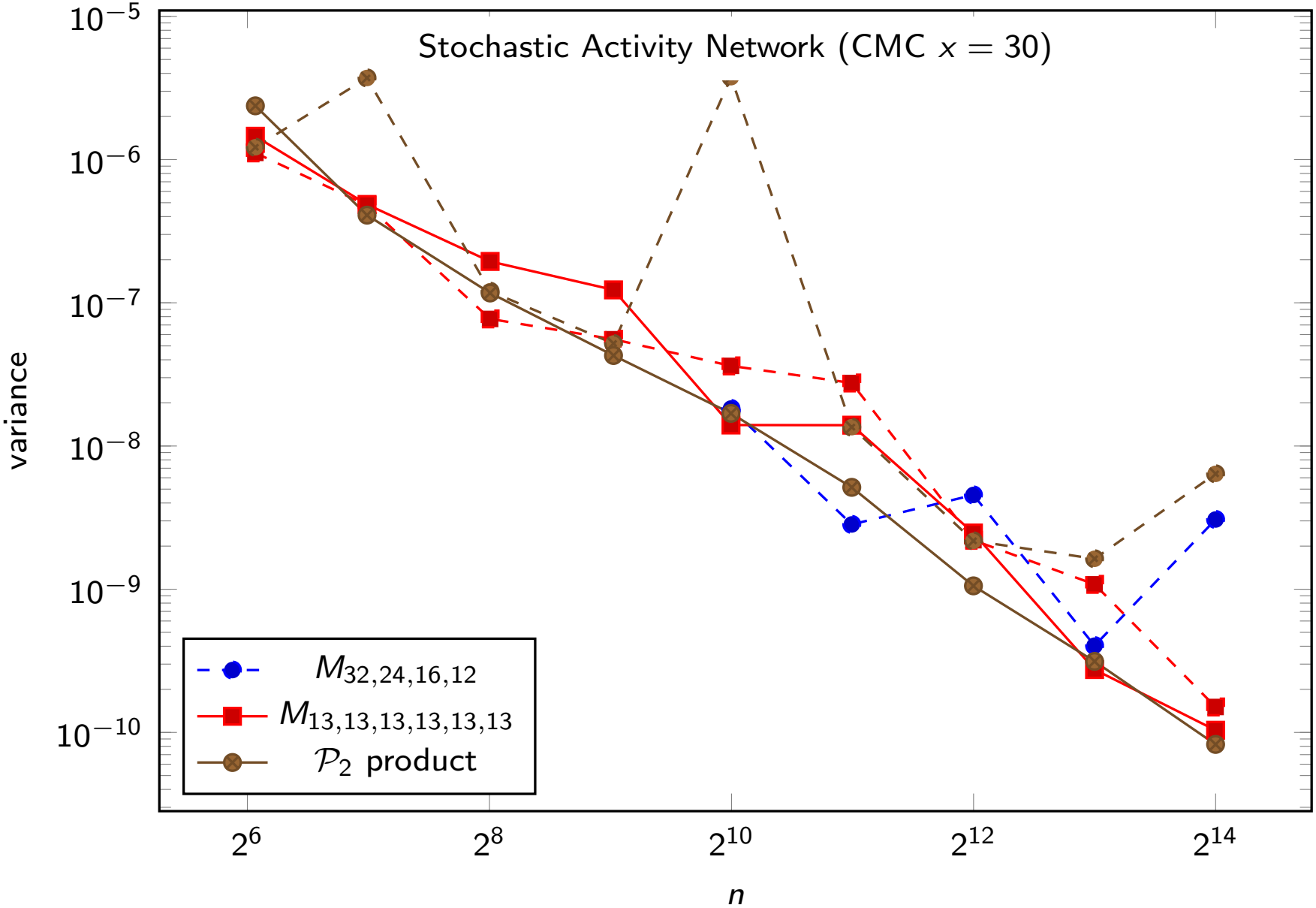
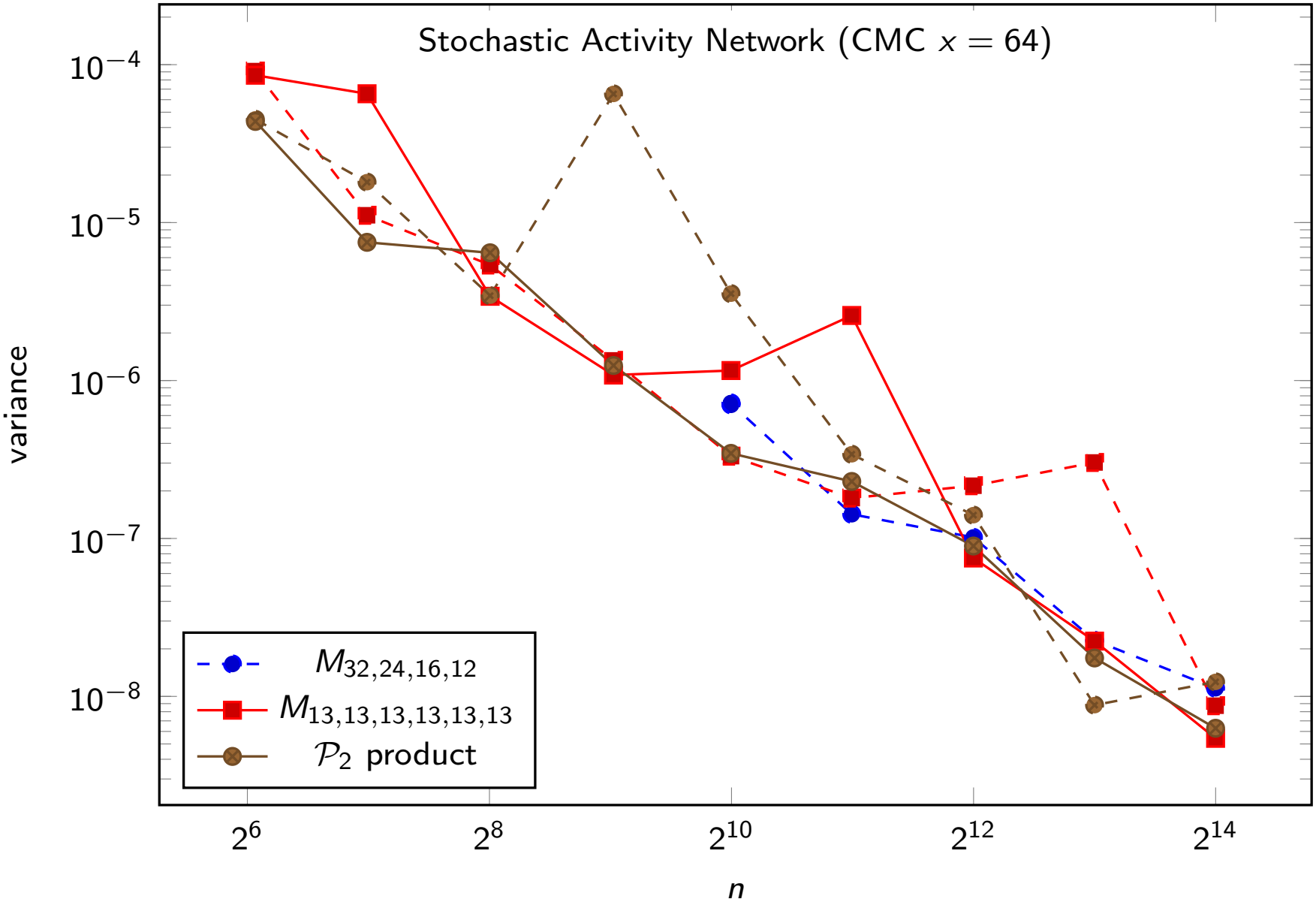


Korobov vs. CBC



Solid: CBC. Dashed: Korobov.

Korobov vs. CBC



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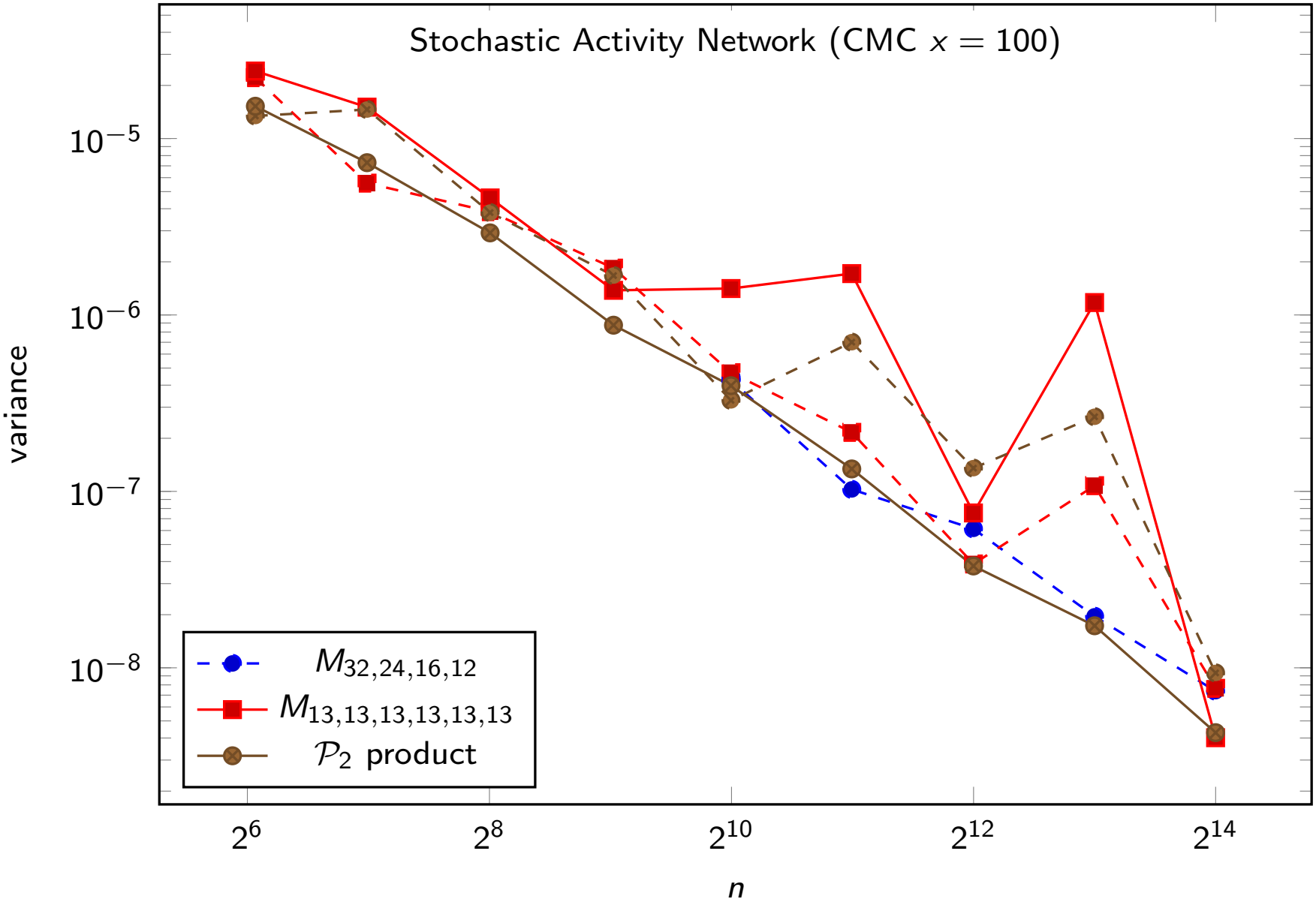

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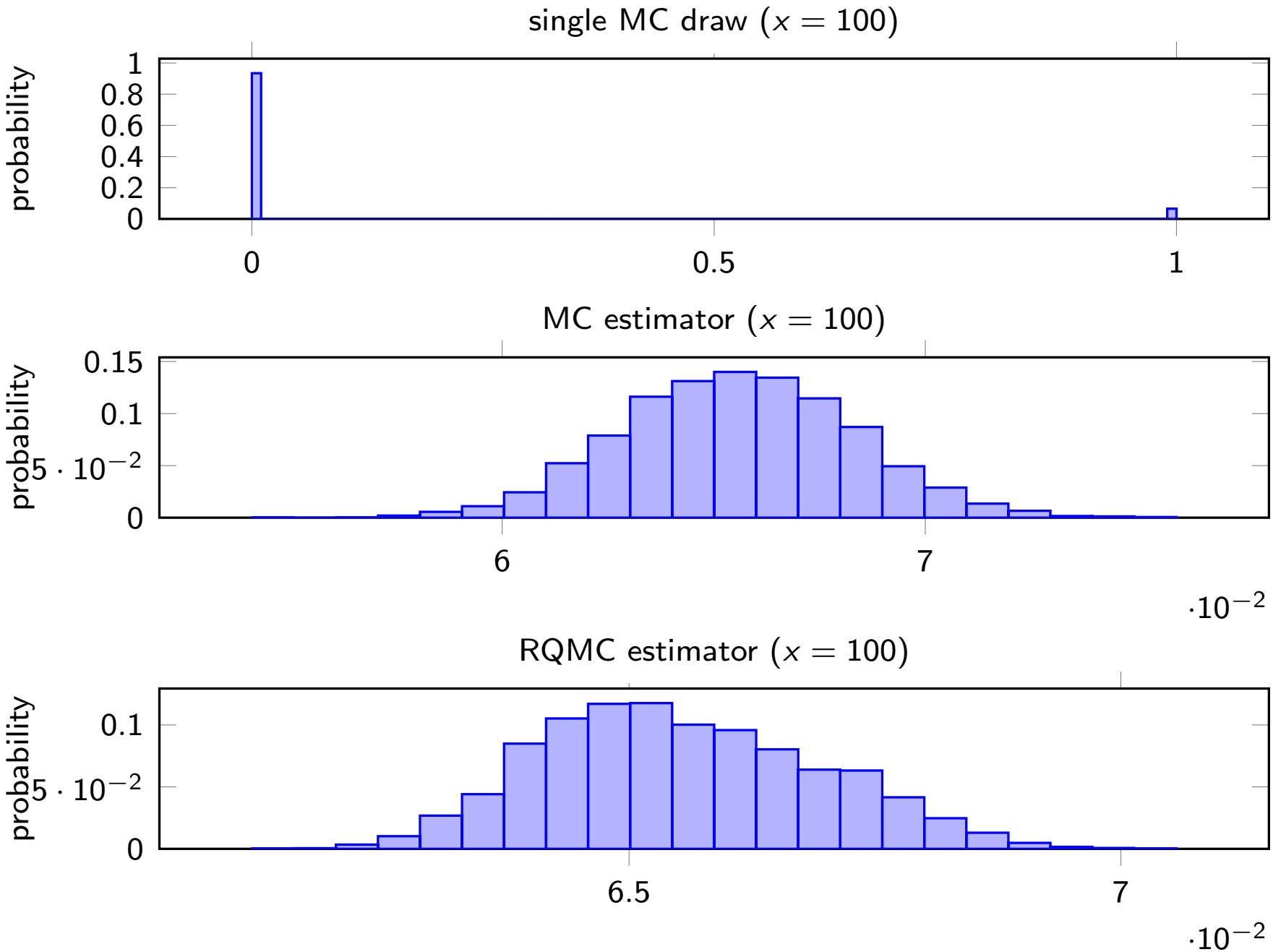
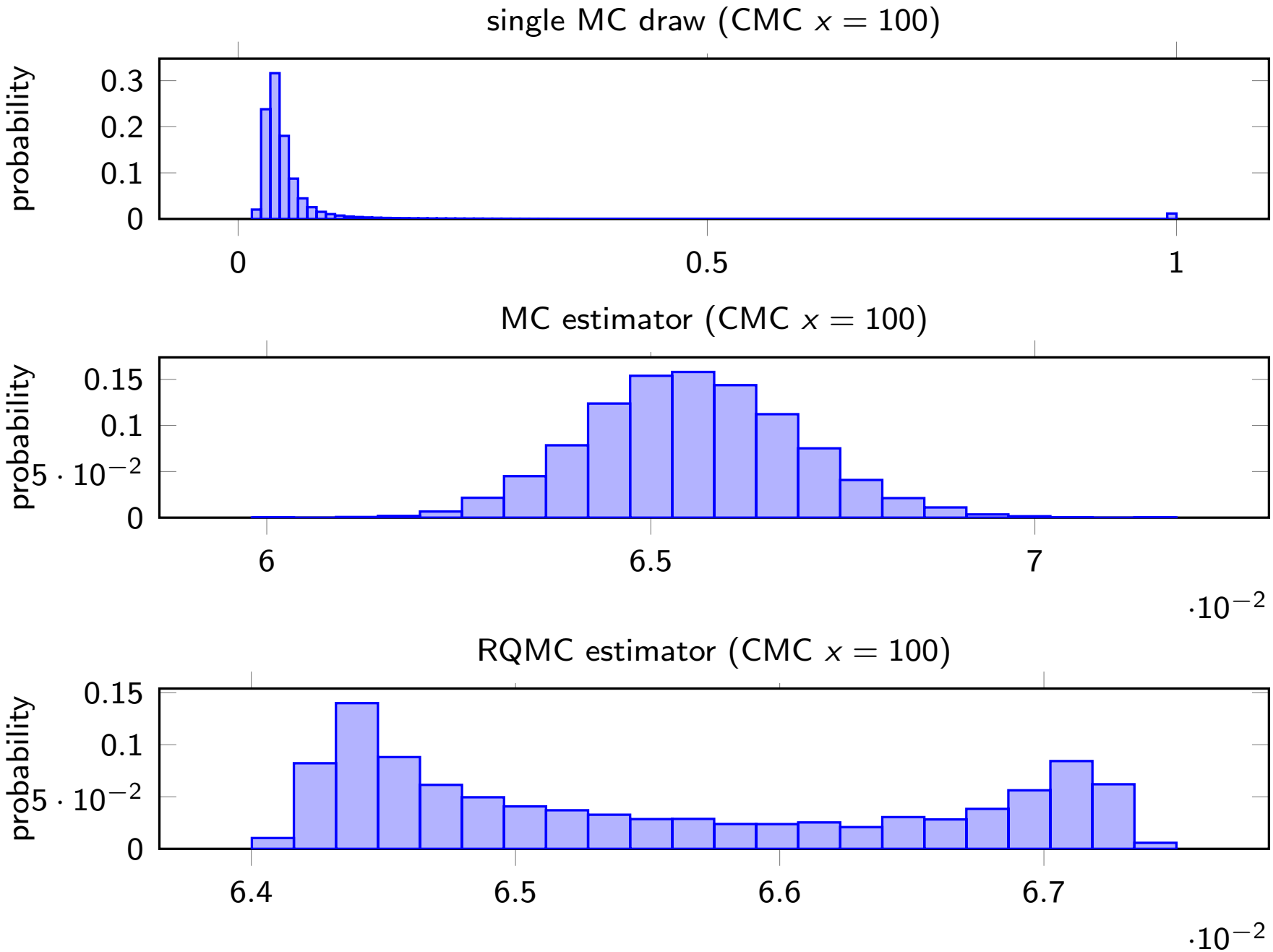
Histograms, for $n = 8191$, $m = 10^4$ replications



Histograms


single MC draw (CMC $x = 100$)
probability
0.3
0.2
0.1
0
0
0.5
1
MC estimator (CMC $x = 100$)
probability
0.15
0.1
$5 \cdot 10^{-2}$
0
6
6.5
7
$\cdot 10^{-2}$
RQMC estimator (CMC $x = 100$)
probability
0.15
0.1
$5 \cdot 10^{-2}$
0
6.4
6.5
6.6
6.7
$\cdot 10^{-2}$

Function of a Multinormal vector

Let $\mu = E[f(\mathbf{U})] = E[g(\mathbf{Y})]$ where $\mathbf{Y} = (Y_1, \ldots, Y_s) \sim N(\mathbf{0}, \mathbf{\Sigma})$.

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For example, if the payoff of a financial derivative is a function of the values taken by a c-dimensional geometric Brownian motions (GMB) at d observations times $0 < t_1 < \cdots < t_d = T$, then we have $s = cd$.

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To generate $\mathbf{Y}$: Decompose $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^{\mathrm{t}}$, generate $\mathbf{Z} = (Z_1, \ldots, Z_s) = (\Phi^{-1}(U_1), \ldots, \Phi^{-1}(U_s)) \sim N(\mathbf{0}, \mathbf{I})$ and return $\mathbf{Y} = \mathbf{A}\mathbf{Z}$.

Function of a Multinormal vector

Let $\mu = E[f(\mathbf{U})] = E[g(\mathbf{Y})]$ where $\mathbf{Y} = (Y_1, \ldots, Y_s) \sim N(\mathbf{0}, \mathbf{\Sigma})$.

For example, if the payoff of a financial derivative is a function of the values taken by a c-dimensional geometric Brownian motions (GMB) at d observations times $0 < t_1 < \cdots < t_d = T$, then we have $s = cd$.

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Choice of $\mathbf{A}$?

Cholesky factorization: $\mathbf{A}$ is lower triangular.

Principal component decomposition (PCA):
$\mathbf{A} = \mathbf{P}\mathbf{D}^{1/2}$ where $\mathbf{D} = \mathrm{diag}(\lambda_s, \ldots, \lambda_1)$ (eigenvalues of $\mathbf{\Sigma}$ in decreasing order) and the columns of $\mathbf{P}$ are the corresponding unit-length eigenvectors.

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Brownian bridge (BB) sampling: Suppose $d = 2^m$.
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Each of these methods corresponds to some matrix $\mathbf{A}$.
Choice has large impact on the ANOVA decomposition of f.

Example: Pricing an Asian option

Single asset, s observation times $t_1, \ldots, t_s$. Want to estimate $\mathbb{E}[f(\mathbf{U})]$, where

$$f(\mathbf{U}) = e^{-rt_s} \max\left[0,\ \frac{1}{s}\sum_{j=1}^{s} S(t_j) - K\right]$$

and $\{S(t),\ t \geq 0\}$ is a geometric Brownian motion.
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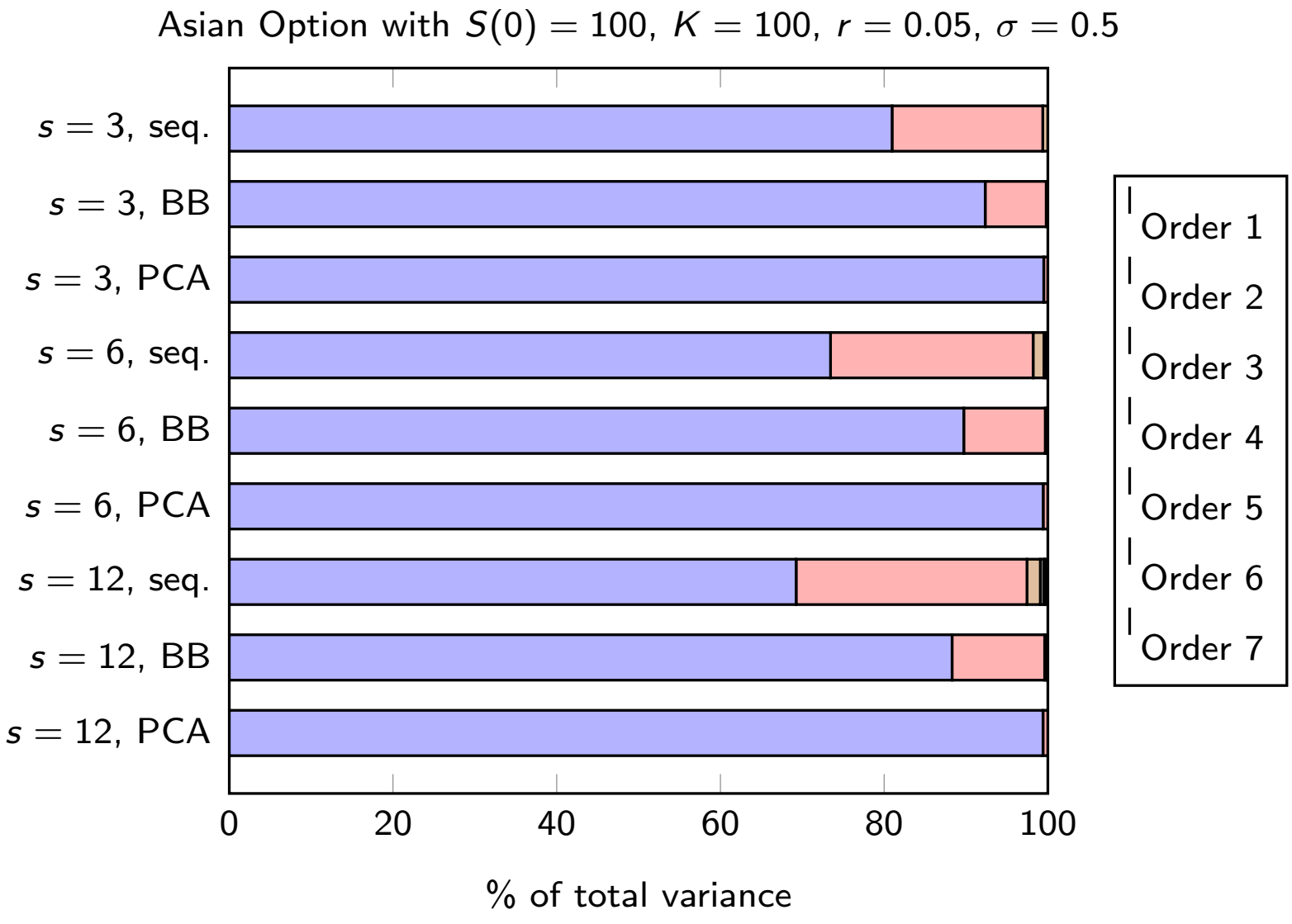
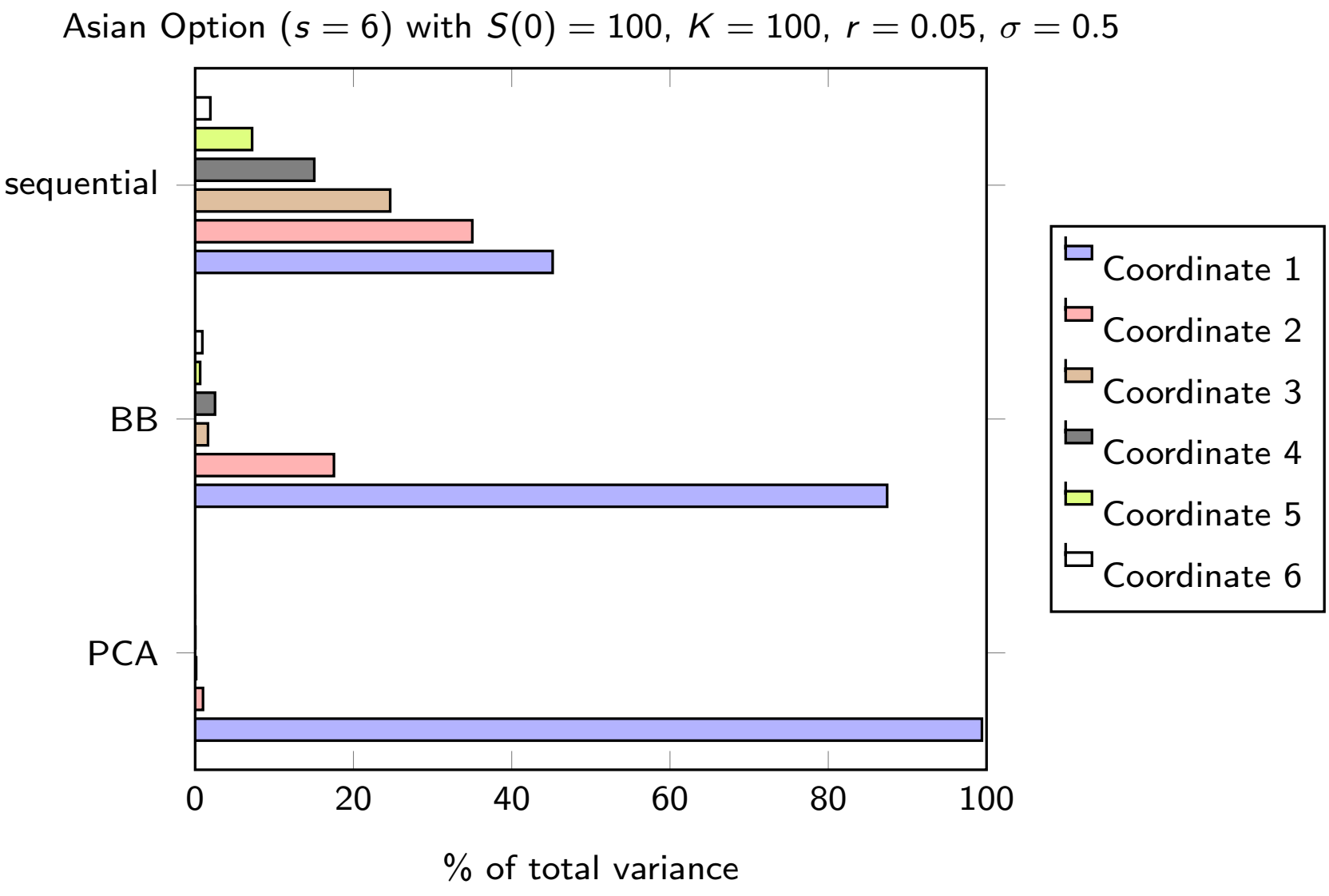
Let $S(0) = 100$, $K = 100$, $r = 0.05$, $t_s = 1$, and $t_j = jT/s$ for $1 \leq j \leq s$.
We consider $\sigma = 0.2$, 0.5 and $s = 3, 6, 12$.

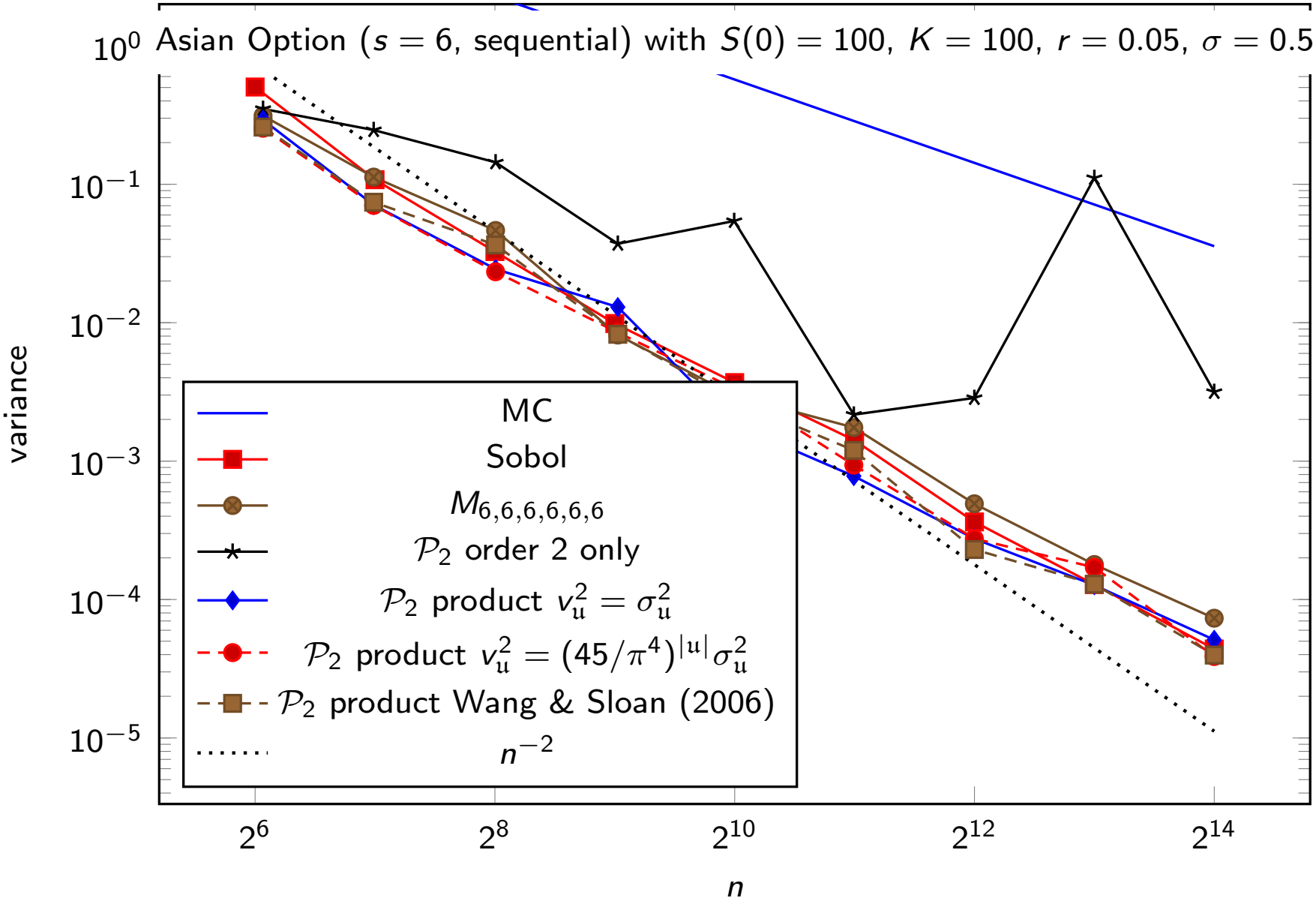
ANOVA Variances for the Asian Option



Total Variance per Coordinate for the Asian Option



Lattices of Rank 1 with CBC



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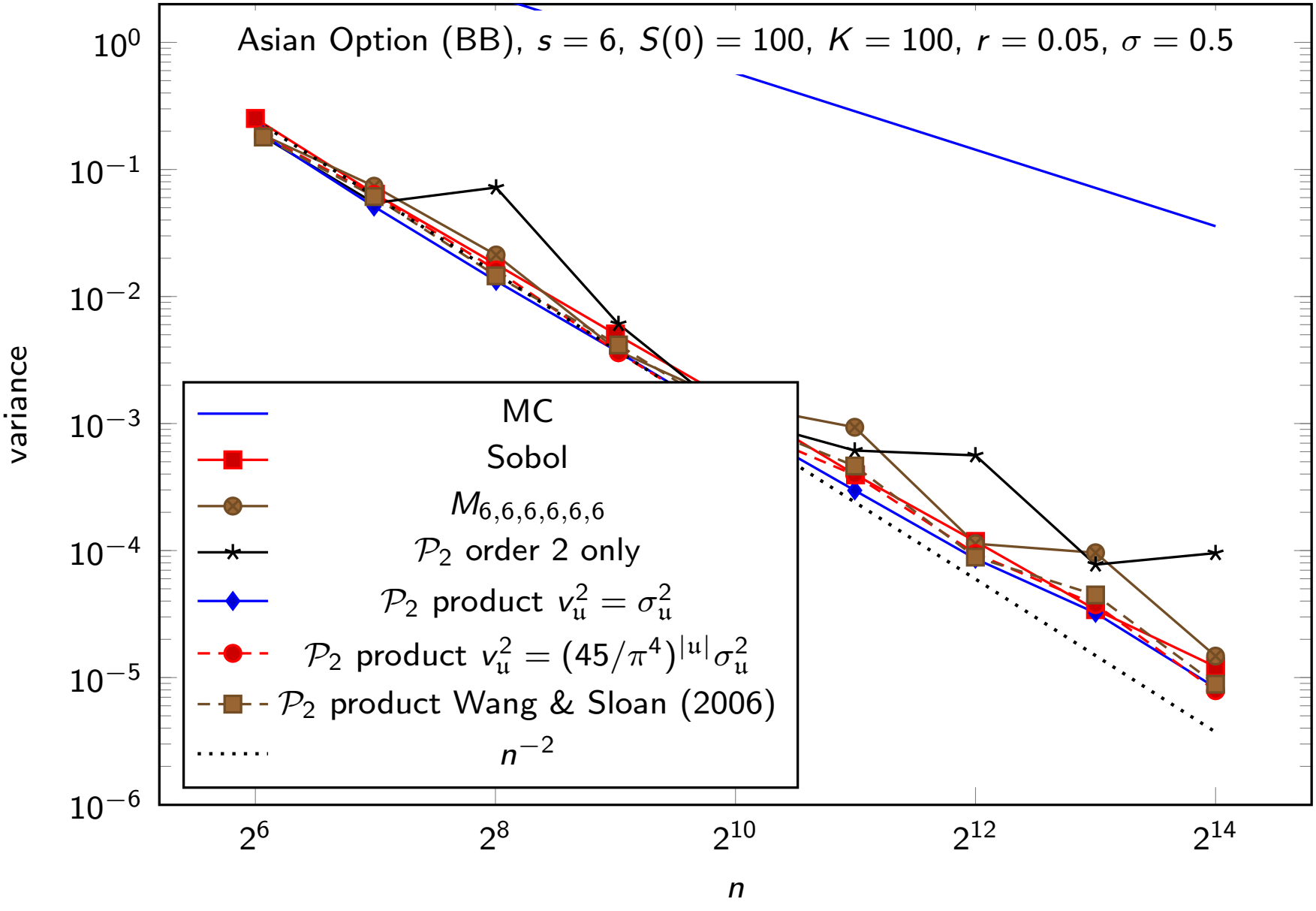

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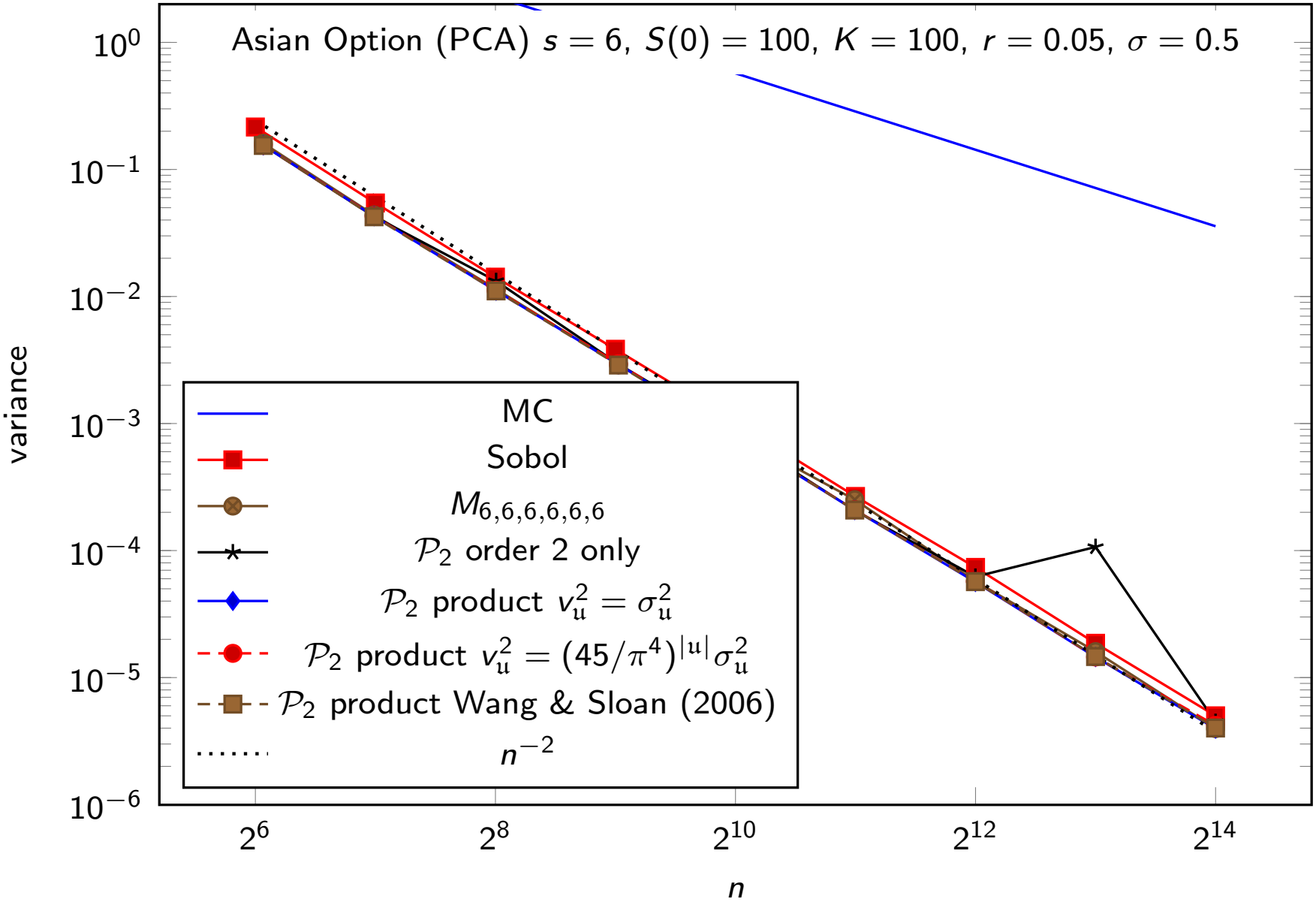

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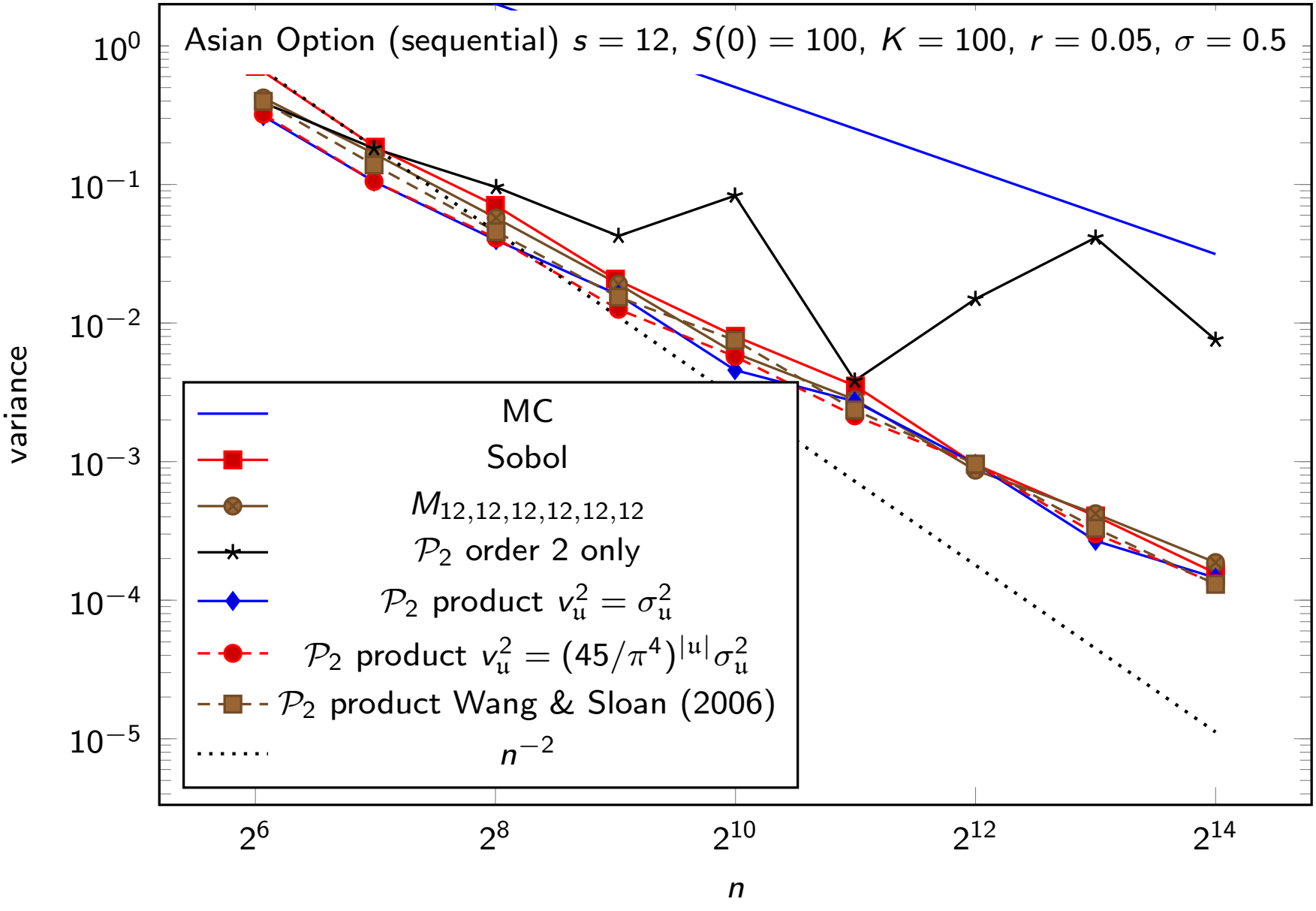

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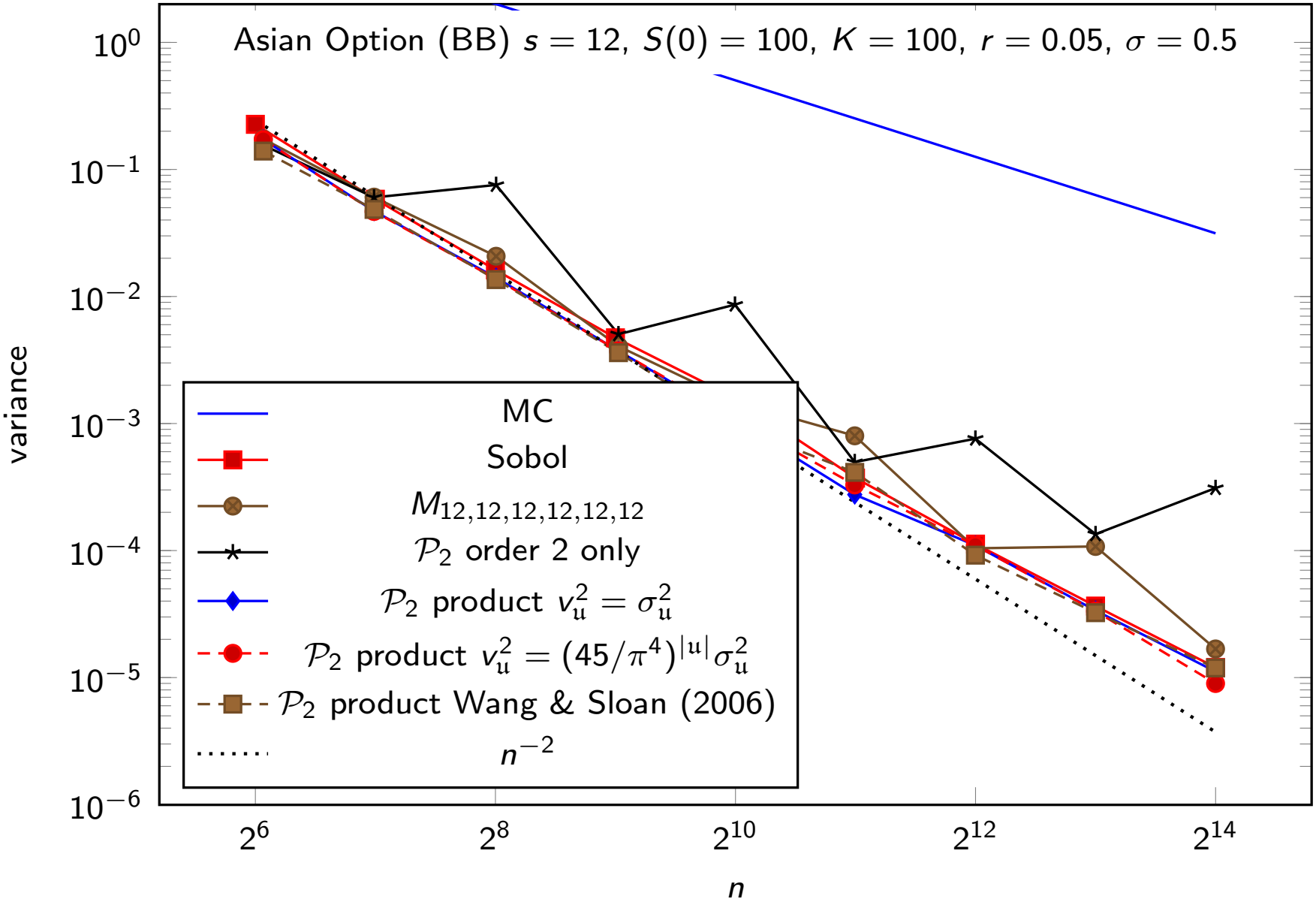

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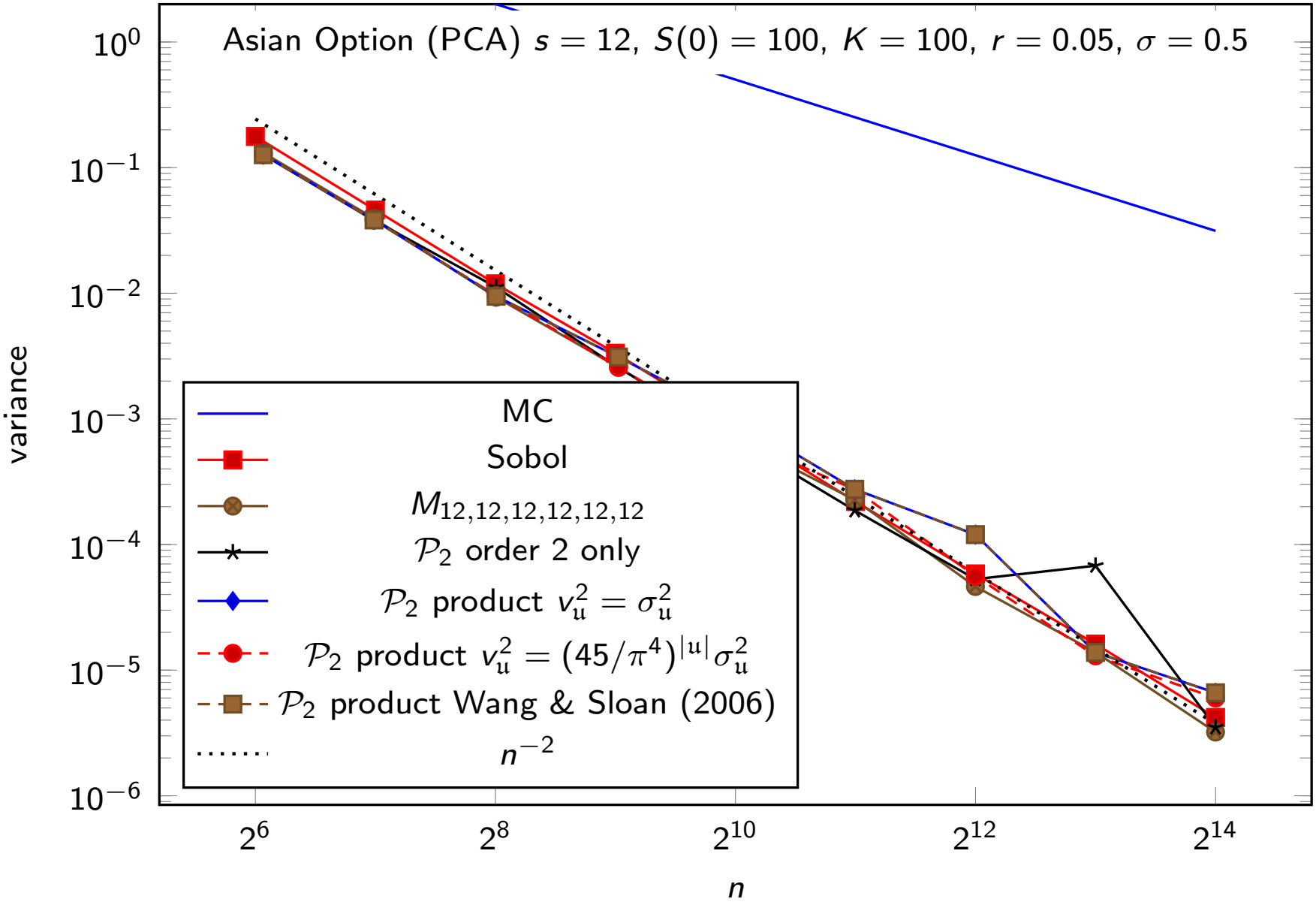
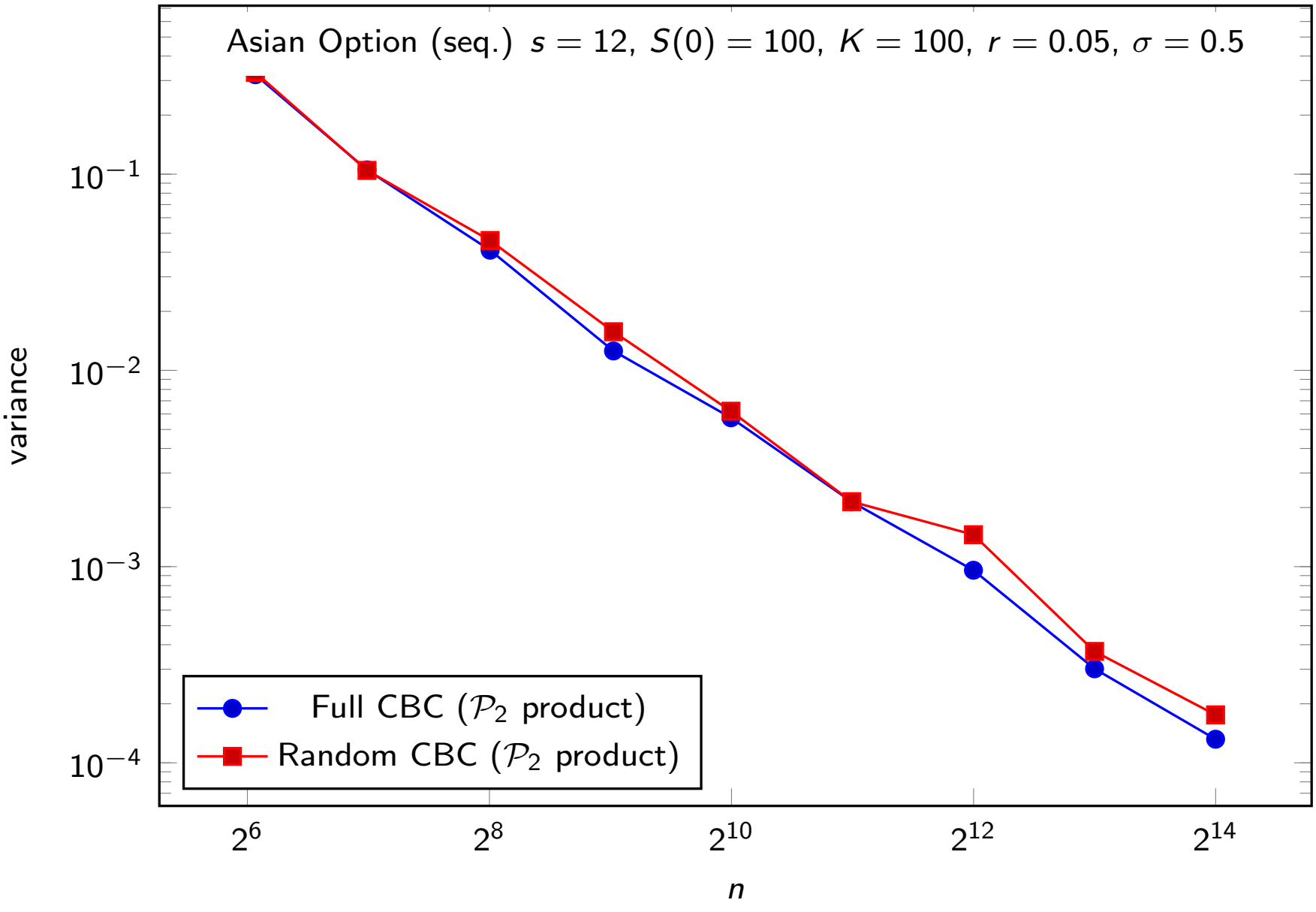
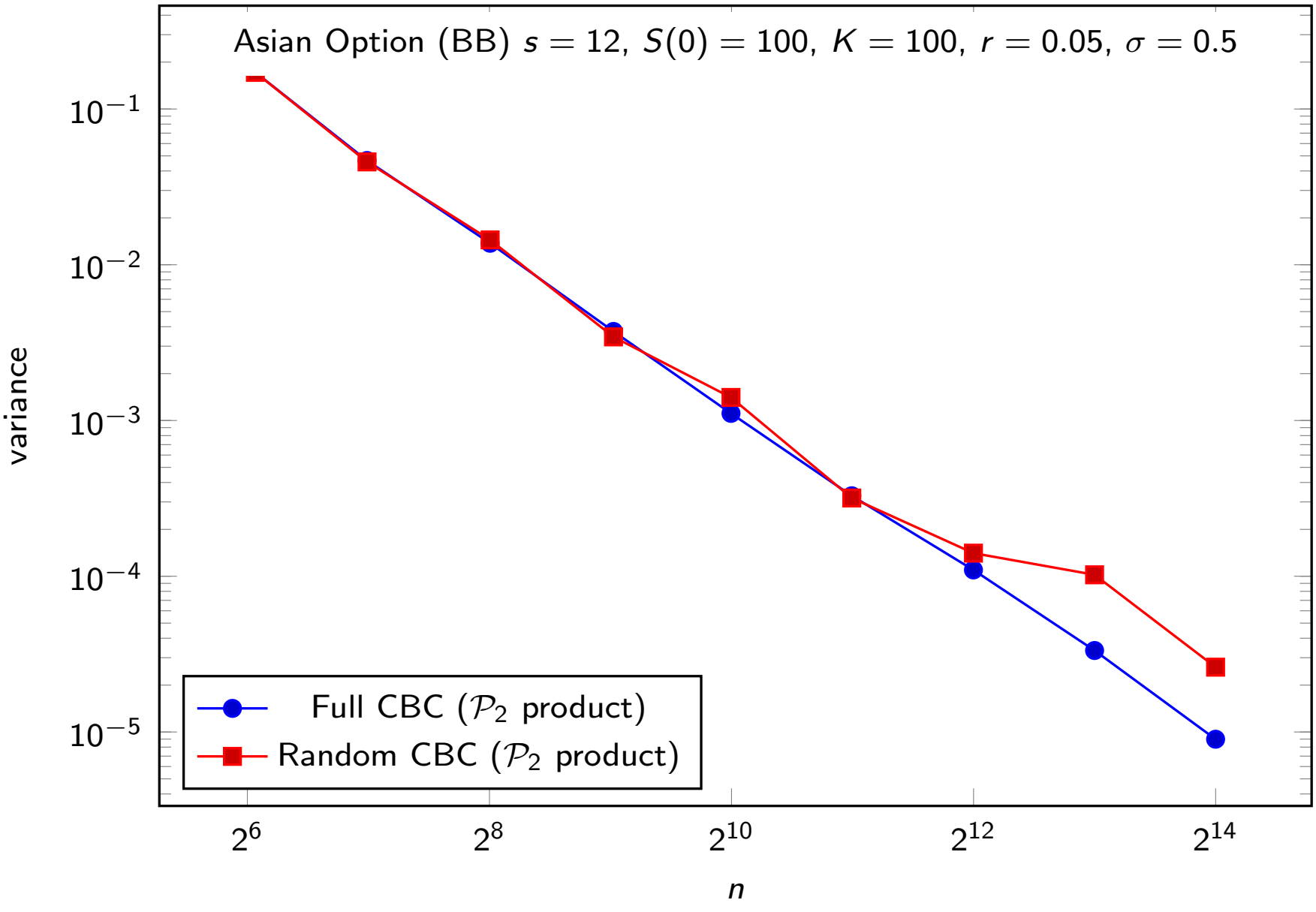


Random vs. Full CBC



Random vs. Full CBC



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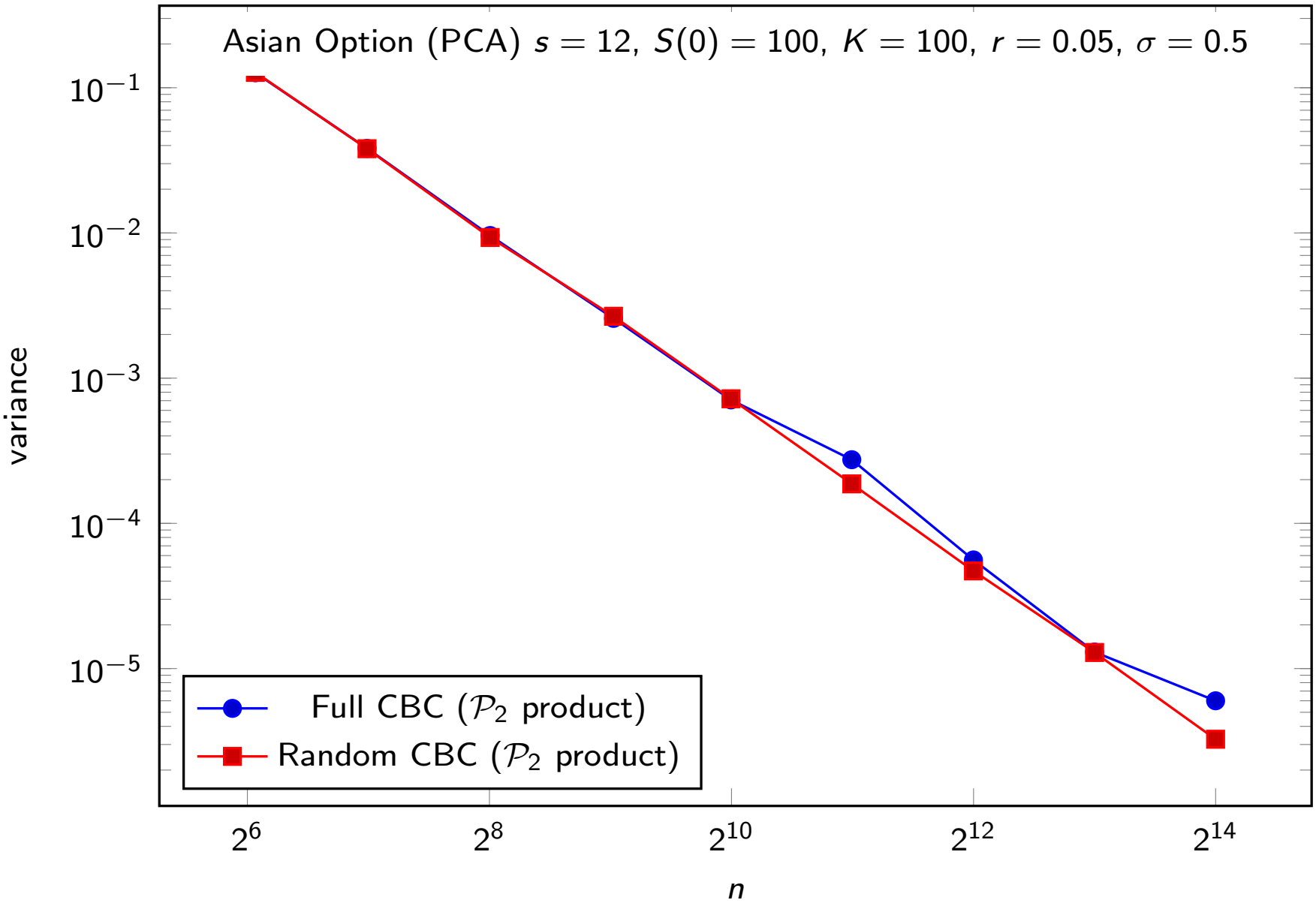
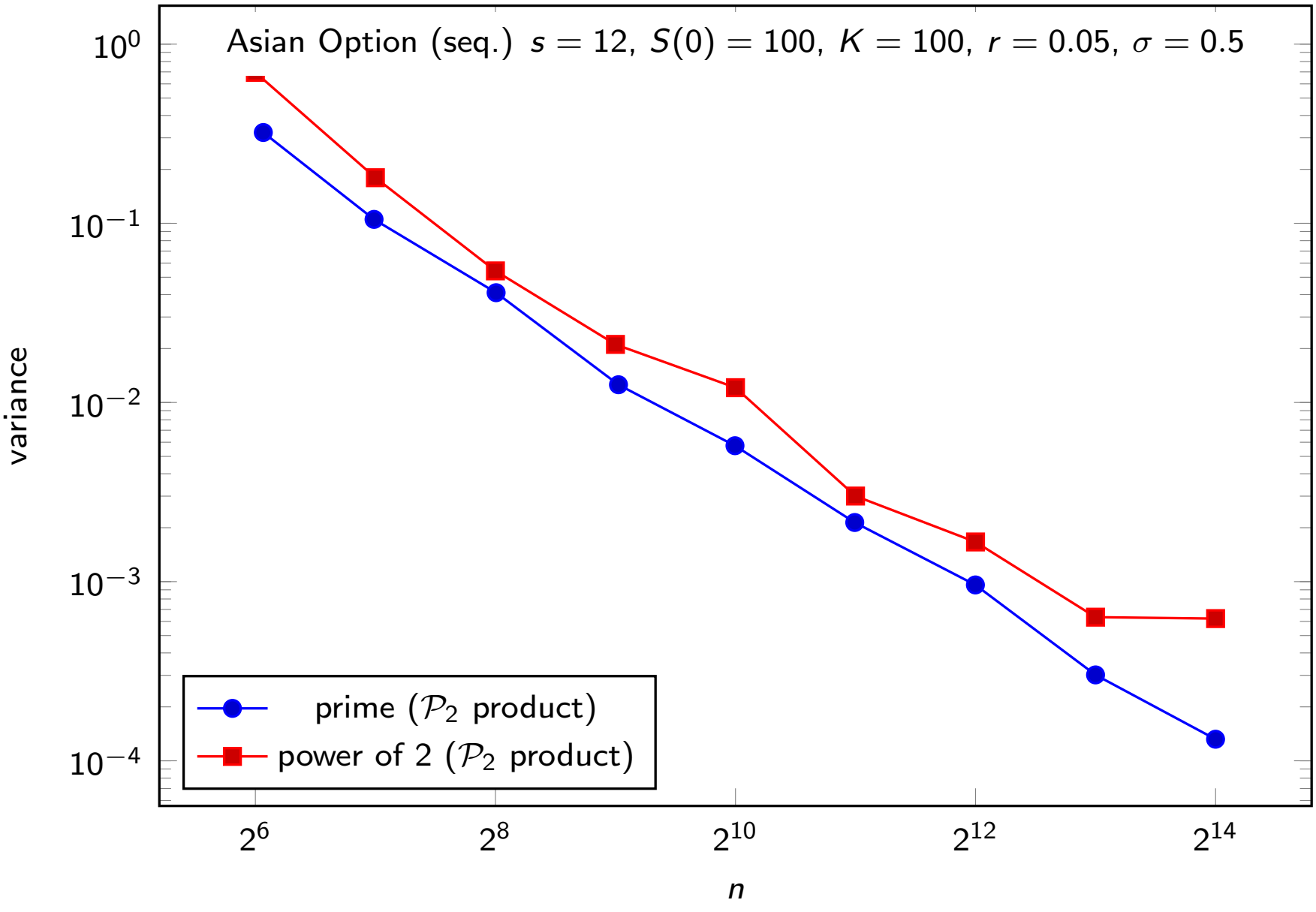


Prime vs. Power-of-2 Number of Points



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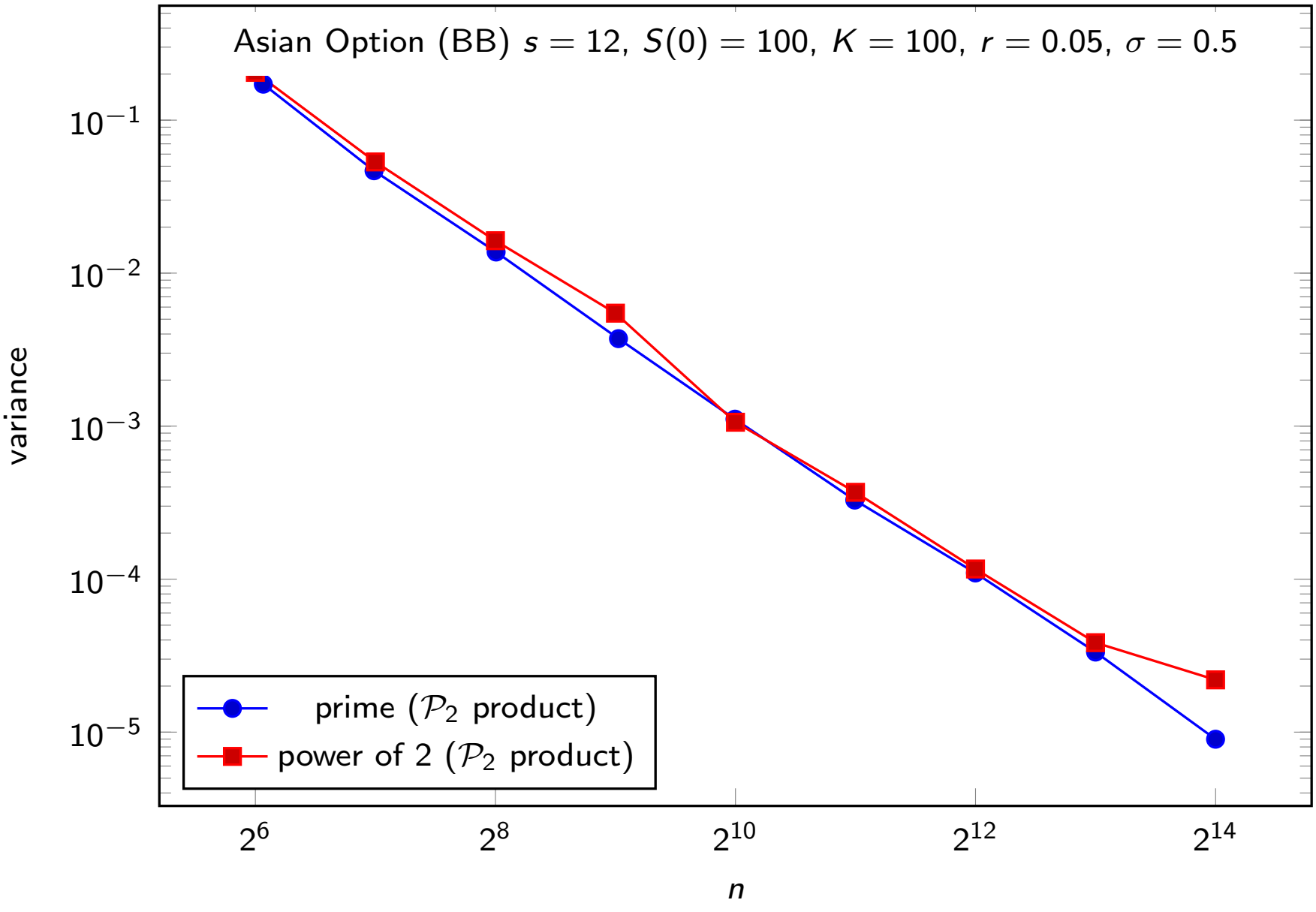

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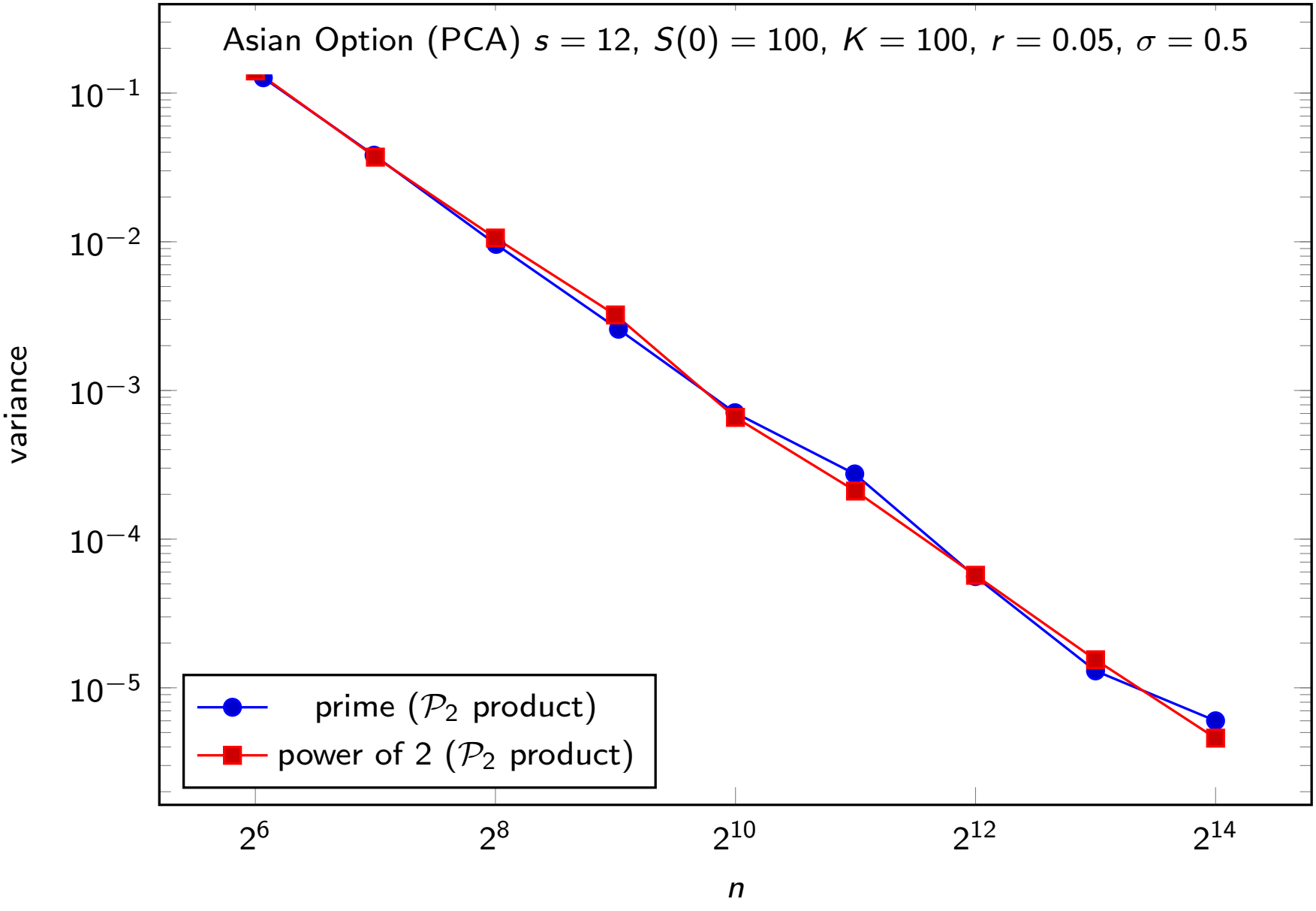
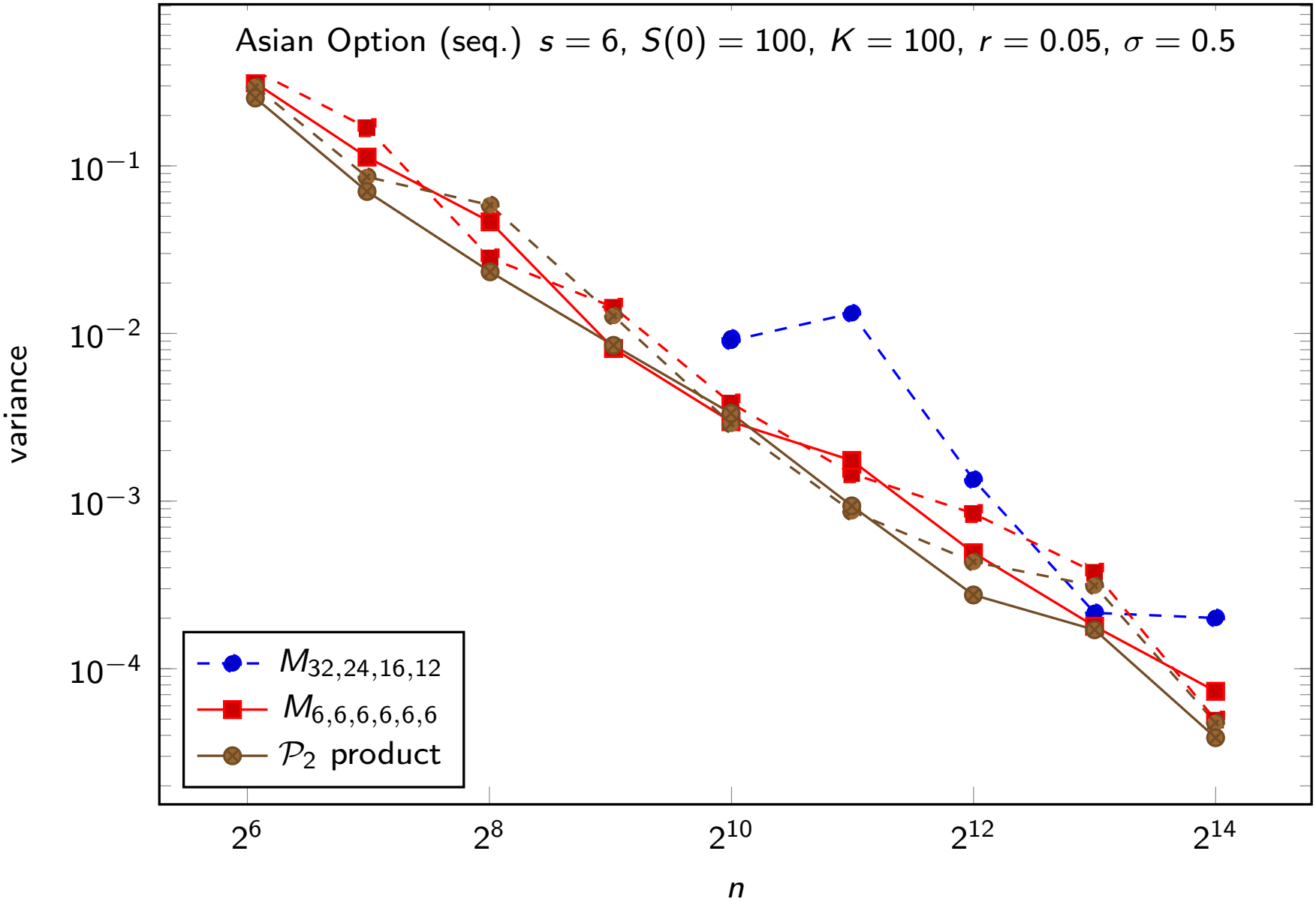


Korobov vs. CBC



Solid: CBC. Dashed: Korobov.

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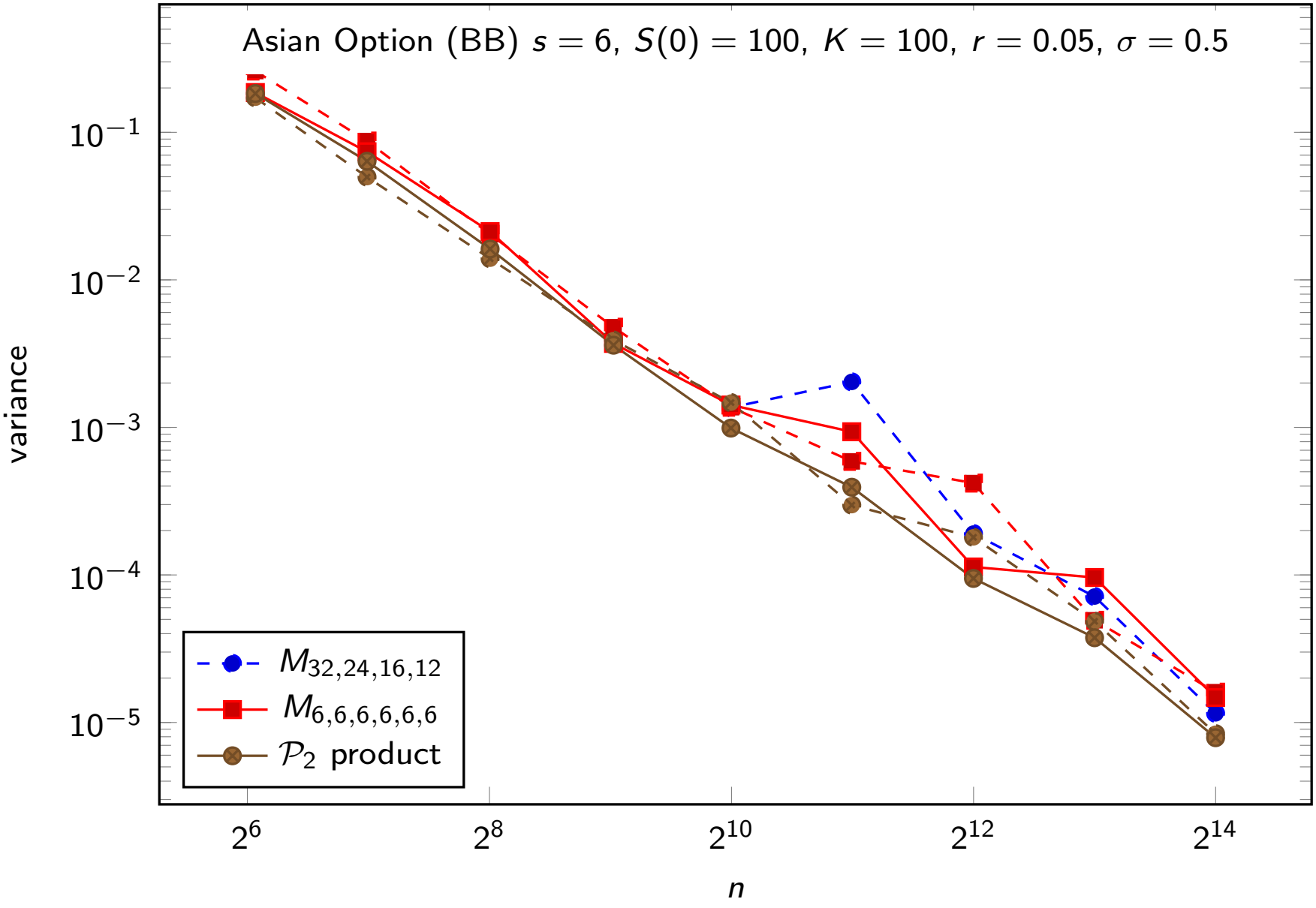

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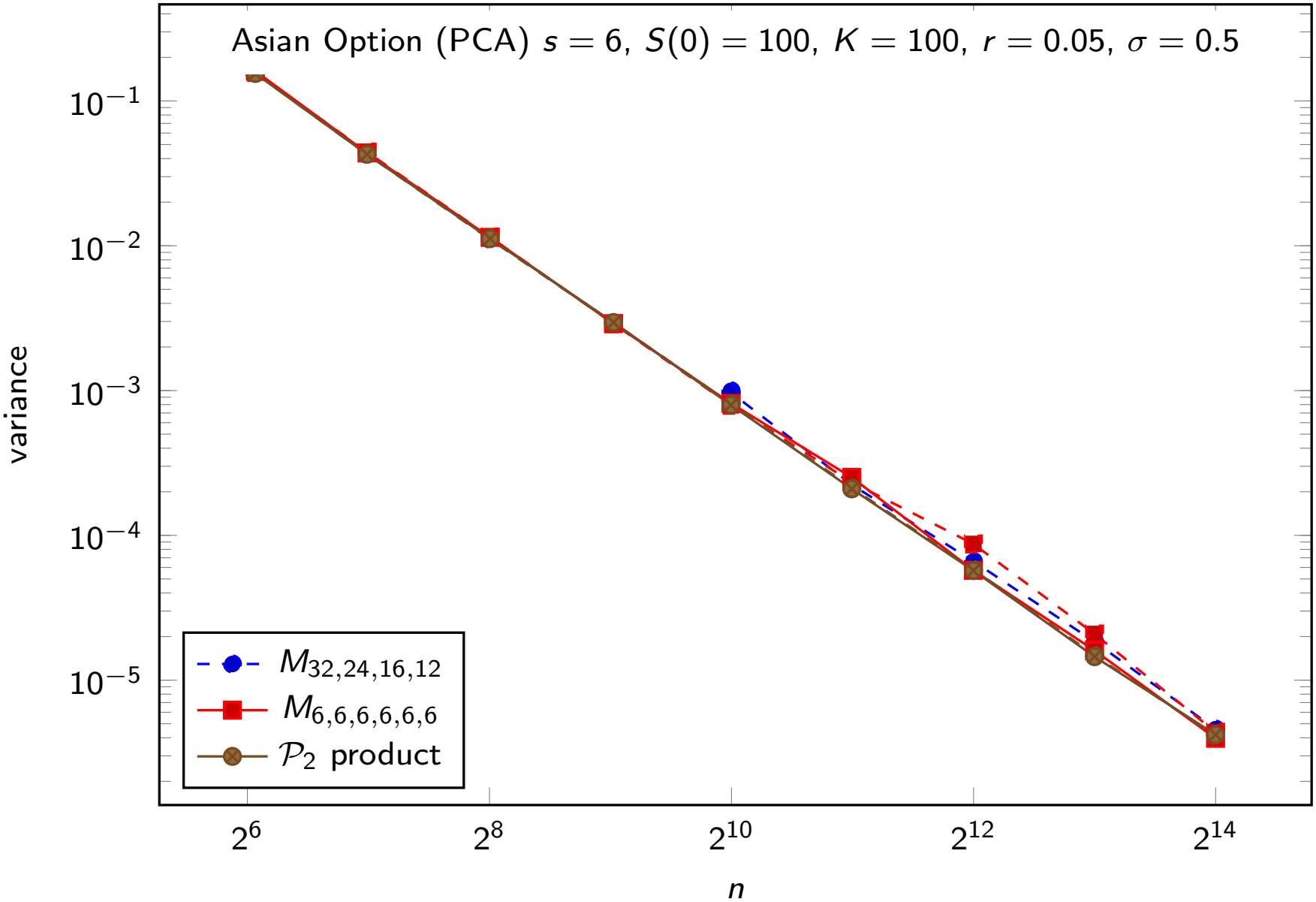
Korobov vs. CBC



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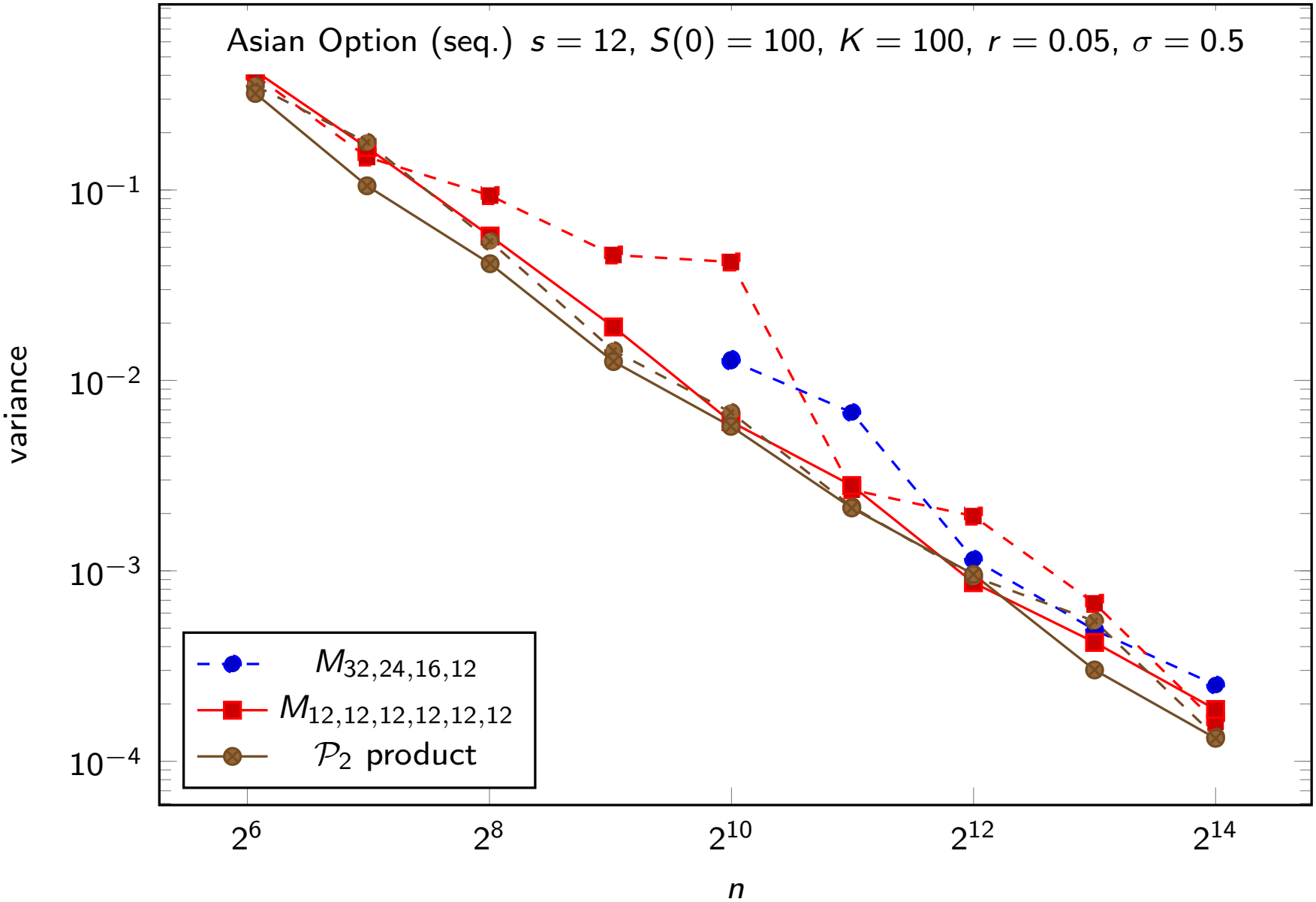
Korobov vs. CBC



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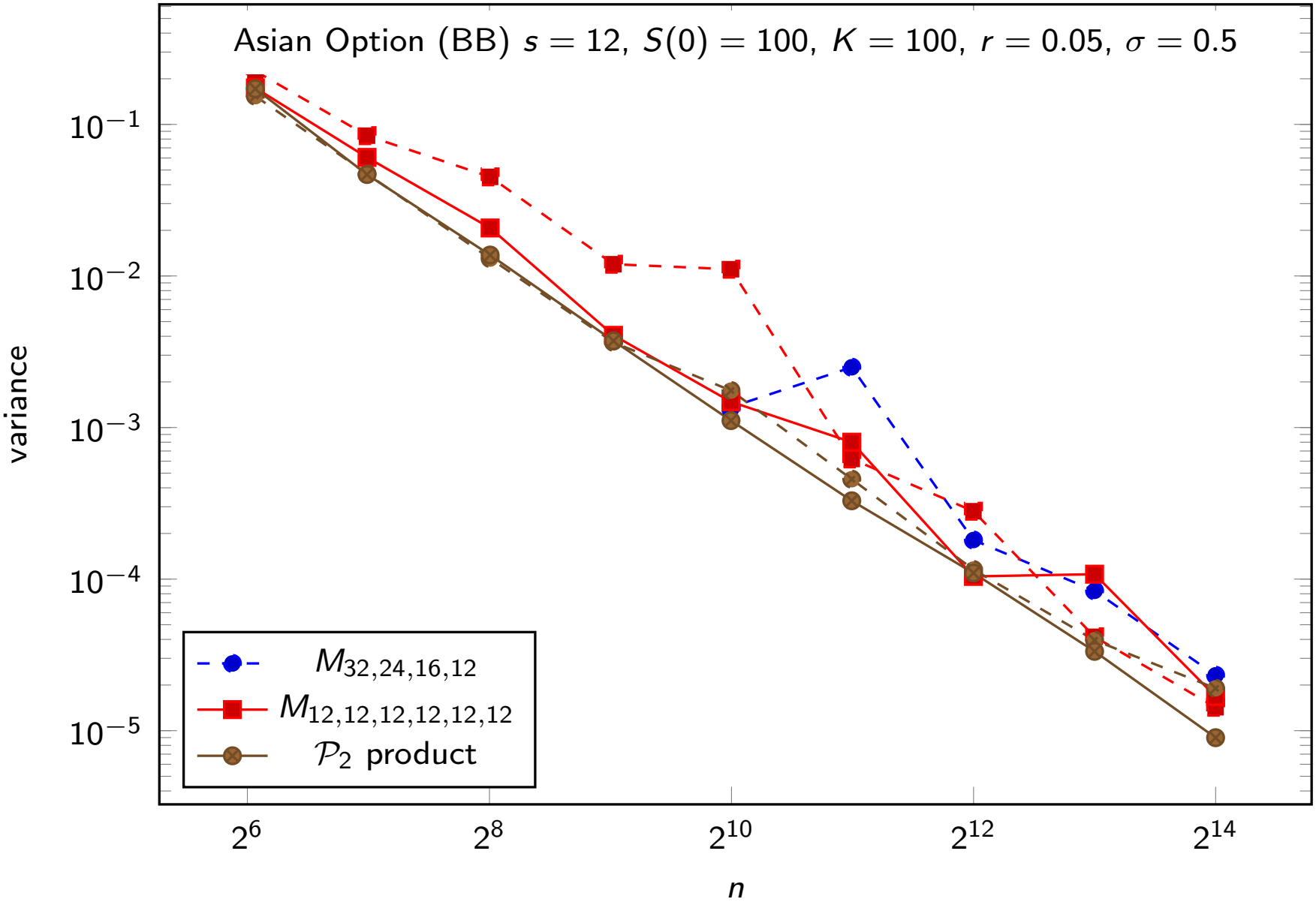

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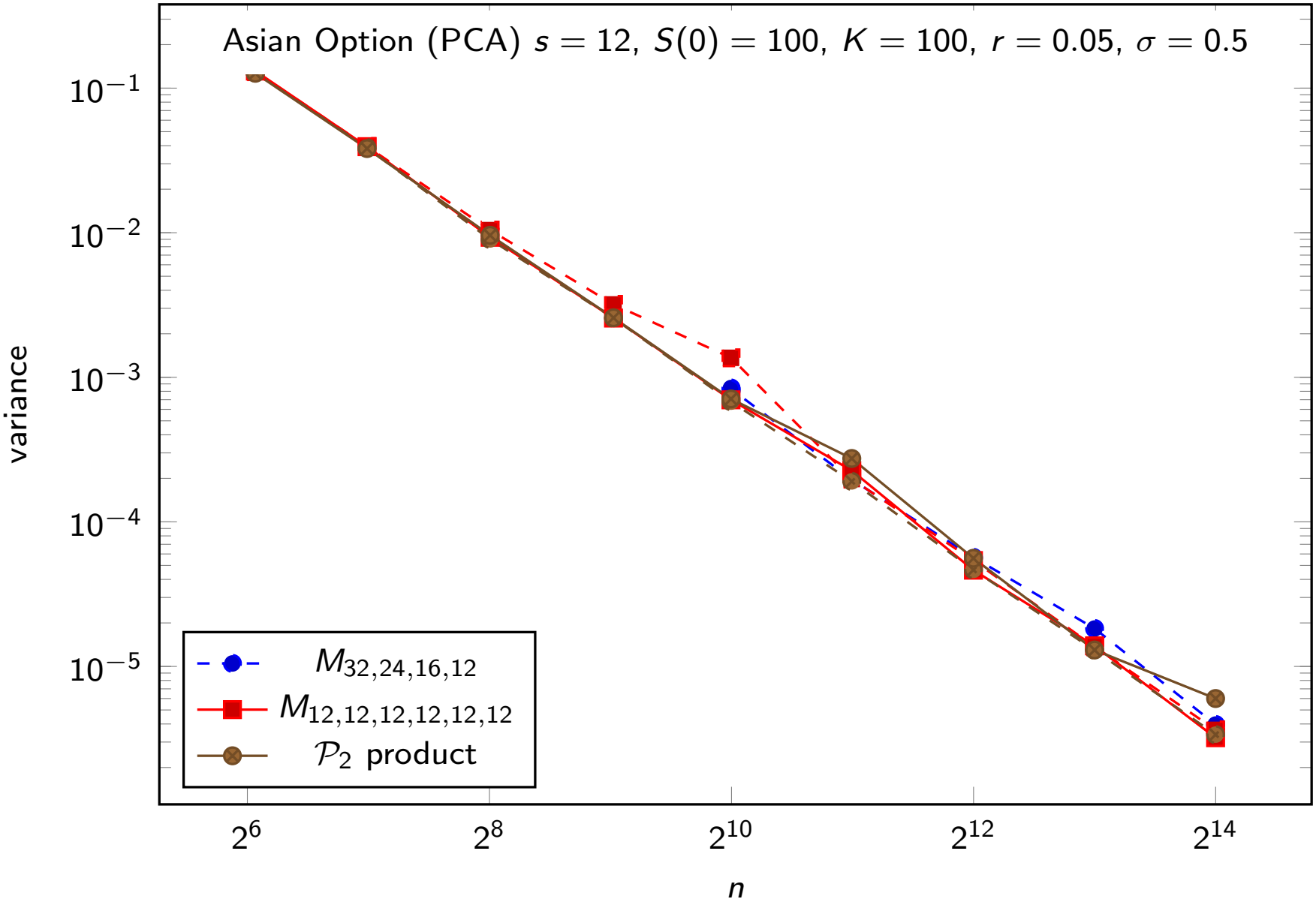
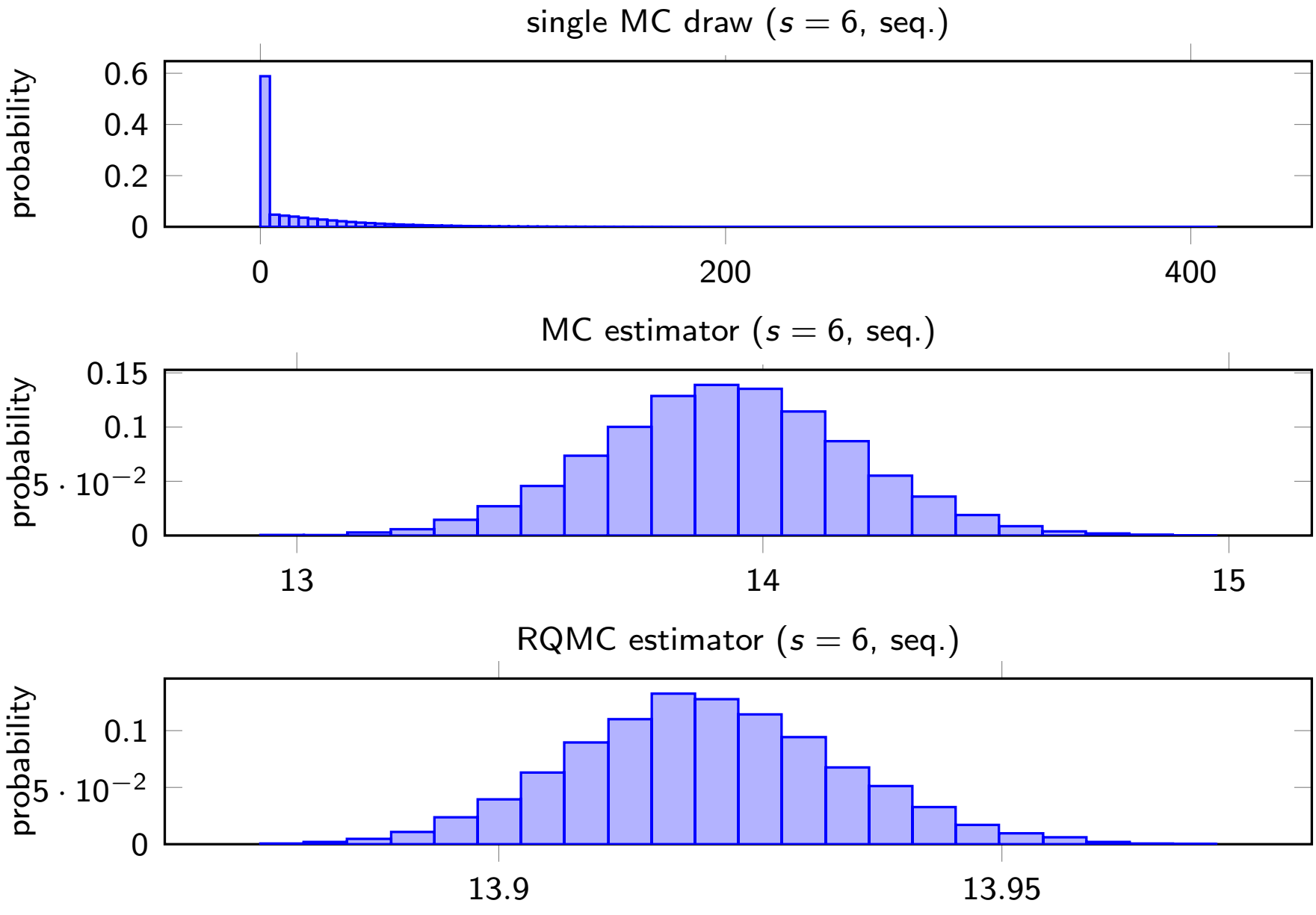
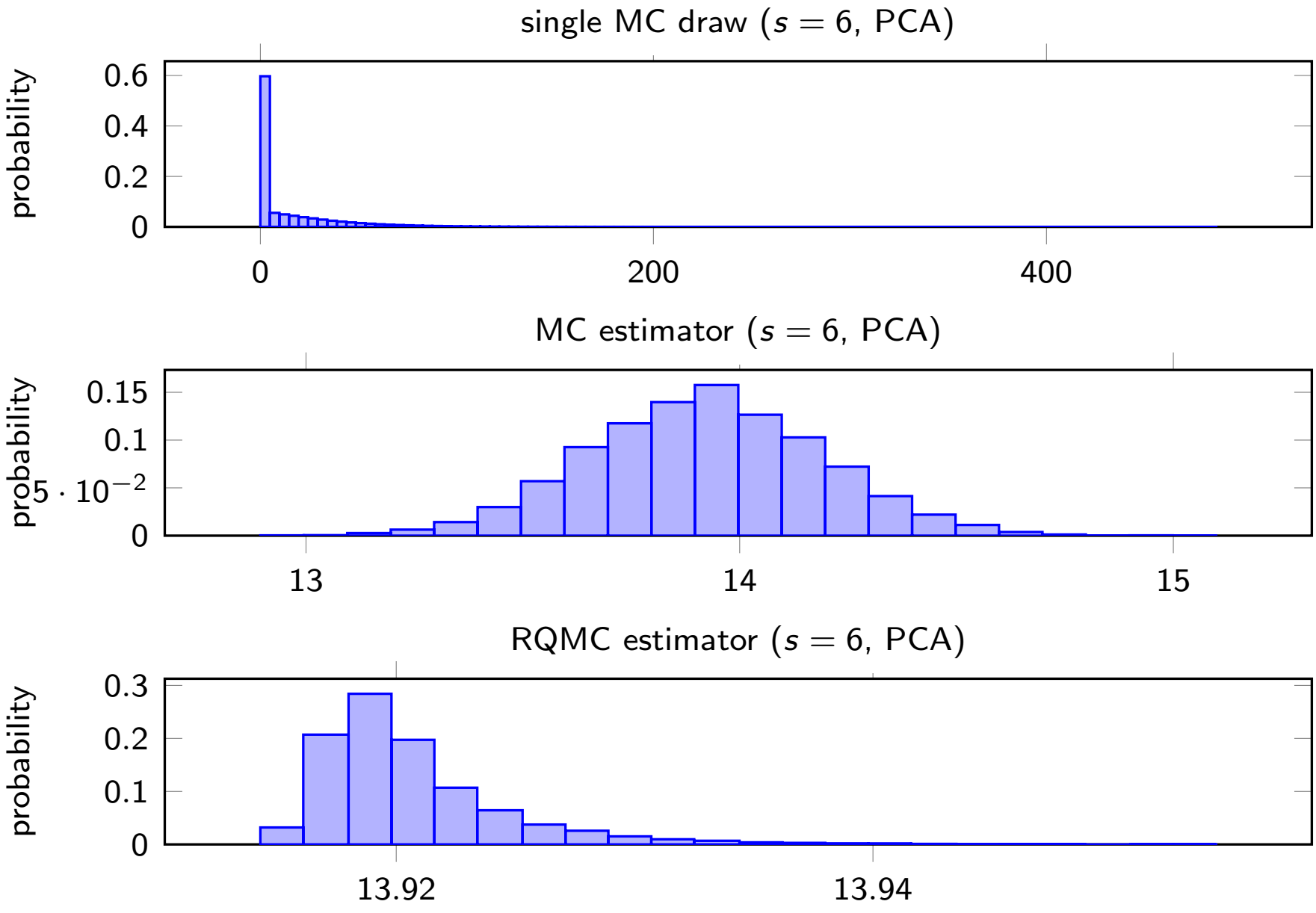


Solid: CBC. Dashed: Korobov.

Histograms for the Asian Option, $s = 6$, sequential



Histograms for the Asian option, $s = 6$, PCA

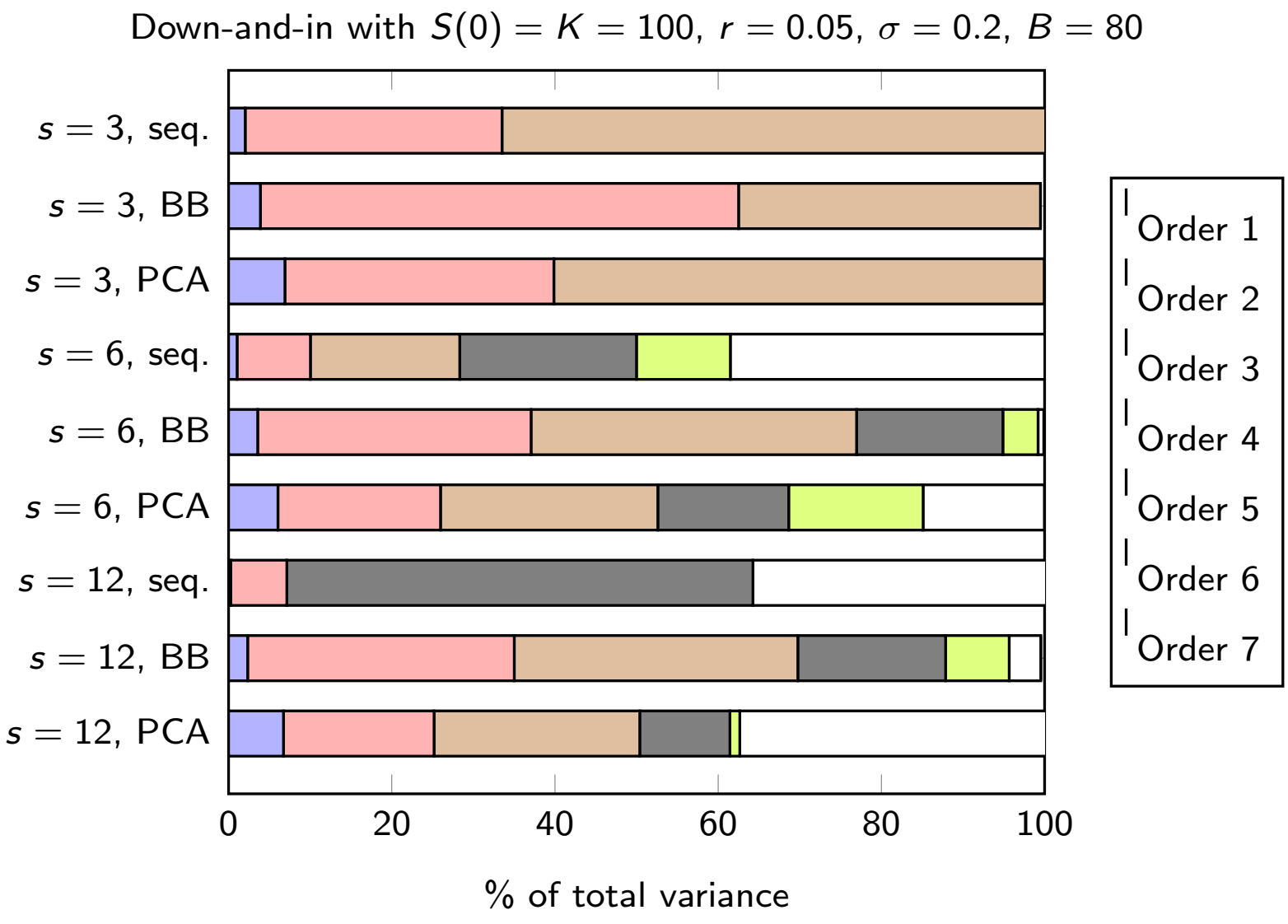


A down-and-in Asian option with barrier B

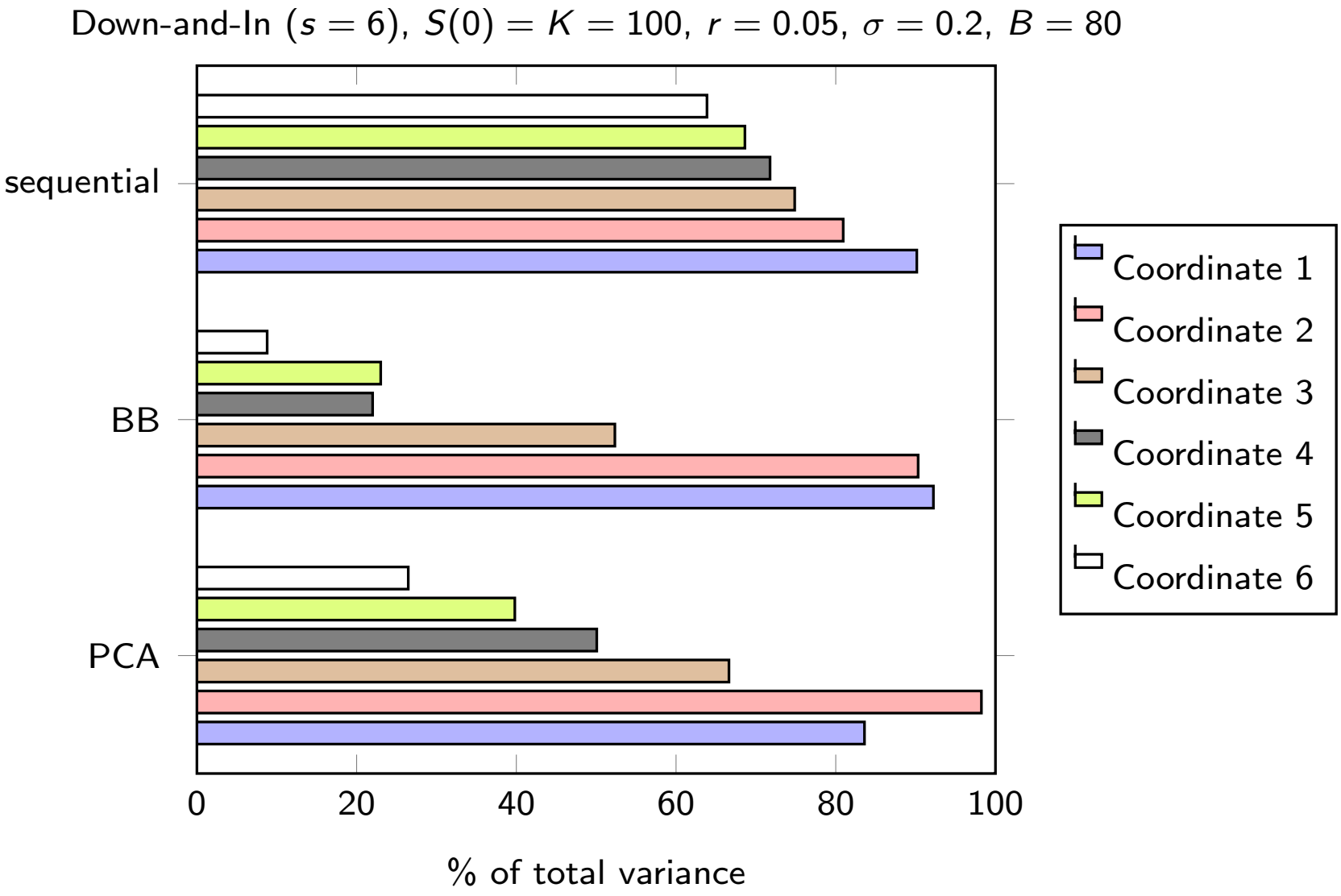
Same as for Asian option, except that payoff is zero unless

$$\min_{1 \le j \le s} S(t_j) \le 80.$$

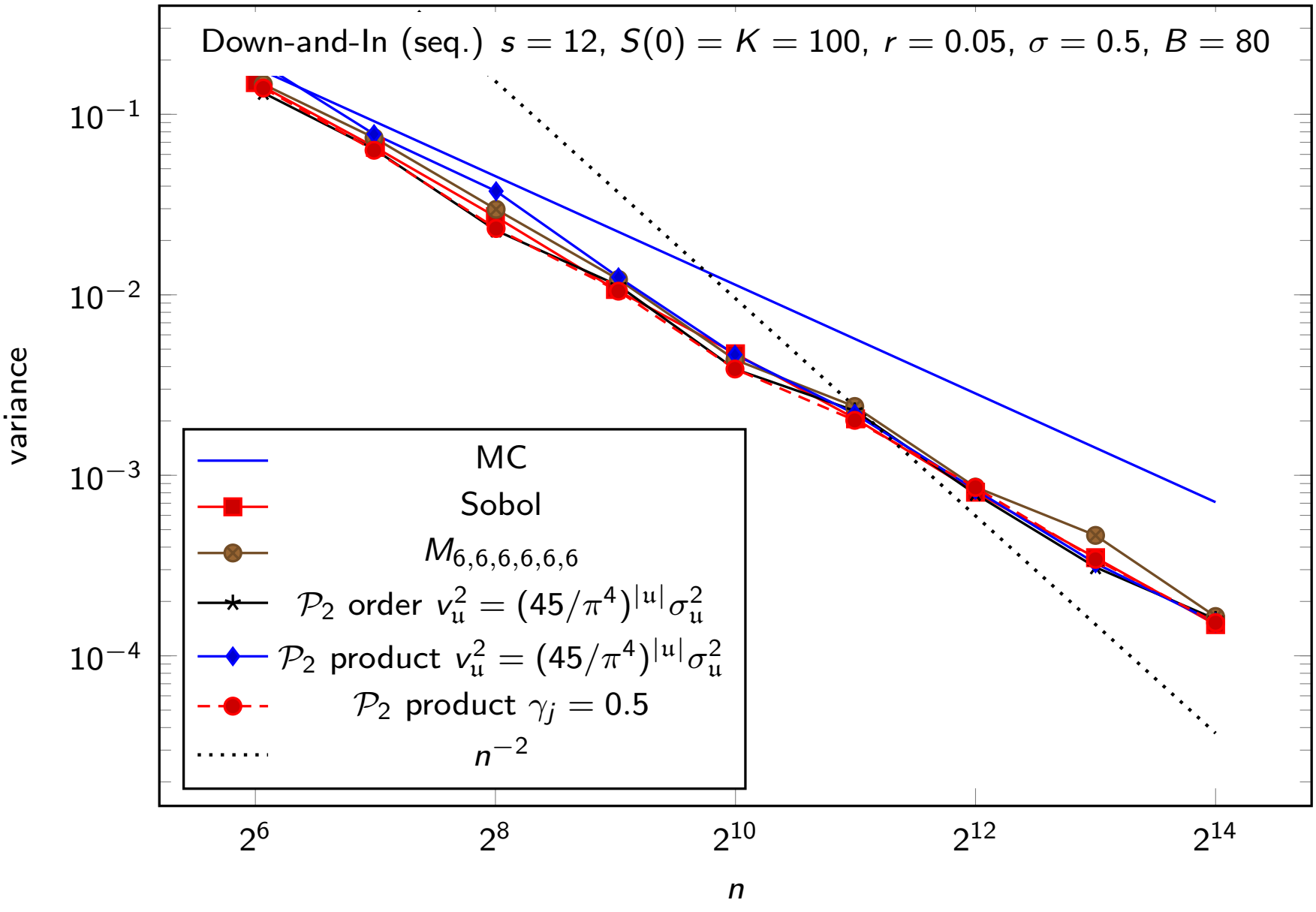
ANOVA Variances for the down-and-in Asian Option



Total Variance per Coordinate for the down-and-in Asian Option



Lattices of Rank 1 with CBC



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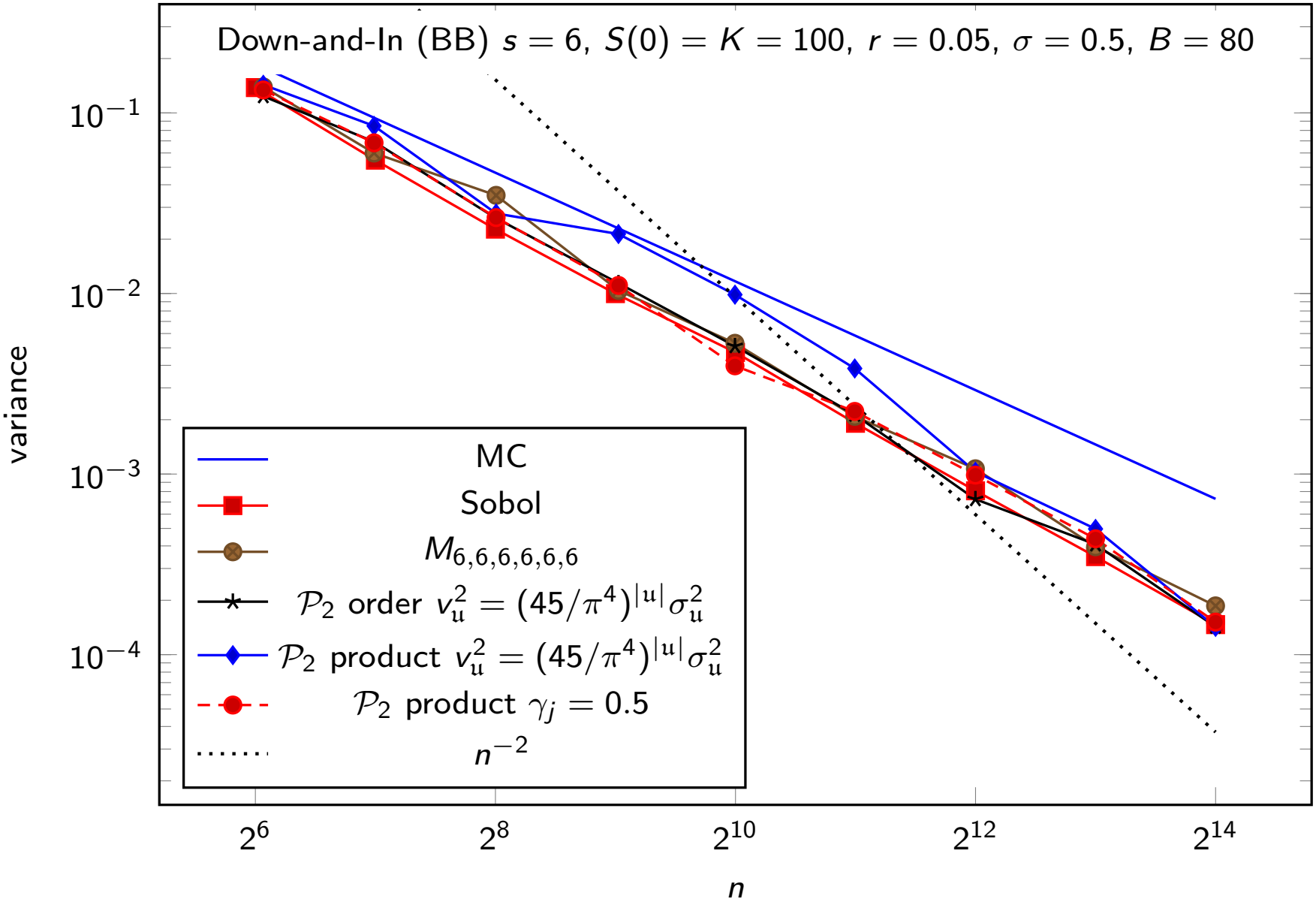

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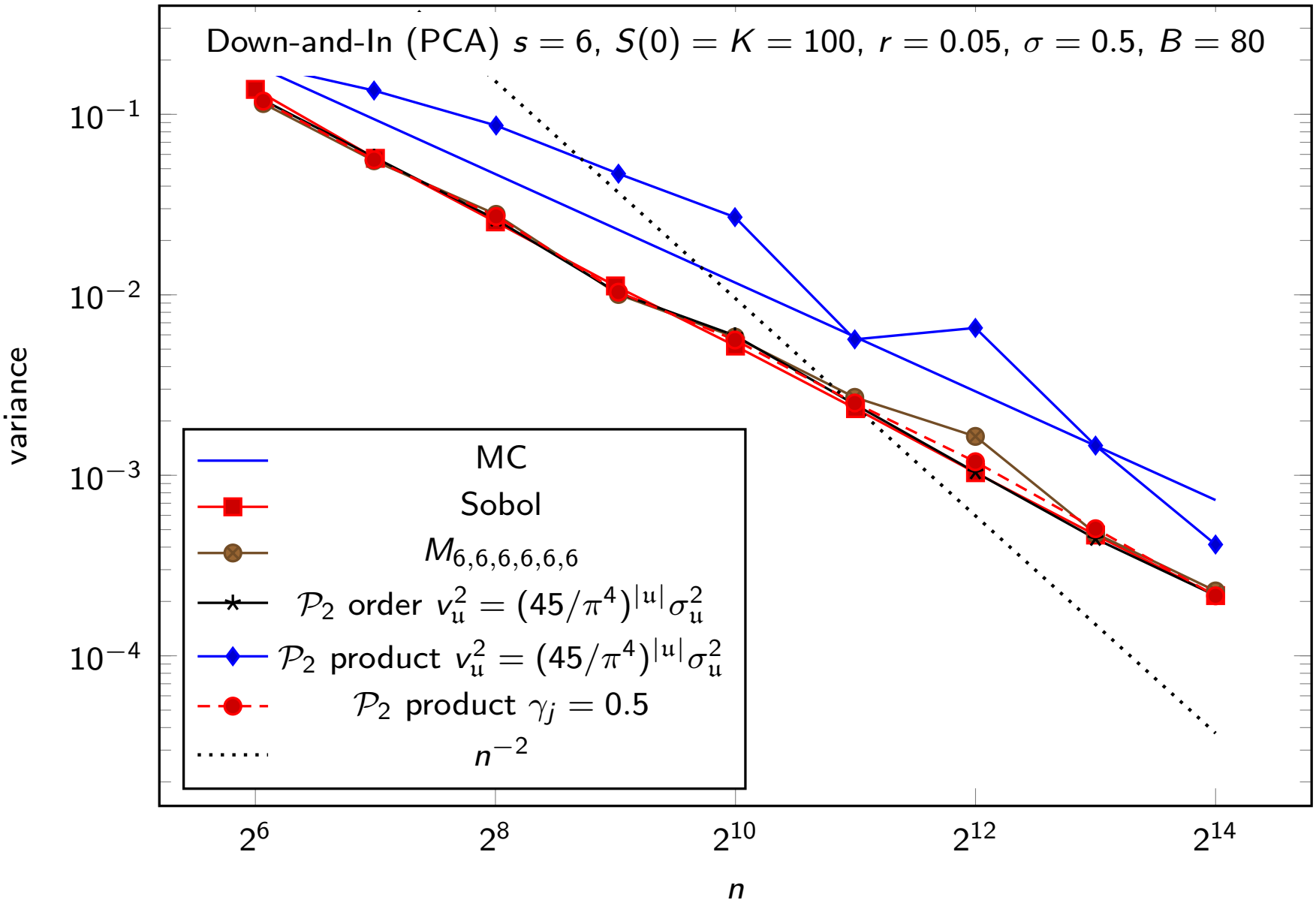
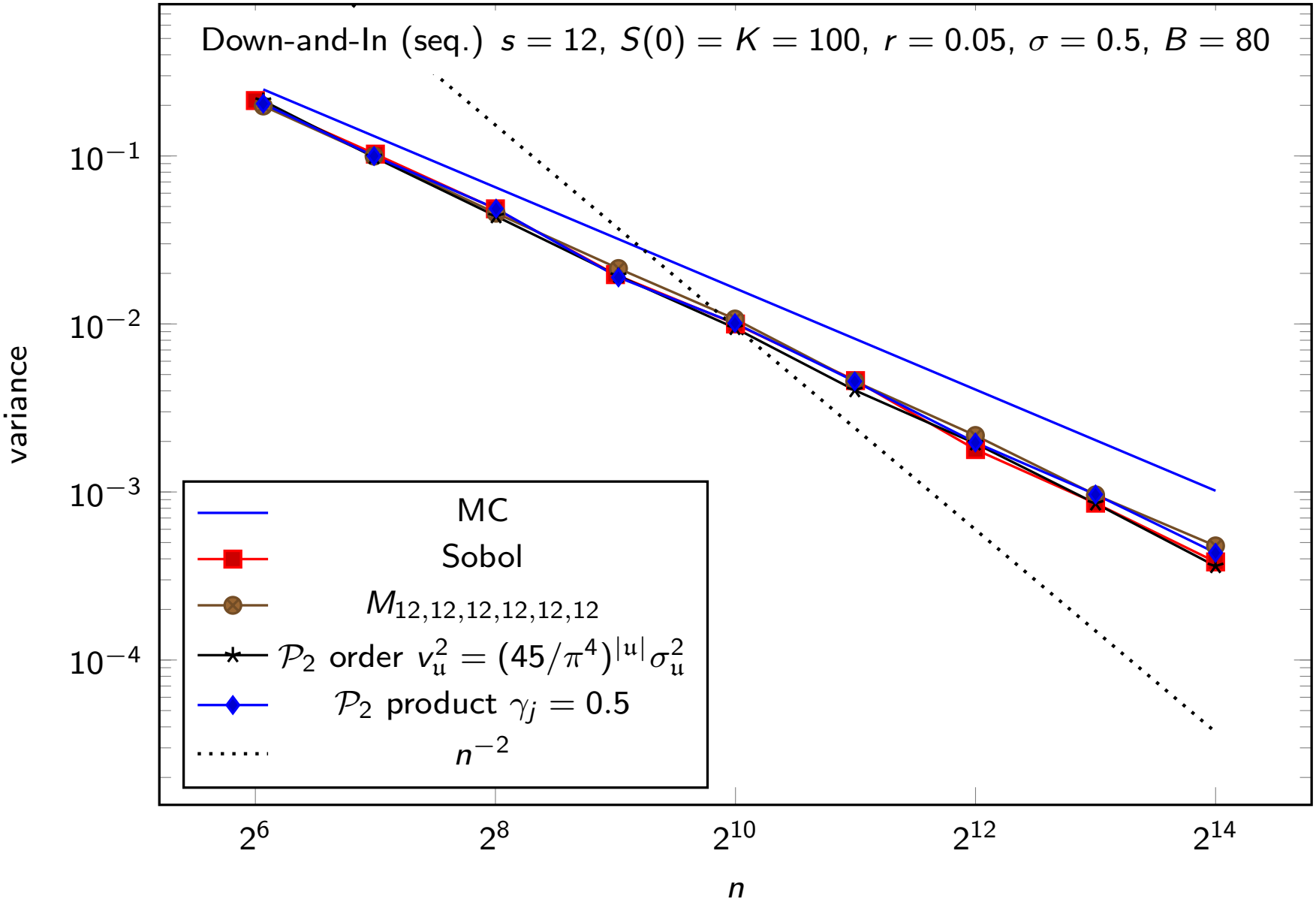


Lattices of Rank 1 with CBC



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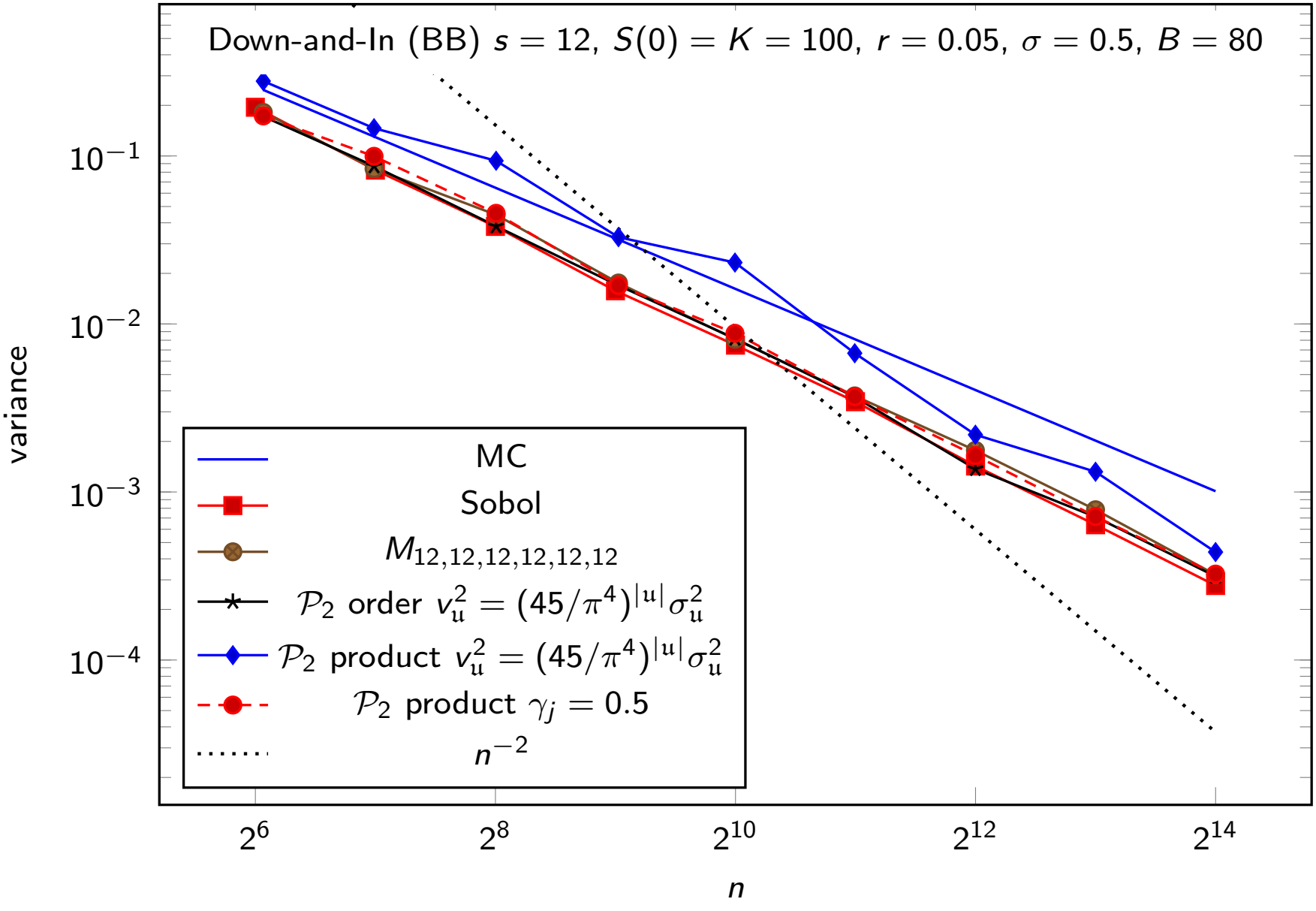

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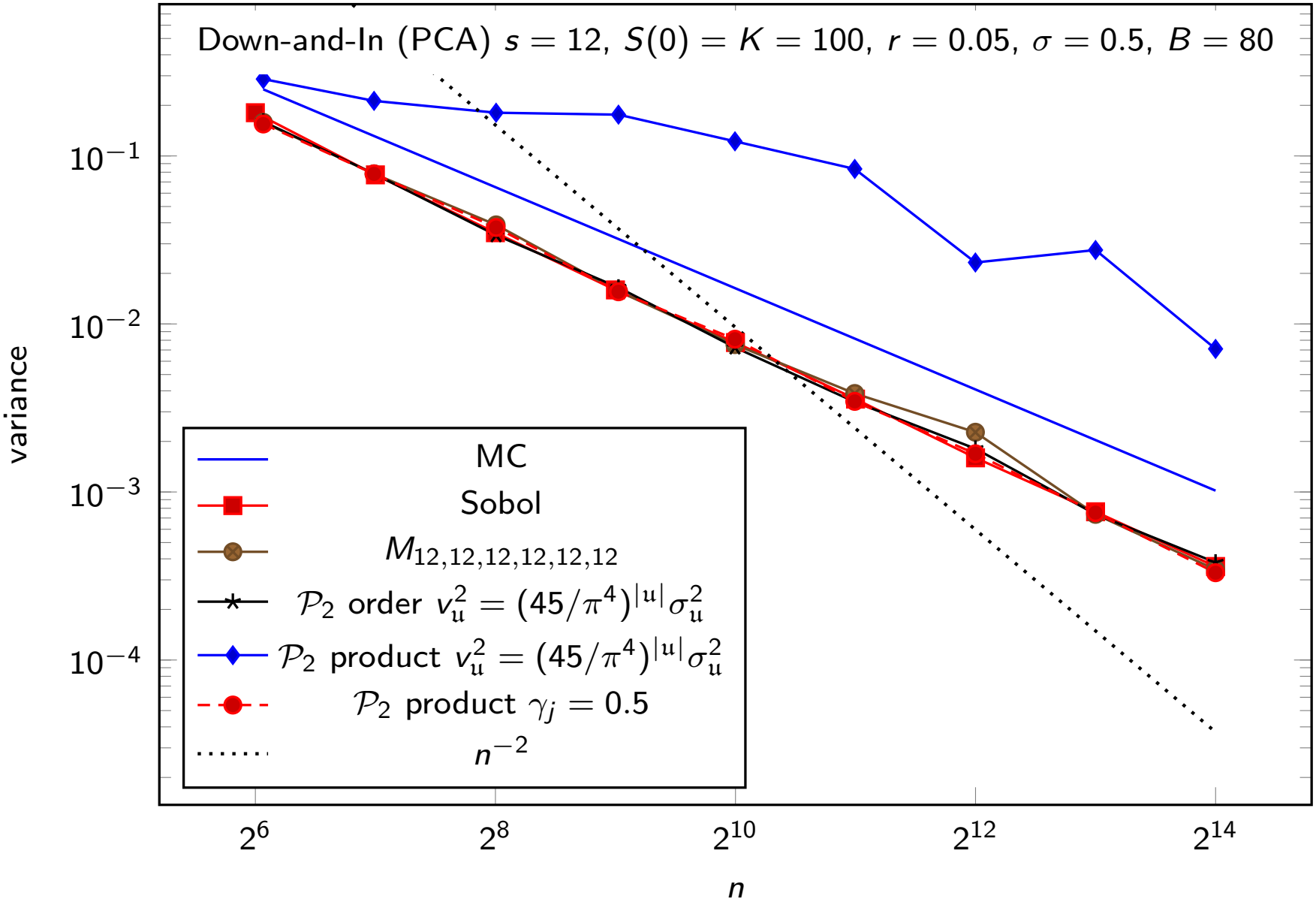
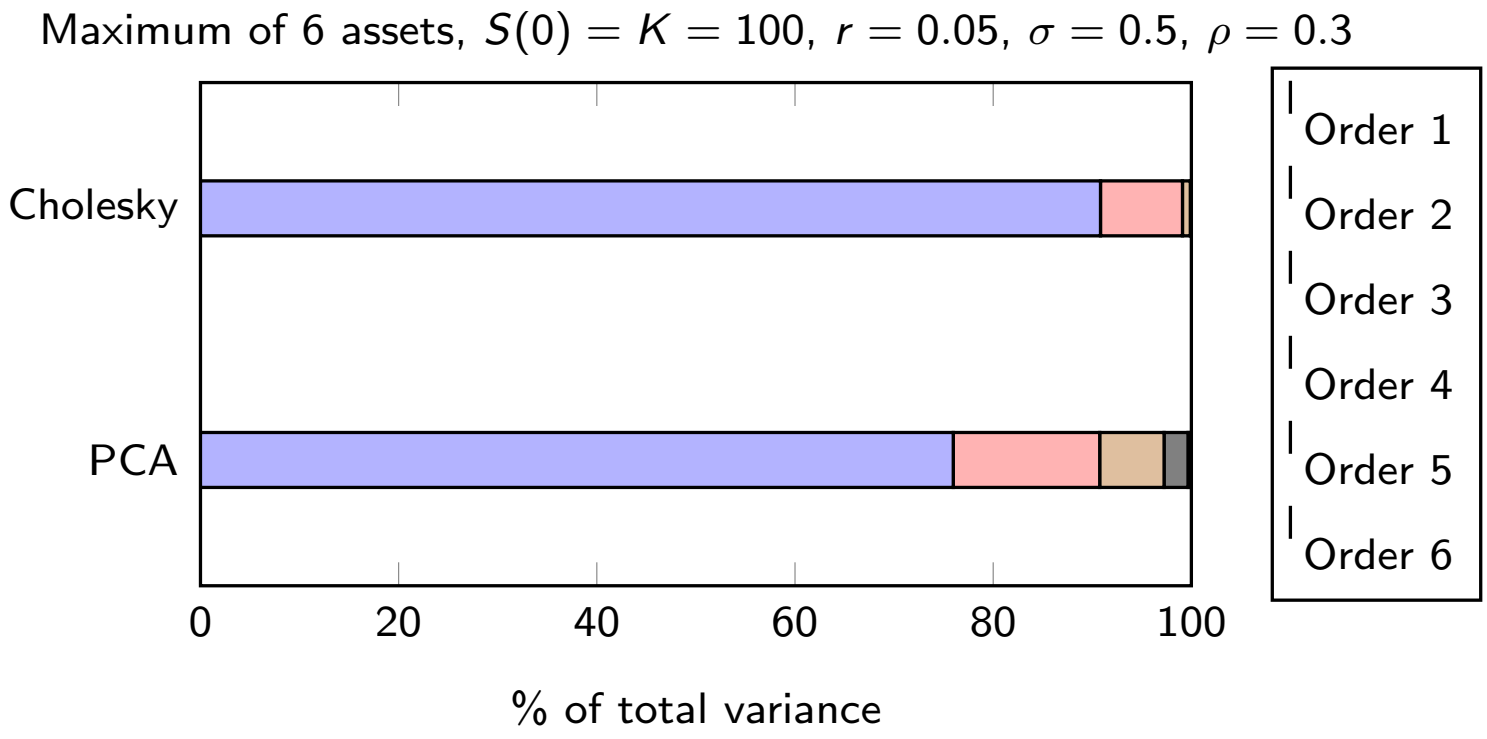


Call on the maximum of 6 assets

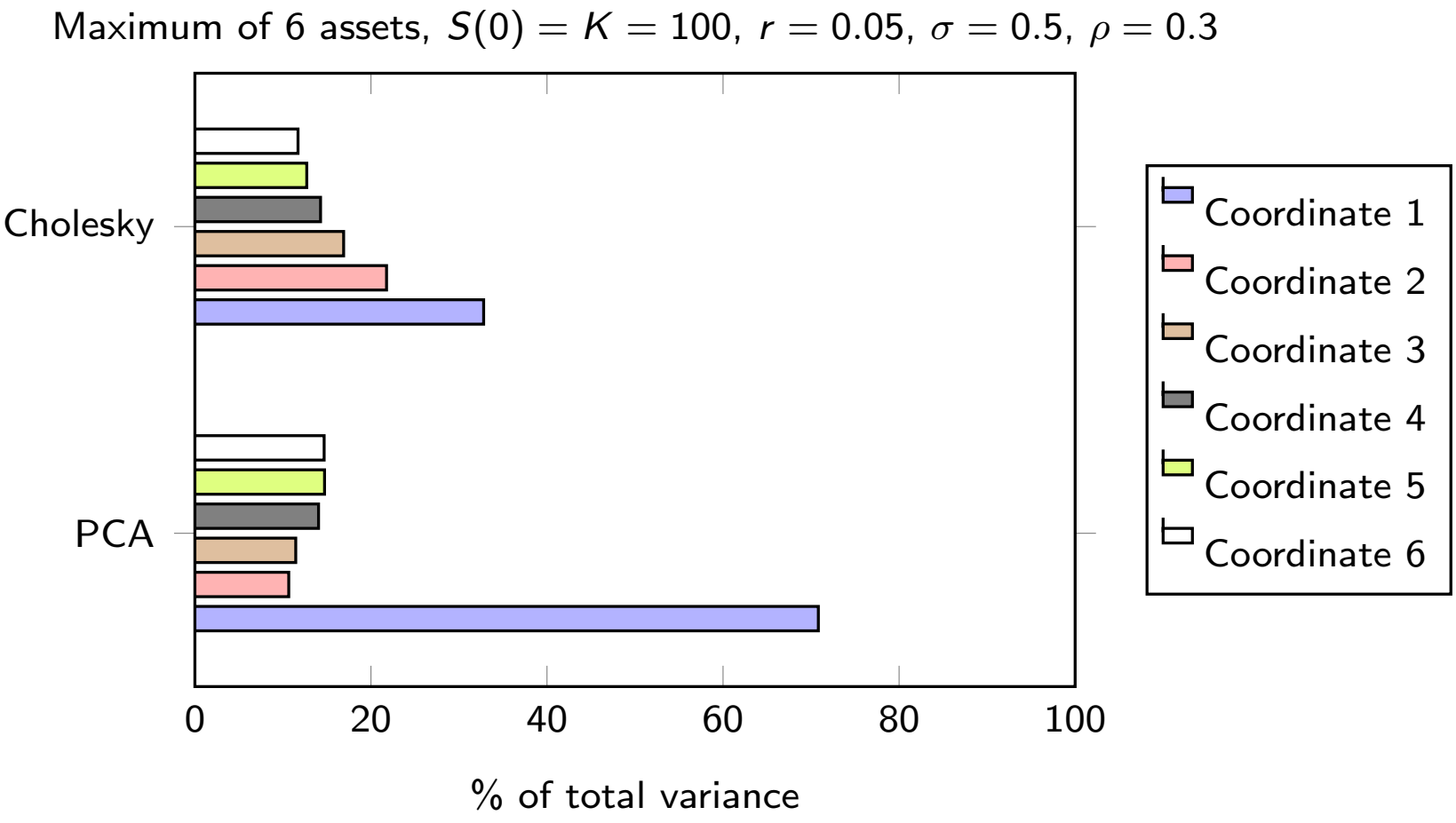
Each of 6 asset prices obeys a GBM with $s_0 = 100$, $r = 0.05$, $\sigma = 0.2$. The pairwise correlation between Brownian motions is 0.3.

The assets pay a dividend at rate 0.10, which means that the effective risk-free rate can be taken as $r' = 0.05 - 0.10 = -0.05$.

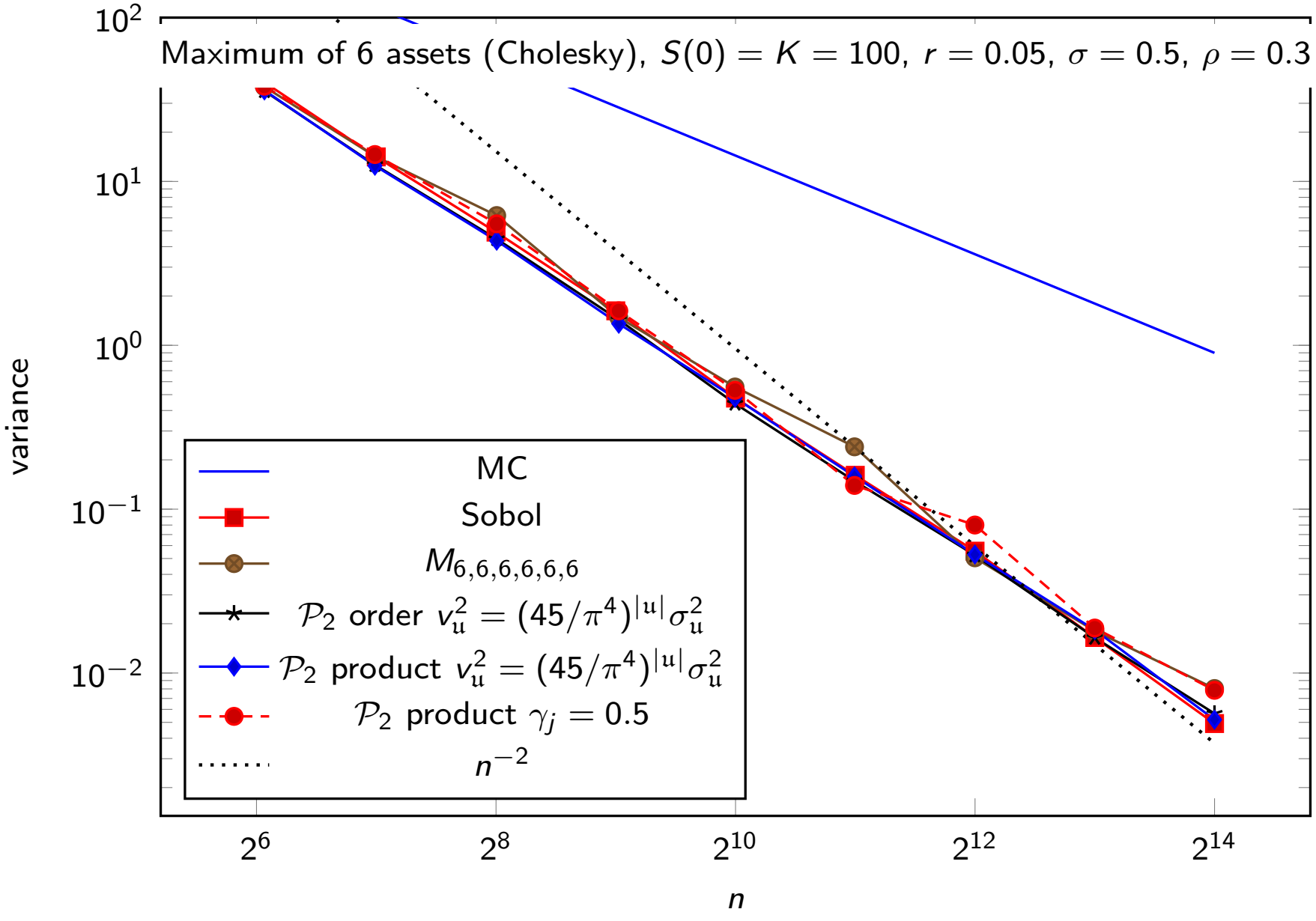
ANOVA variances for the maximum of 6 assets



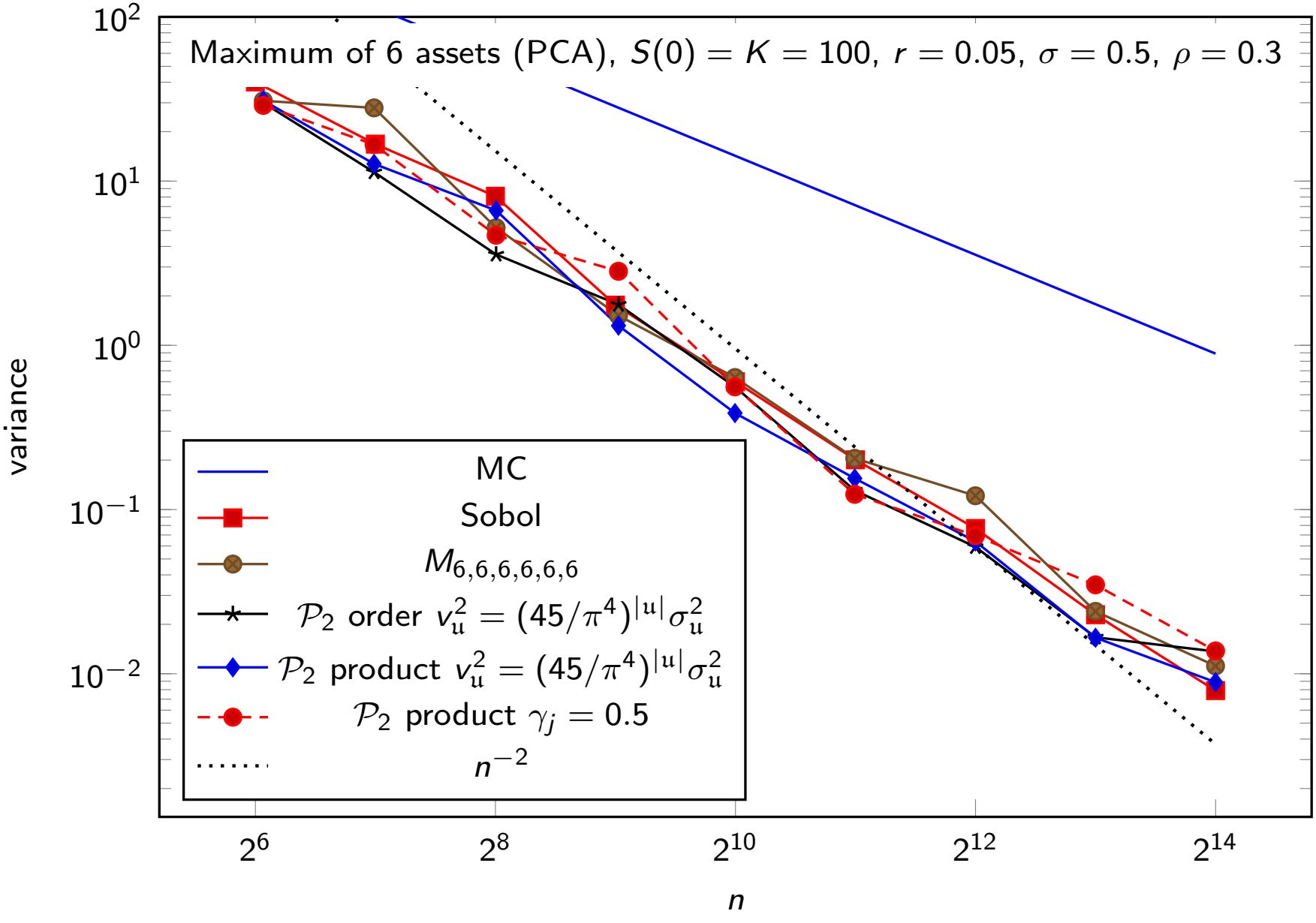
Total Variance per Coordinate for max of 6 assets



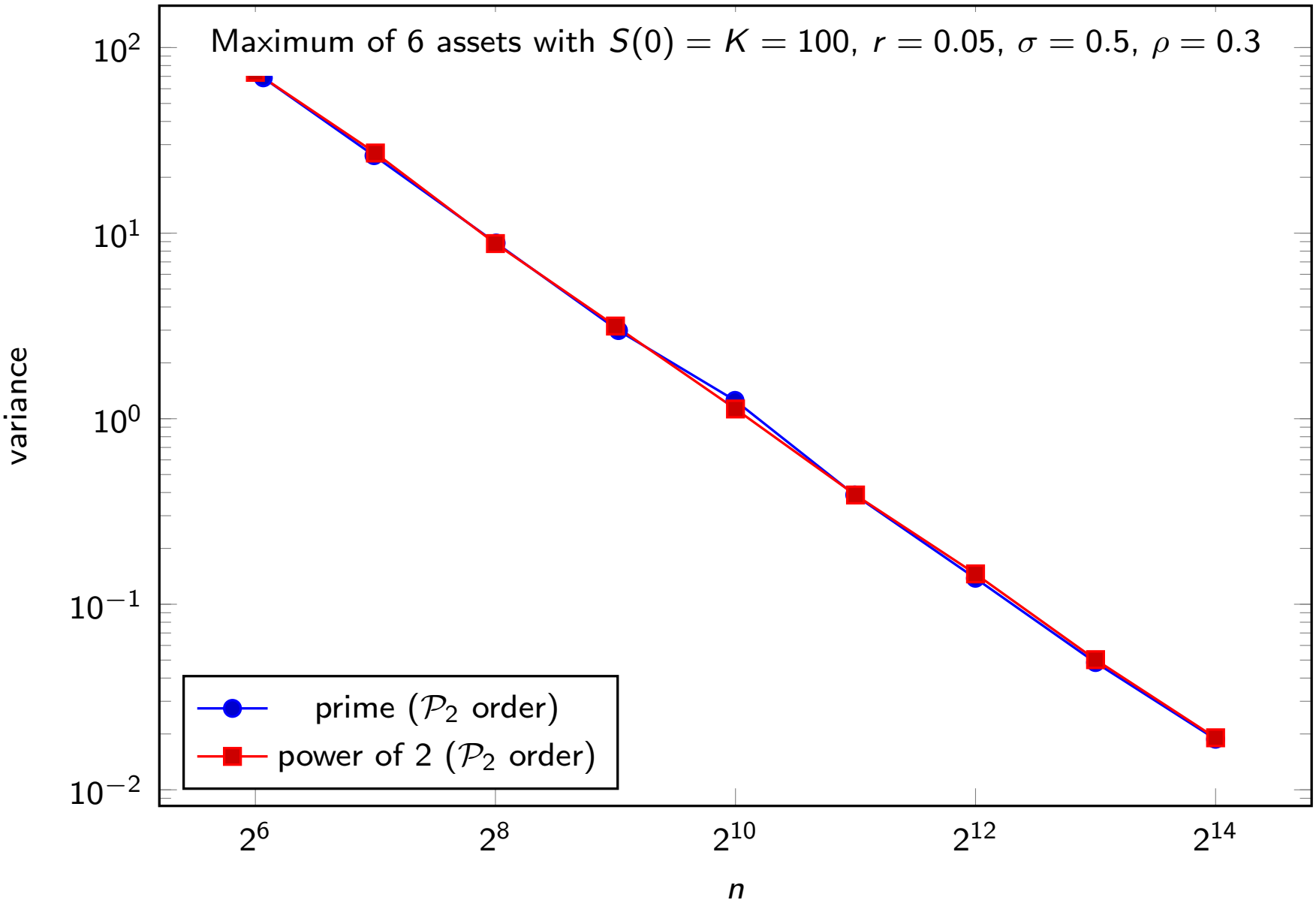
Lattices of Rank 1 with CBC



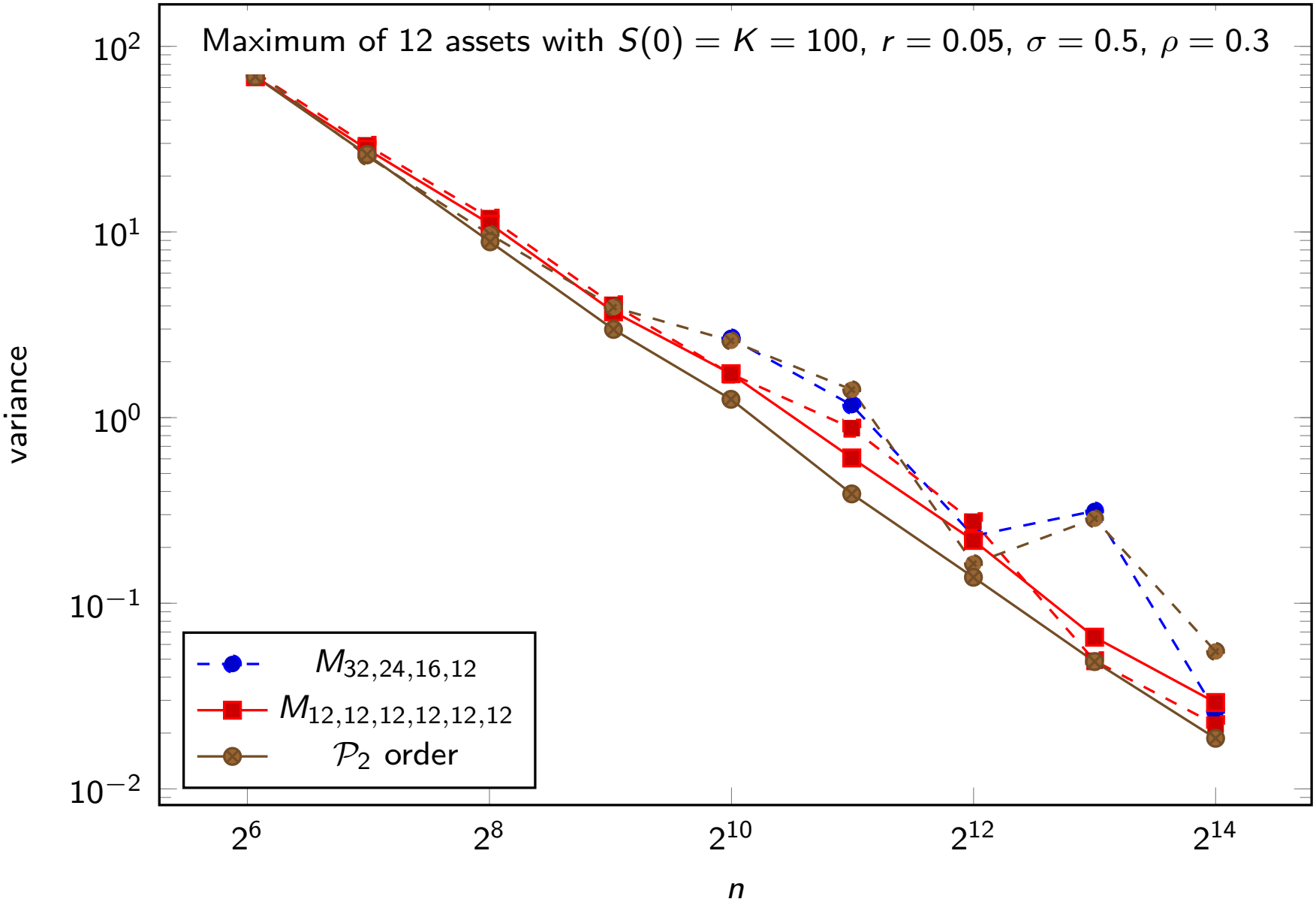
Lattices of Rank 1 with CBC



Prime vs. Power-of-2 Number of Points



Korobov vs. CBC



Solid: CBC. Dashed: Korobov.

Discrete choice with multinomial mixed logit probability, max likelihood estimation

Utility of alternative j for individual q is

$$
\begin{aligned}
U_{q,j} &= \boldsymbol{\beta}_q^{\mathsf{t}} \mathbf{x}_{q,j} + \epsilon_{q,j} = \sum_{\ell=1}^{s} \beta_{q,\ell} x_{q,j,\ell} + \epsilon_{q,j}, \text{ where} \\
\boldsymbol{\beta}_q^{\mathsf{t}} &= (\beta_{q,1}, \ldots, \beta_{q,s}) \text{ gives the tastes of individual } q, \\
\mathbf{x}_{q,j}^{\mathsf{t}} &= (x_{q,j,1}, \ldots, x_{q,j,s}) \text{ attributes of alternative } j \text{ for individual } q, \\
\epsilon_{q,j} & \quad \text{noise; Gumbel of mean 0 and scale parameter } \lambda = 1.
\end{aligned}
$$

Individual q selects alternative with largest utility $U_{q,j}$.

Can observe the $\mathbf{x}_{q,j}$ and choices y_q, but not the rest.

Logit model: for β_q fixed, j is chosen with probability

$$L_q(j \mid \beta_q) = \frac{\exp[\beta_q^{\mathrm{t}} \mathbf{x}_{q,j}]}{\sum_{a \in \mathcal{A}(q)} \exp[\beta_q^{\mathrm{t}} \mathbf{x}_{q,a}]}$$

where $\mathcal{A}(q)$ are the available alternatives for q.

Logit model: for $\boldsymbol{\beta}_q$ fixed, j is chosen with probability

$$L_q(j \mid \boldsymbol{\beta}_q) = \frac{\exp[\boldsymbol{\beta}_q^{\mathrm{t}} \mathbf{x}_{q,j}]}{\sum_{a \in \mathcal{A}(q)} \exp[\boldsymbol{\beta}_q^{\mathrm{t}} \mathbf{x}_{q,a}]}$$

where $\mathcal{A}(q)$ are the available alternatives for q.

For a random individual, suppose $\boldsymbol{\beta}_q$ is random with density $f_{\boldsymbol{\theta}}$, which depends on (unknown) parameter vector $\boldsymbol{\theta}$. We want to estimate $\boldsymbol{\theta}$ from the data (the $\mathbf{x}_{q,j}$ and y_q).

The unconditional probability of choosing j is

$$p_q(j, \boldsymbol{\theta}) = \int L_q(j \mid \boldsymbol{\beta}) f_{\boldsymbol{\theta}}(\boldsymbol{\beta}) \boldsymbol{\beta}.$$

It depends on $\mathcal{A}(q)$, j, and $\boldsymbol{\theta}$.

Maximum likelihood: Maximize the log of the joint probability of the sample, w.r.t. $\boldsymbol{\theta}$:

$$\ln L(\boldsymbol{\theta}) = \ln \prod_{q=1}^{m} p_q(y_q, \boldsymbol{\theta}) = \sum_{q=1}^{m} \ln p_q(y_q, \boldsymbol{\theta}).$$

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No formula for $p_q(j, \boldsymbol{\theta})$, but can use MC or RQMC, for each q and fixed $\boldsymbol{\theta}$.

Generate n realizations of $\boldsymbol{\beta}$ from $f_{\boldsymbol{\theta}}$, say $\boldsymbol{\beta}_q^{(1)}(\boldsymbol{\theta}), \ldots, \boldsymbol{\beta}_q^{(n)}(\boldsymbol{\theta})$, and estimate $p_q(y_q, \boldsymbol{\theta})$ by

$$\hat{p}_q(y_q, \boldsymbol{\theta}) = \frac{1}{n} \sum_{i=1}^{n} L_q(j, \boldsymbol{\beta}_q^{(i)}(\boldsymbol{\theta})).$$

Then we can find the maximizer $\hat{\boldsymbol{\theta}}$ of $\ln \prod_{q=1}^{m} \hat{p}_q(y_q, \boldsymbol{\theta})$ w.r.t. $\boldsymbol{\theta}$.

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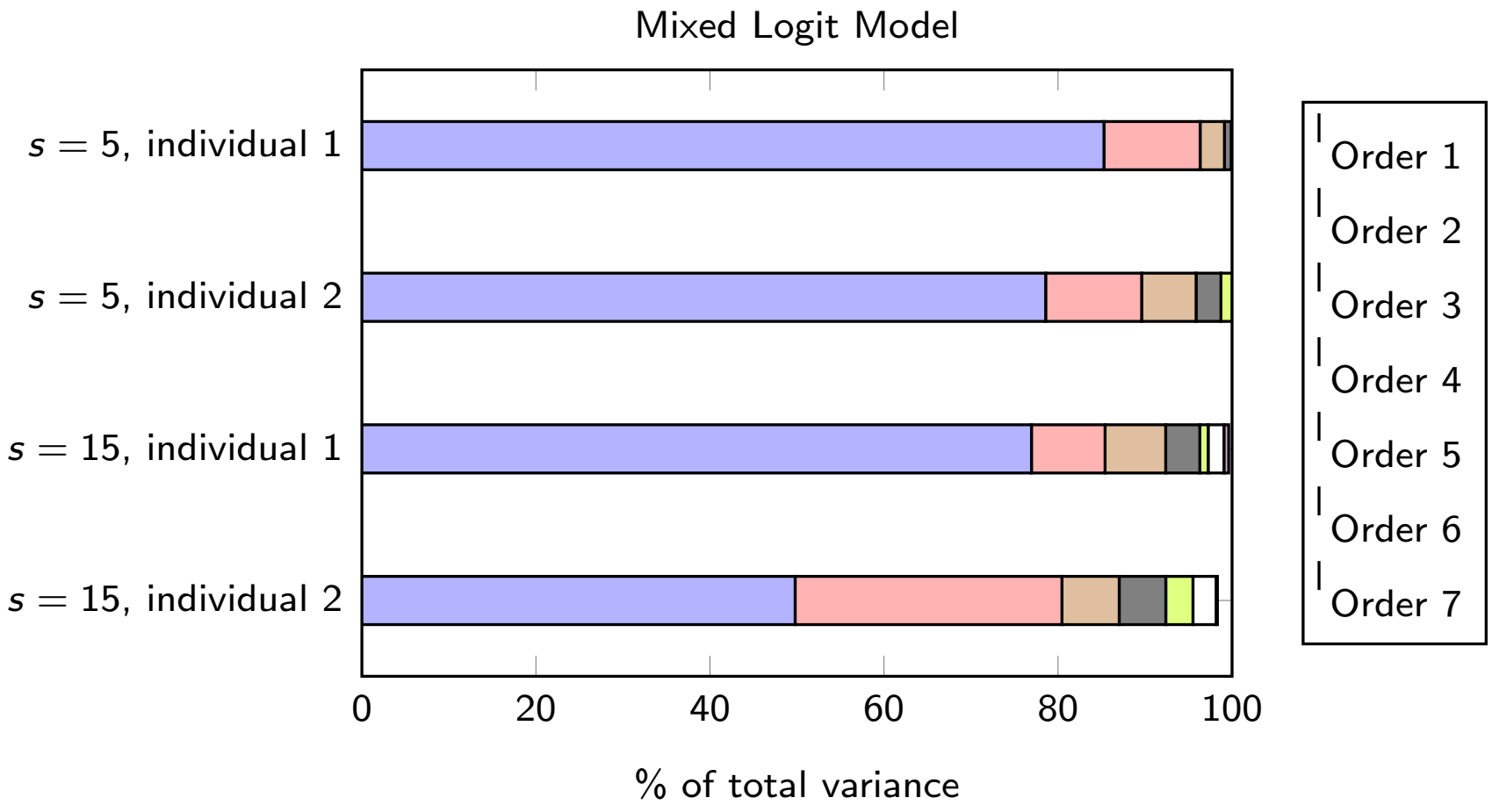
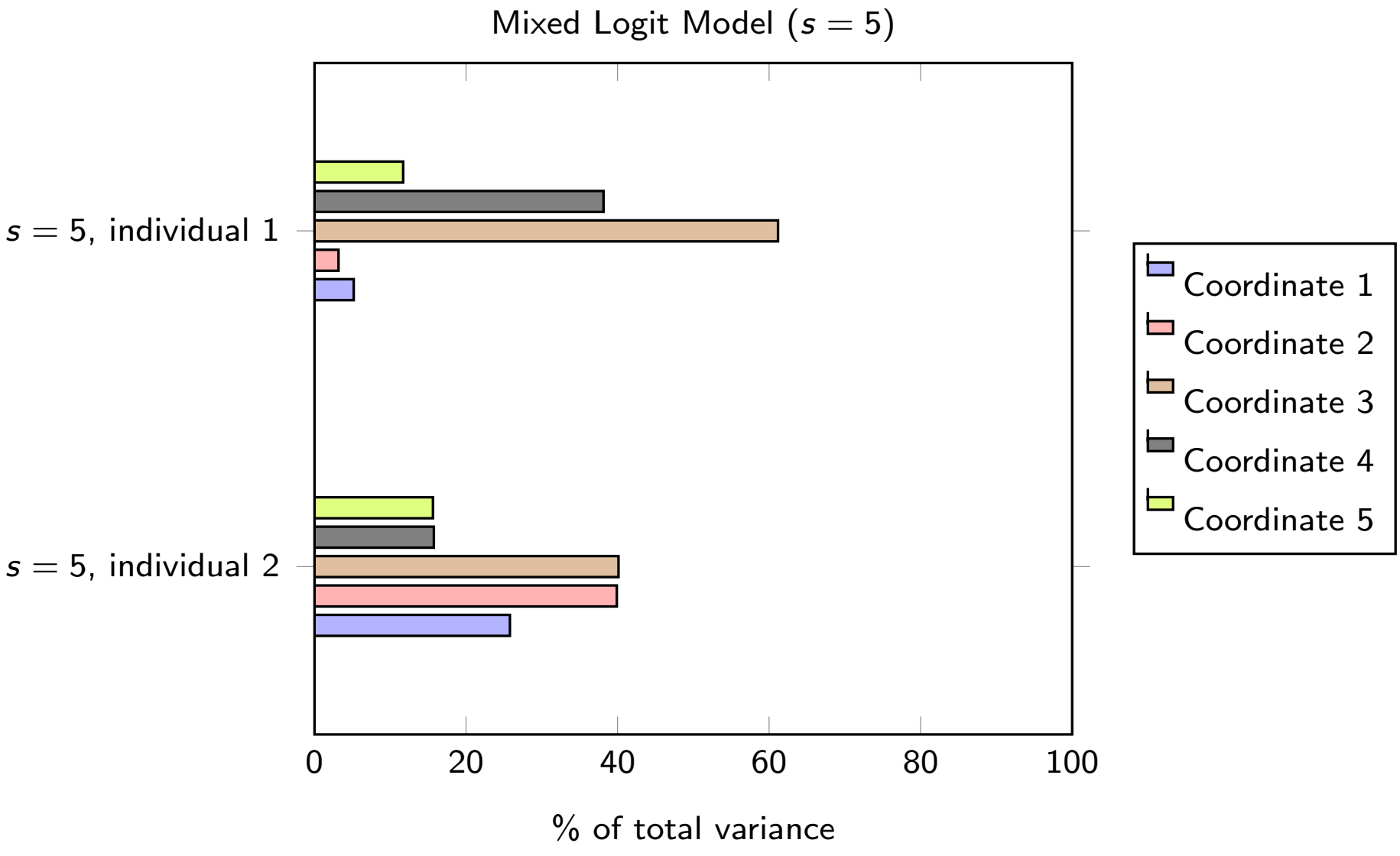
We take 4 alternatives, with indep. attributes, resp. $N(1,1)$, $N(1,1)$, $N(0.5,1)$, $N(0.5,1)$. We try $s = 5$, 10, 15.

$\boldsymbol{\beta}_q$ is a vector of s indep. $N(1,1)$ random variables.

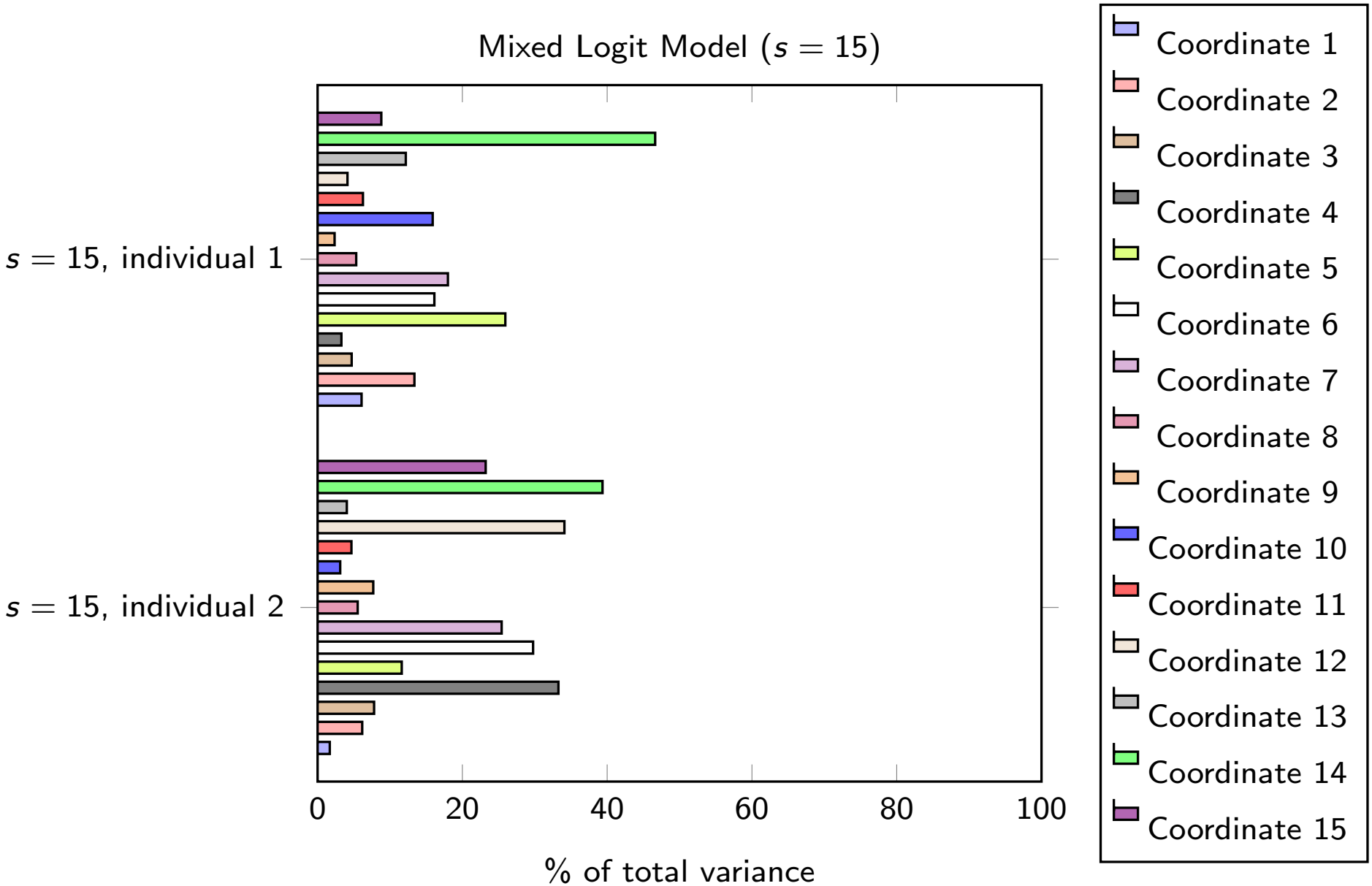
ANOVA Variances for the Mixed Logit Model



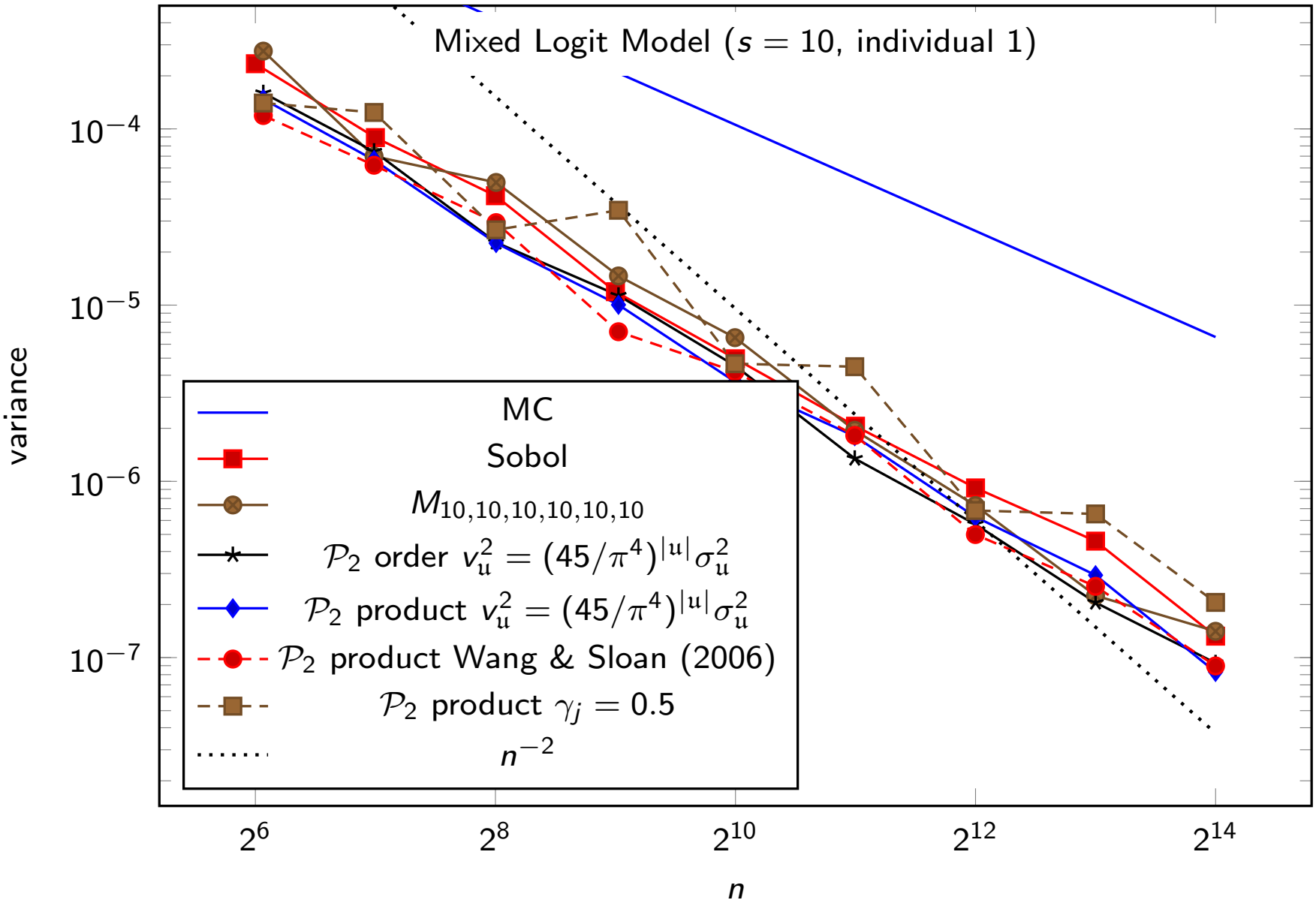
Total variance per coordinate



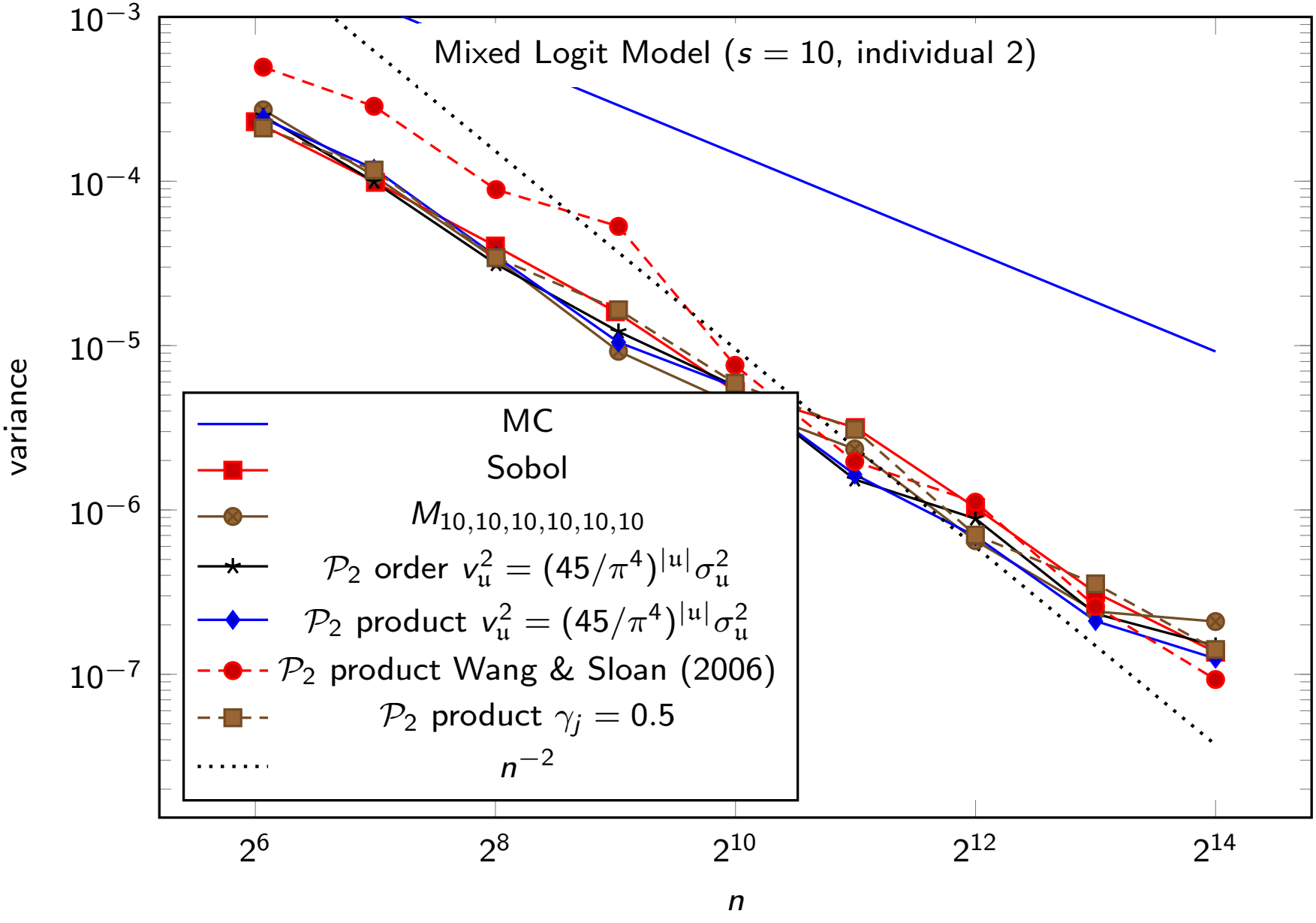
Total variance per coordinate



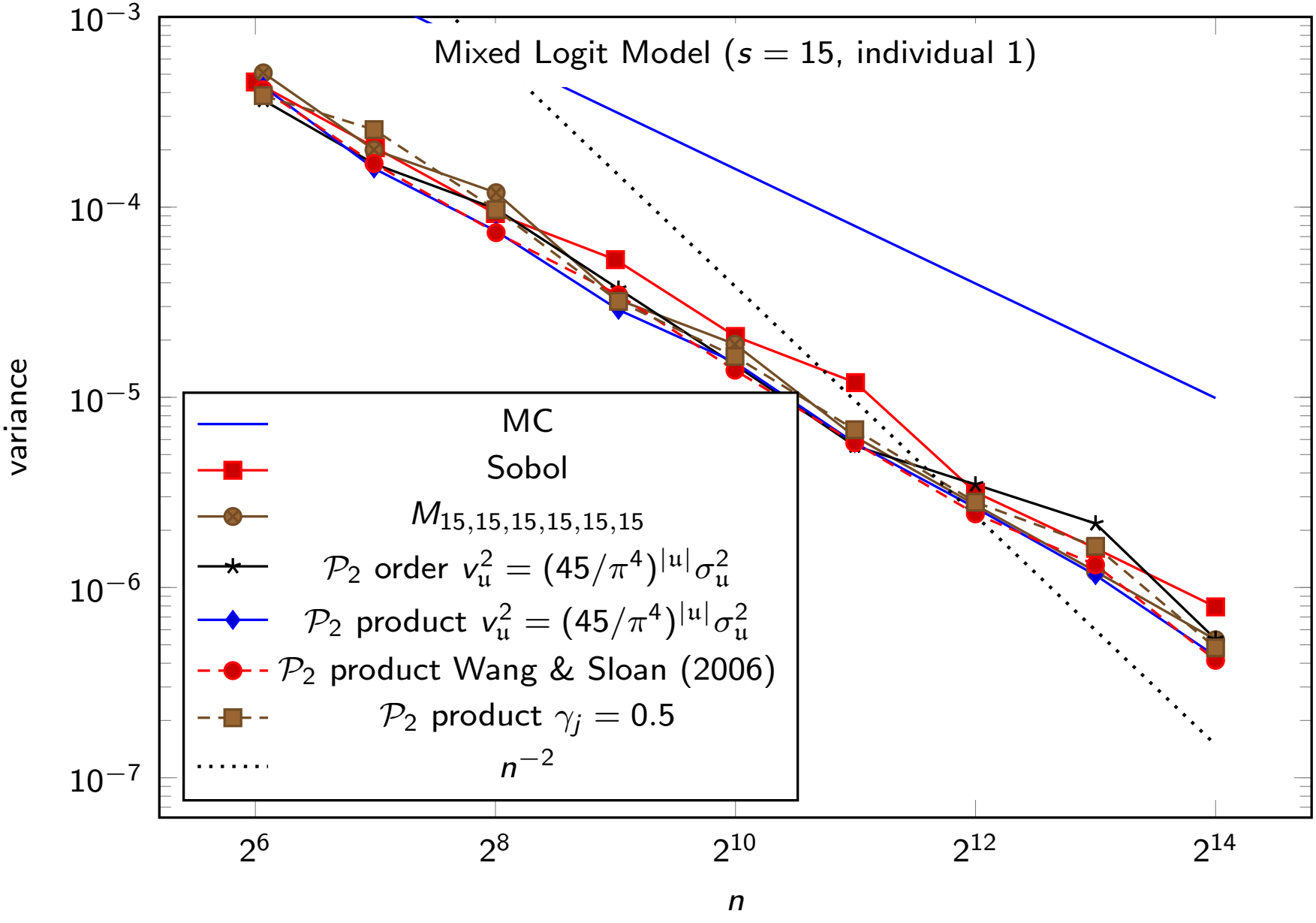
Lattices of Rank 1 with CBC



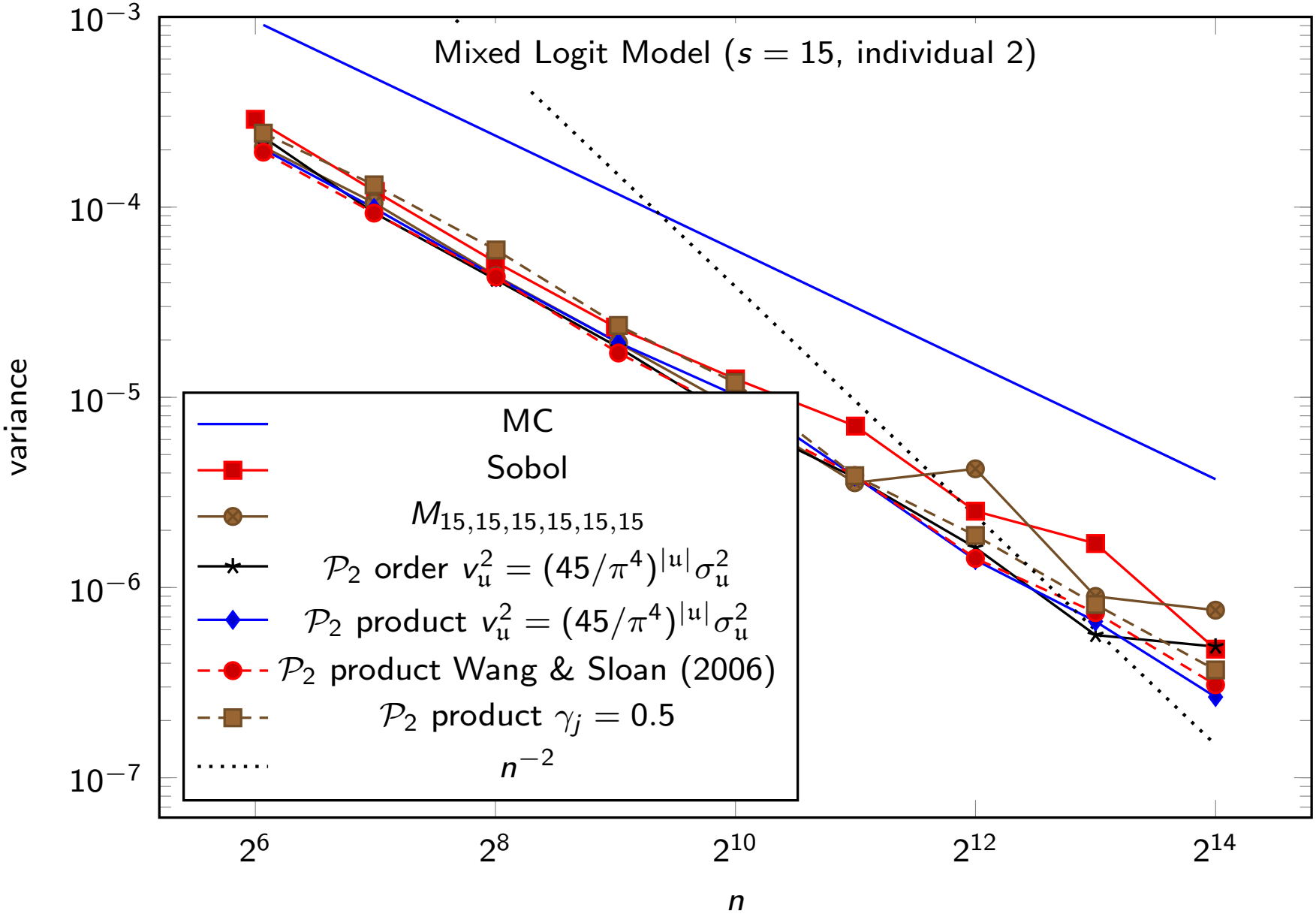
Lattices of Rank 1 with CBC



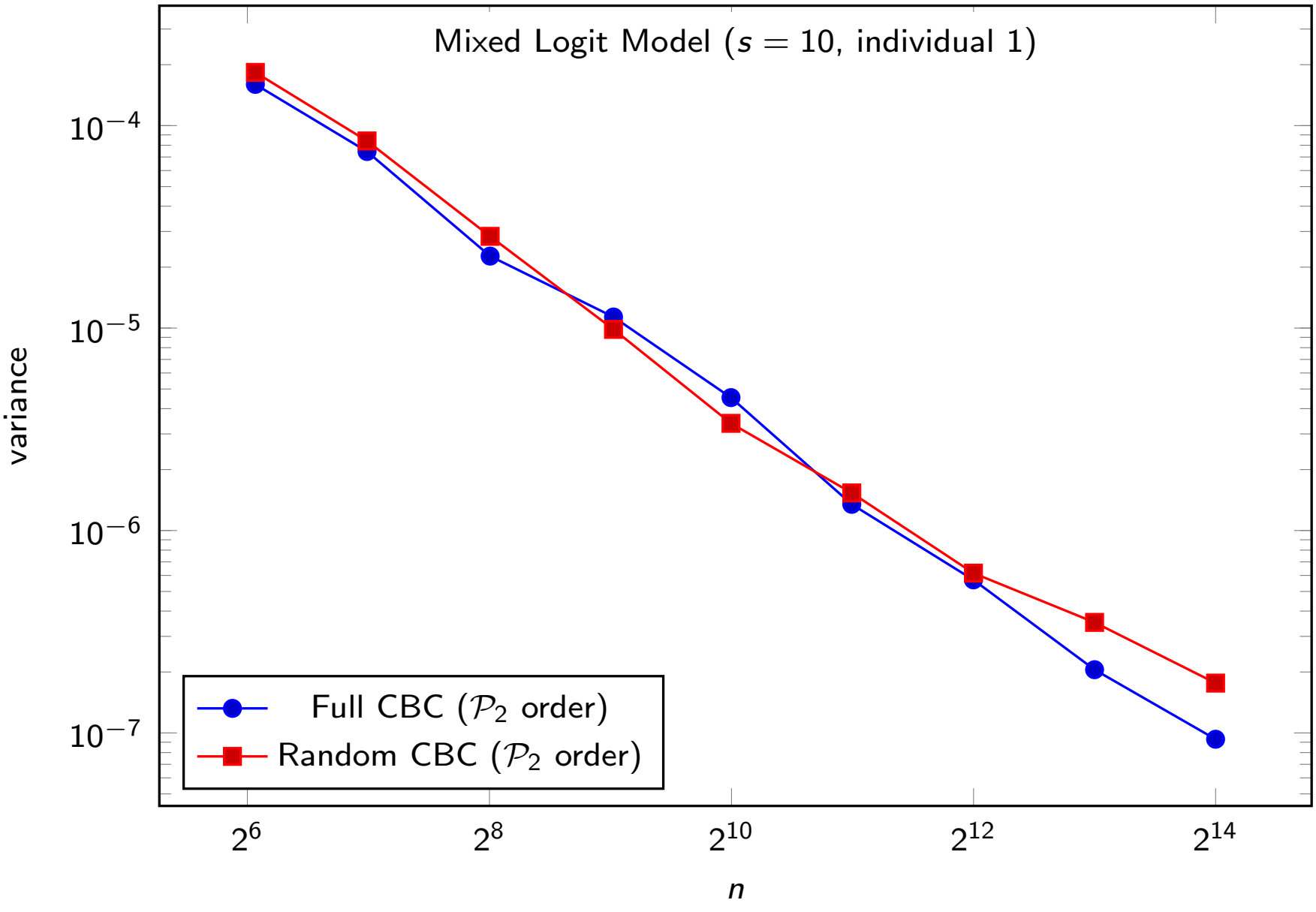
Lattices of Rank 1 with CBC



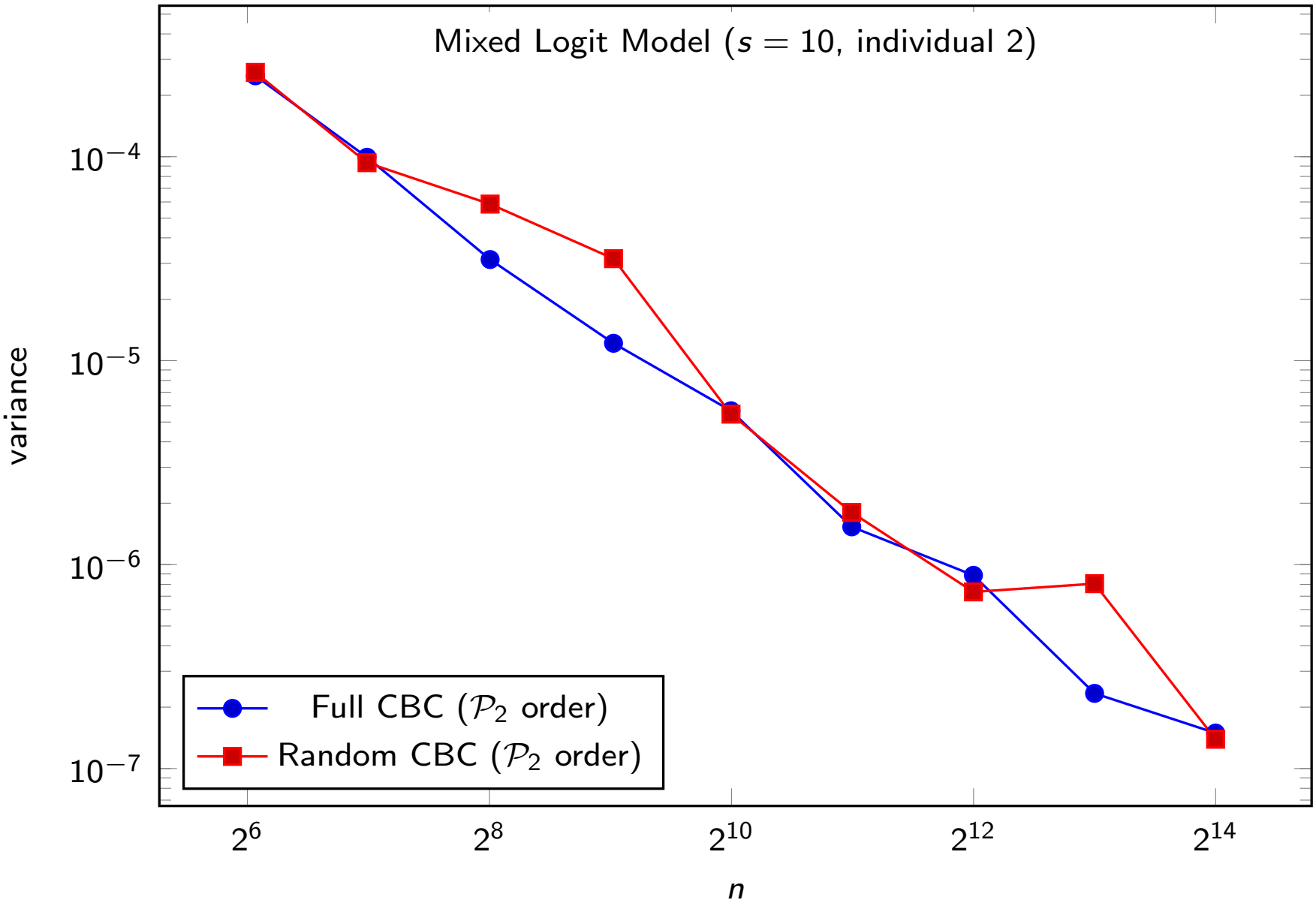
Lattices of Rank 1 with CBC



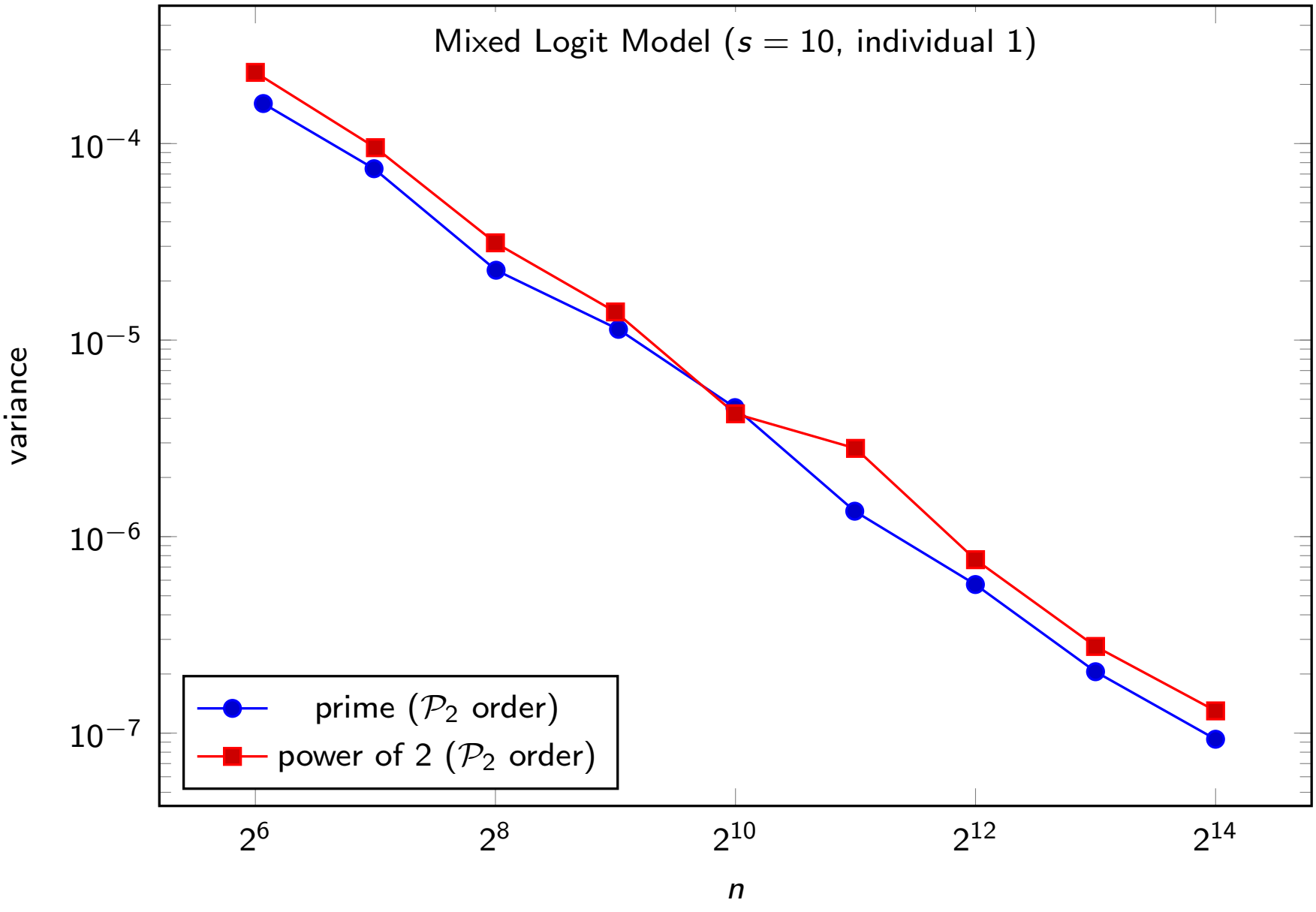
Random vs. Full CBC



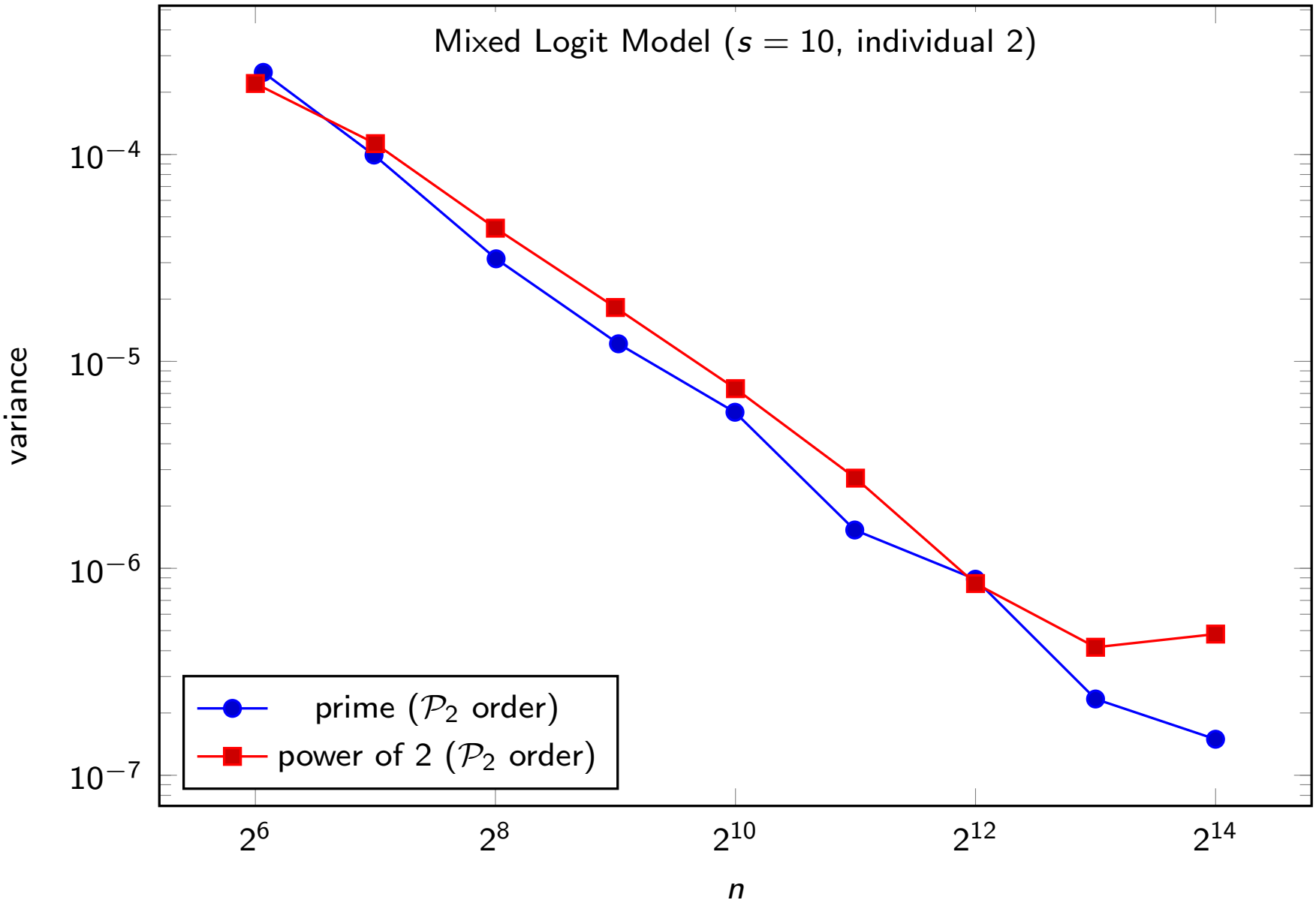
Random vs. Full CBC



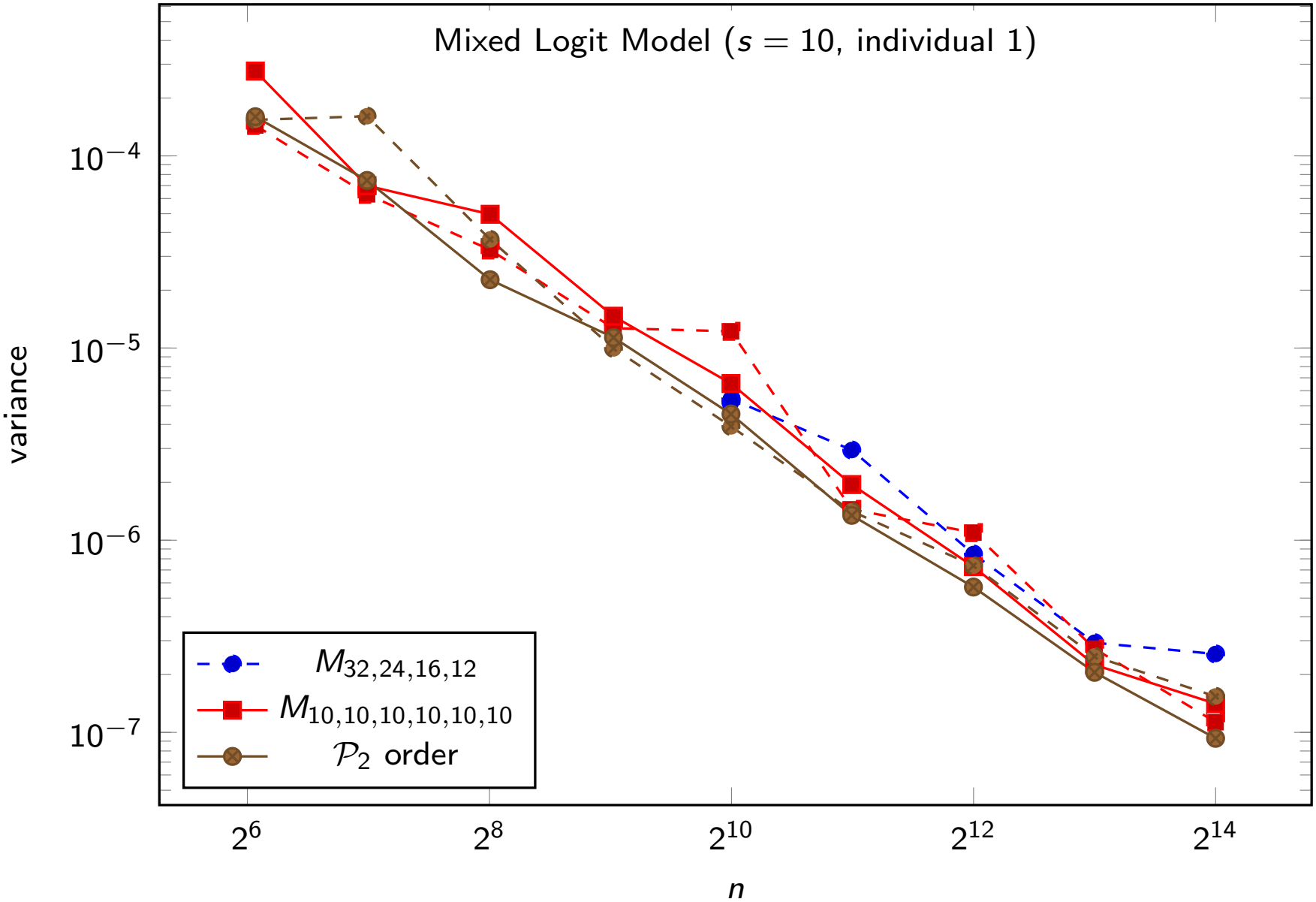
Prime vs. Power-of-2 Number of Points



Prime vs. Power-of-2 Number of Points

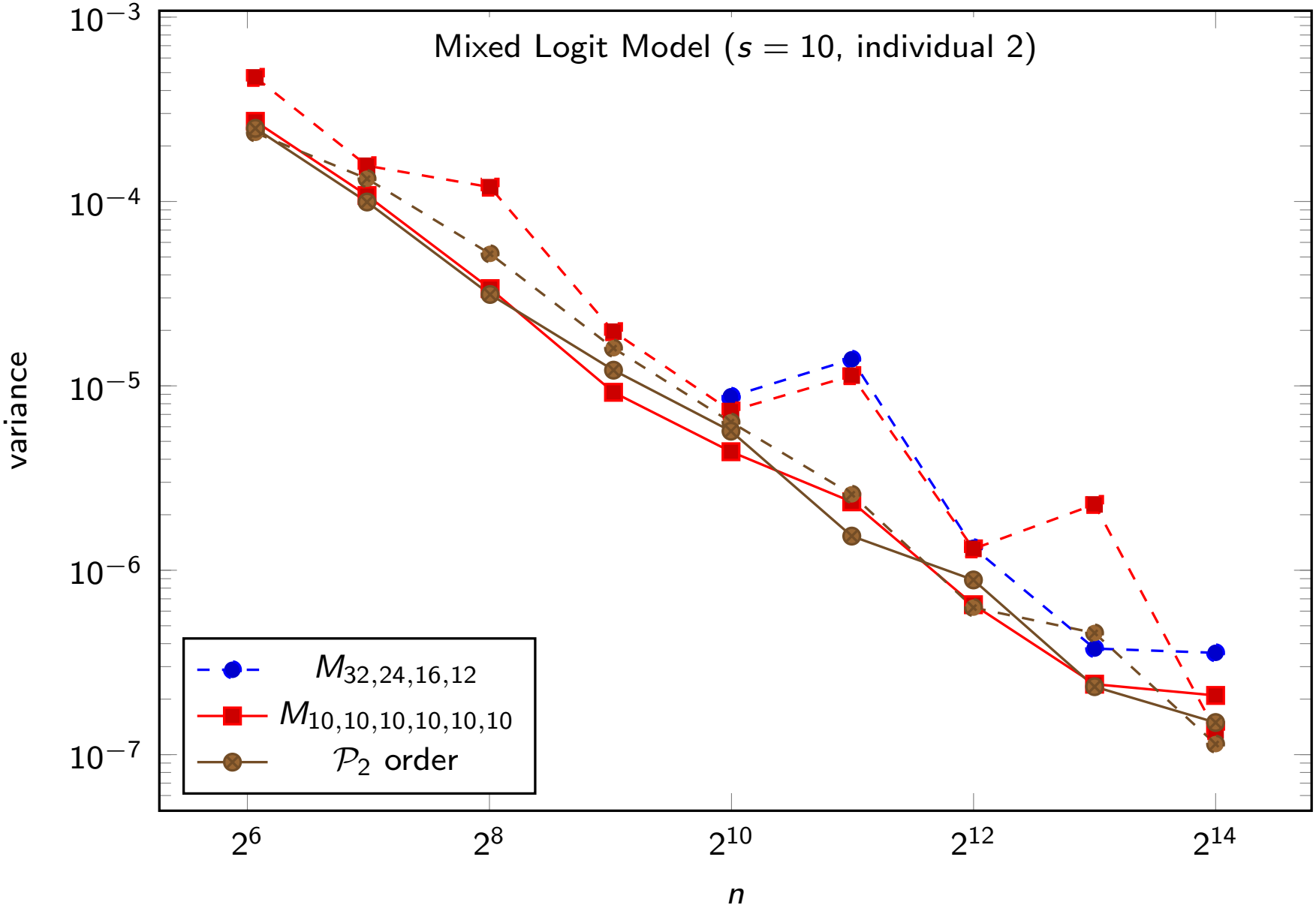


Korobov vs. CBC



Solid: CBC. Dashed: Korobov.

Korobov vs. CBC



Solid: CBC. Dashed: Korobov.

References for the material of this talk (some were added afterward):

1. P. L'Ecuyer and D. Munger, "On the Choice of Figure of Merit for Randomly-Shifted Lattice Rules", in Monte Carlo and Quasi Monte Carlo Methods 2010, H. Wozniakowski and L. Plaskota, Eds., Springer-Verlag, Berlin, 2012, 133–159.

2. P. L'Ecuyer and C. Lemieux, "Variance Reduction via Lattice Rules", Management Science **46**, 9 (2000), 1214–1235.

3. P. L'Ecuyer, "Quasi-Monte Carlo Methods in Finance", Proceedings of the 2004 Winter Simulation Conference, IEEE Press, 2004, 1645–1655.

4. P. L'Ecuyer, "Quasi-Monte Carlo Methods with Applications in Finance," Finance and Stochastics, 13, 3 (2009), 307–349.

5. P. L'Ecuyer and D. Munger, "Algorithm 958: LatticeBuilder: A General Software Tool for Constructing Rank-1 Lattice Rules", ACM Transactions on Mathematical Software, **42**, 2, Article 15, 2016.