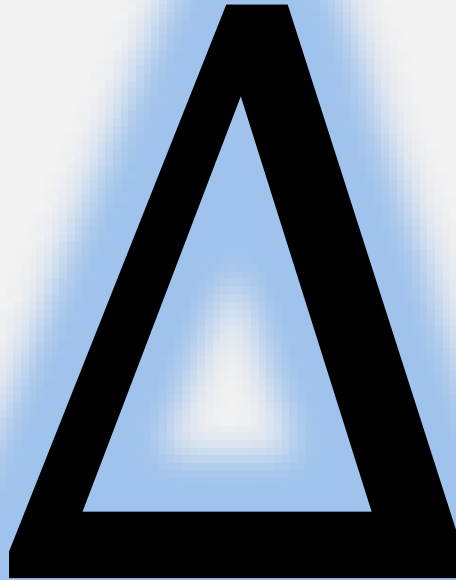


# IFT 6112

## 07 – CONTINUOUS LAPLACIAN

<http://www-labs.iro.umontreal.ca/~bmpix/teaching/6112/2018/>



Mikhail Bessmeltsev

# Famous Motivation

## CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

“La Physique ne nous donne pas seulement l’occasion de résoudre des problèmes . . . , elle nous fait sentir la solution.” H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

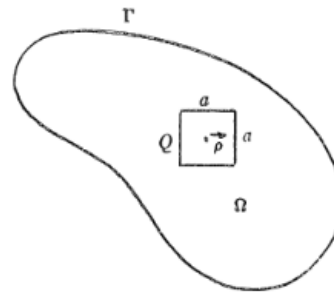


FIG. 1

1. And now to the theme and the title.

It has been known for well over a century that if a membrane  $\Omega$ , held fixed along its boundary  $\Gamma$  (see Fig. 1), is set in motion its displacement (in the direction perpendicular to its original plane)

$$F(x, y; t) \equiv F(\vec{\rho}; t)$$

obeys the wave equation

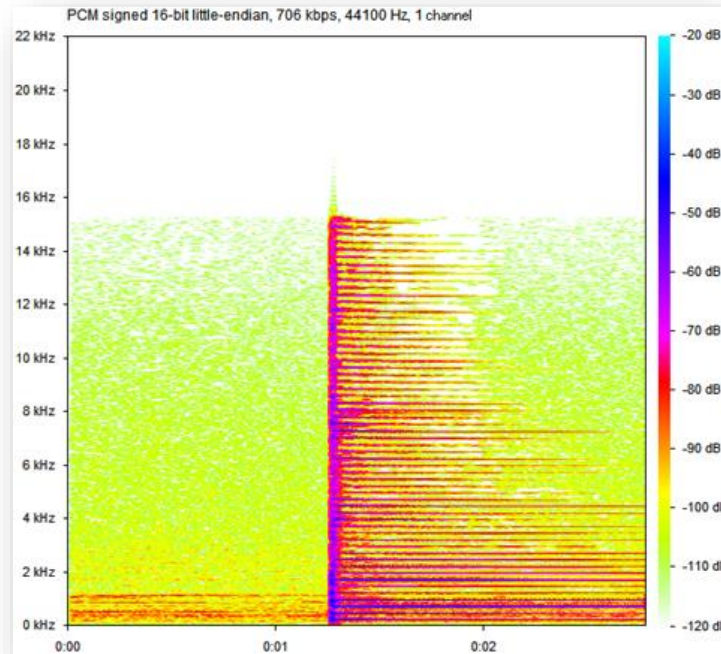
$$\partial^2 F$$

# An Experiment



**Is this possible?**

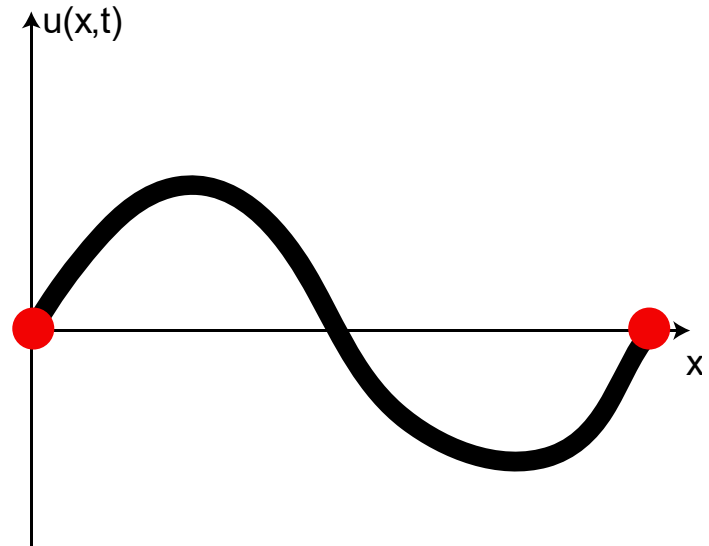
# Unreasonable to Ask?



**Length  
of string**

# 1D: length of a string

$u(x, t)$  - string height

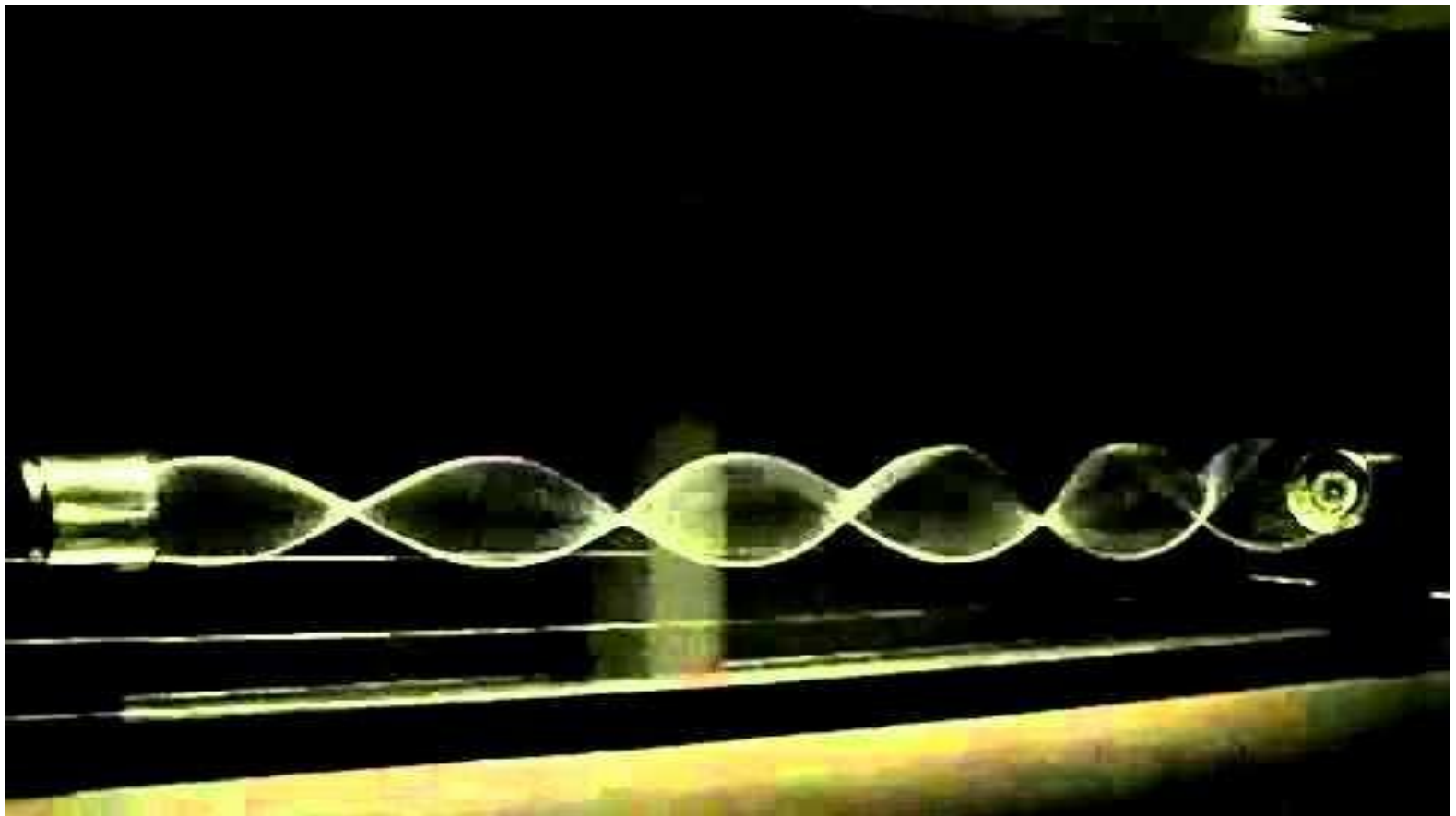


Boundary conditions:  $u(0, t) = 0 = u(l, t)$

$$\text{PDE: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

# Standing waves?

(on the board)

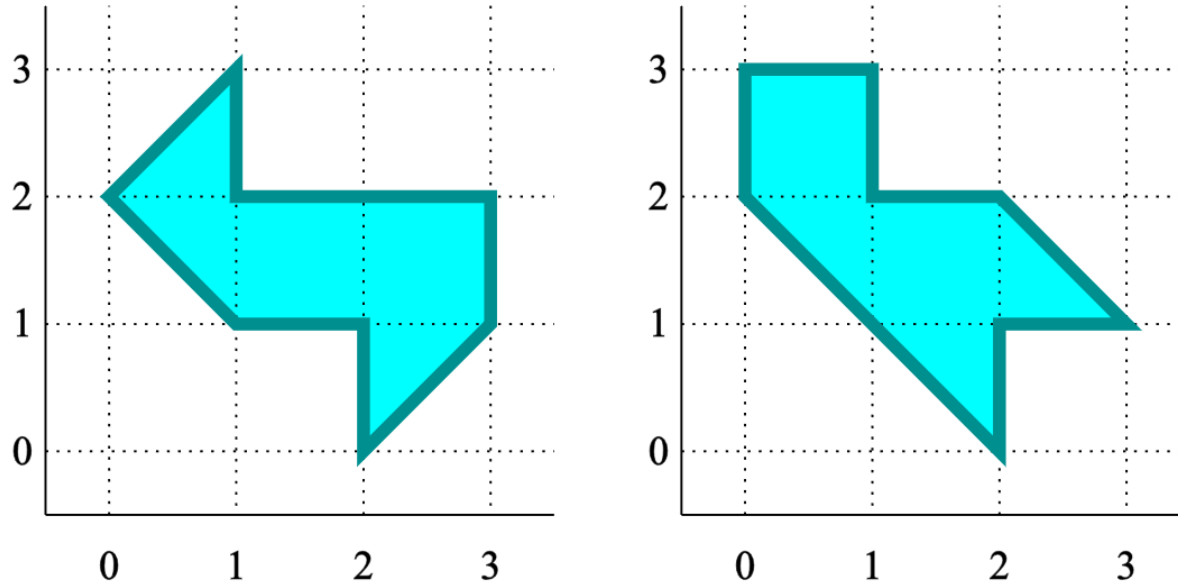


Can you hear the length of an interval?

$$\lambda_k = \left( \frac{\pi k}{l} \right)^2$$

*Yes!*

# Spoiler Alert



**“No, but...”**

- **Has to be a weird drum**
- **Spectrum tells you a lot!**



# Rough Intuition

[http://pngimg.com/upload/hammer\\_PNG3886.png](http://pngimg.com/upload/hammer_PNG3886.png)



**You can learn a lot  
about a shape by  
hitting it (lightly)  
with a hammer!**

# Spectral Geometry

What can you learn about its shape  
from

vibration frequencies and  
oscillation patterns?

$$\Delta f = \lambda f$$

# Objectives

- Make “vibration modes” more **precise**
- **Progressively more complicated** domains
  - Line segments
  - Regions in  $\mathbb{R}^2$
  - Graphs
  - Surfaces/manifolds
- Next time: **Discretization, applications**

*Review:*

# Vector Spaces and Linear Operators

$$\mathcal{L}[\vec{x} + \vec{y}] = \mathcal{L}[\vec{x}] + \mathcal{L}[\vec{y}]$$

$$\mathcal{L}[c\vec{x}] = c\mathcal{L}[\vec{x}]$$

*Review:*

# In Finite Dimensions

$A$        $\vec{x}$   
matrix    vector

$\vec{x} \mapsto A\vec{x}$   
linear operator

# Recall: Spectral Theorems in $\mathbb{C}^n$

**Theorem.** Suppose  $A \in \mathbb{C}^{n \times n}$  is Hermitian. Then,  $A$  has an orthogonal basis of  $n$  eigenvectors. If  $A$  is positive semidefinite, the corresponding eigenvalues are nonnegative.

# Our Progression

- **Line segments**
- **Regions in  $\mathbb{R}^2$** 
  - **Graphs**
- **Surfaces/manifolds**

# Minus Second Derivative Operator

“Dirichlet boundary conditions”

$$\{f(\cdot) \in C^\infty([a, b]) : f(0) = f(\ell) = 0\}$$

$$\mathcal{L}[f(\cdot)] := -f''(\cdot)$$

Eigenfunctions:

$$f_k(x) = \sin\left(\frac{\pi k x}{\ell}\right), \quad \lambda_k = \left(\frac{\pi k}{\ell}\right)^2$$



# Physical Intuition: Wave Equation

Minus second derivative operator!

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$



# Observation

$$\{f(\cdot) \in C^\infty([a, b]) : f(0) = f(\ell) = 0\}$$

$$\begin{aligned}\langle f, \mathcal{L}[f] \rangle &= - \int_0^\ell f(x) f''(x) dx \\ &= - [f(x) f'(x)]_0^\ell + \underbrace{\int_0^\ell f'(x)^2 dx}_{\geq 0}\end{aligned}$$

# Hilbert-Schmidt Theorem

**Theorem.** Let  $H \neq 0$  be an infinite-dimensional, separable Hilbert space and let  $K \in L(H)$  be compact and self-adjoint. Then, there exists a countable orthonormal basis of  $H$  consisting of eigenvectors of  $K$ .



**Hilbert space:** Space with inner product

**Separable:** Admits countable, dense subset

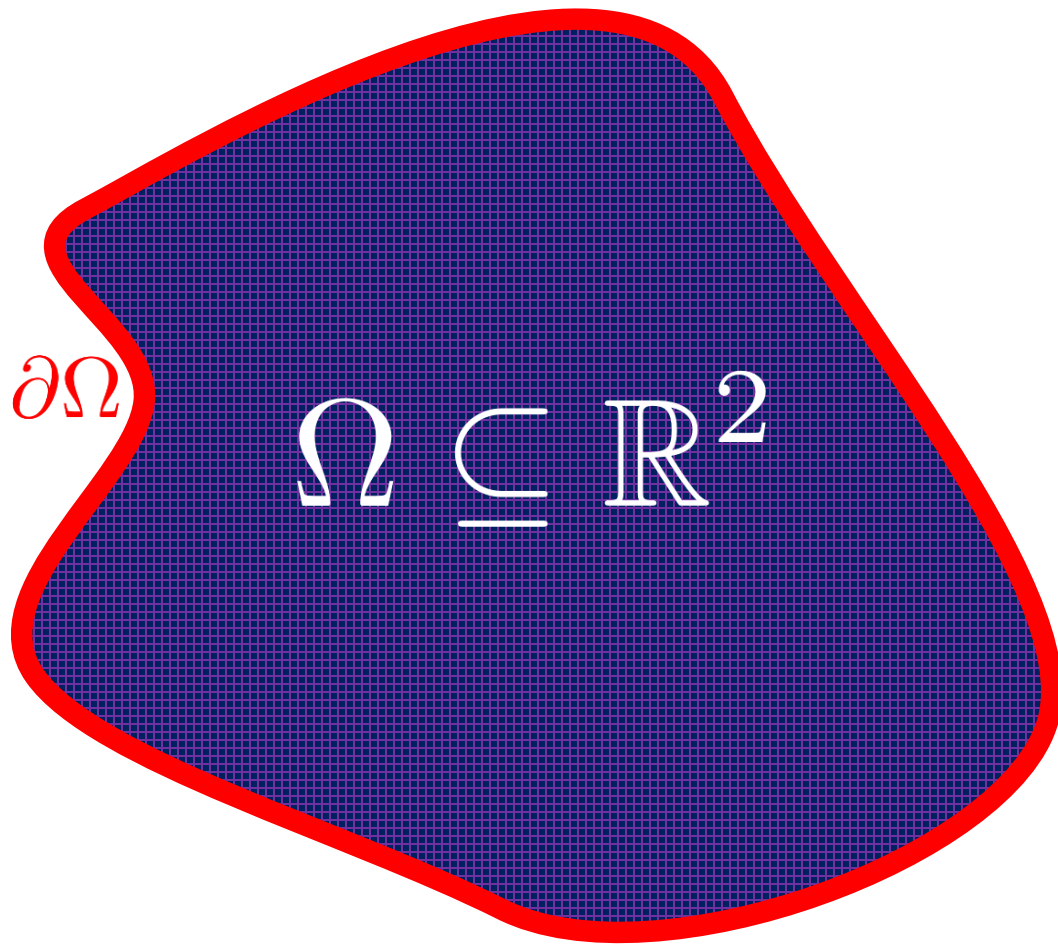
**Compact operator:** Bounded sets to relatively compact sets

**Self-adjoint:**  $\langle Kv, w \rangle = \langle v, Kw \rangle$

# Our Progression

- **Line segments**
- **Regions in  $\mathbb{R}^2$**
- **Graphs**
- **Surfaces/manifolds**

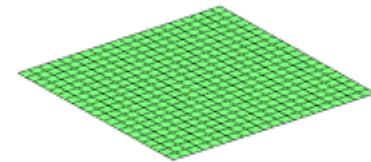
# Planar Region



**Wave equation:**

$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

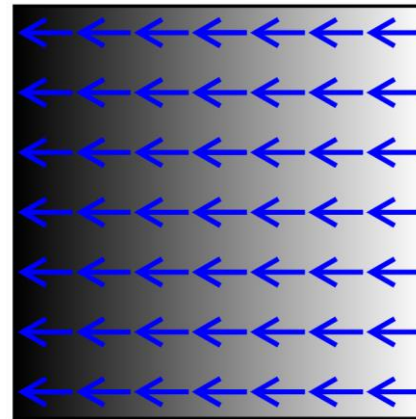
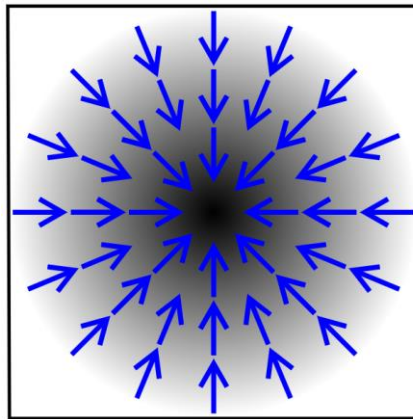
$$\Delta := \sum_i \frac{\partial^2}{\partial x_i^2}$$



# Typical Notation

“  $\Delta = \underbrace{\nabla \cdot}_{\text{divergence}} \underbrace{\nabla}_{\text{gradient}}$  ”

*More later...*



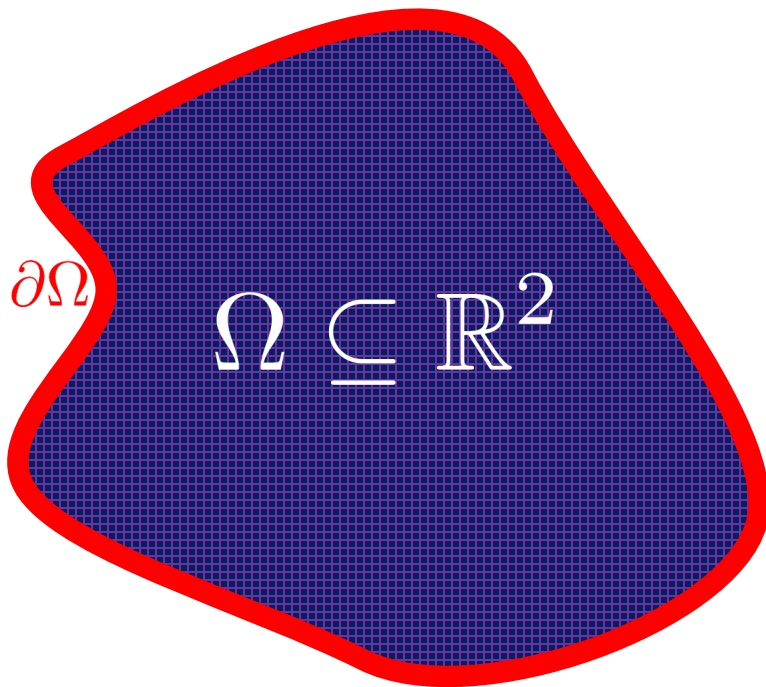
**Gradient operator:**

$$\nabla := \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

# Positivity, Self-Adjointness

$$\{f(\cdot) \in C^\infty(\Omega) : f|_{\partial\Omega} \equiv 0\}$$

“Dirichlet boundary conditions”



$$\mathcal{L}[f] := -\Delta f$$

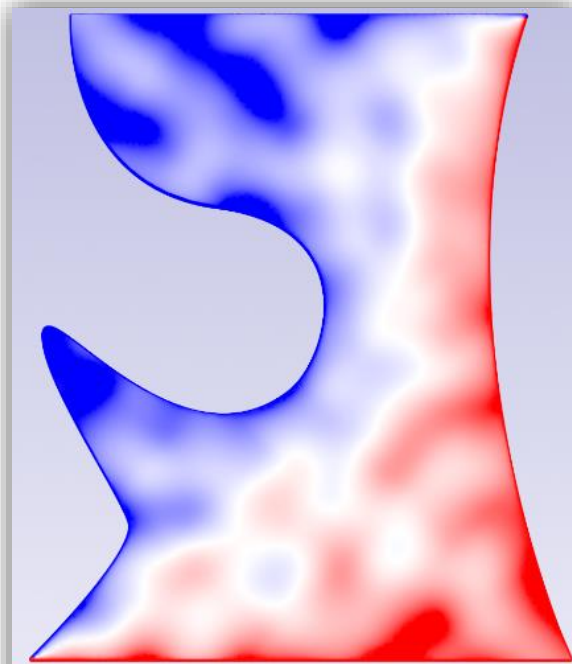
$$\langle f, g \rangle := \int_{\Omega} f(x)g(x) dx$$

**On board:**

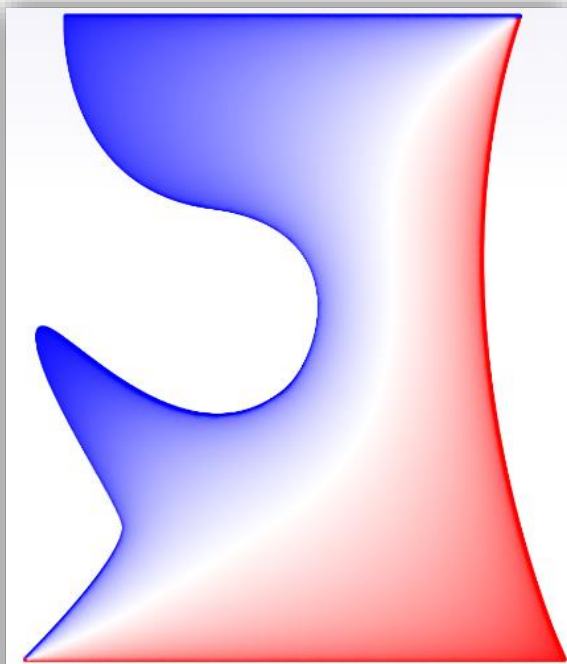
1. **Positive:**  $\langle f, \mathcal{L}[f] \rangle \geq 0$
2. **Self-adjoint:**  $\langle f, \mathcal{L}[g] \rangle = \langle \mathcal{L}[f], g \rangle$

# Dirichlet Energy

$$E[f] := \int_{\Omega} \langle \nabla f, \nabla f \rangle dA$$



non-smooth  $f(x)$



solution  $\Delta f = 0$

**On board:**

$$\begin{aligned} \min_f E[f] \\ \text{s.t. } f|_{\partial\Omega} = g \end{aligned}$$

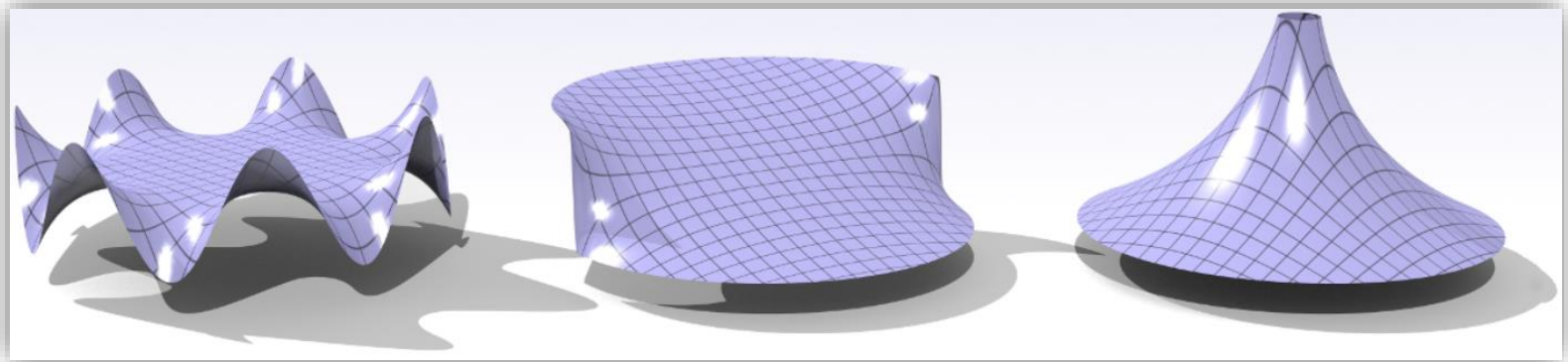
$$\Delta f \equiv 0$$

“Laplace equation”  
“Harmonic function”



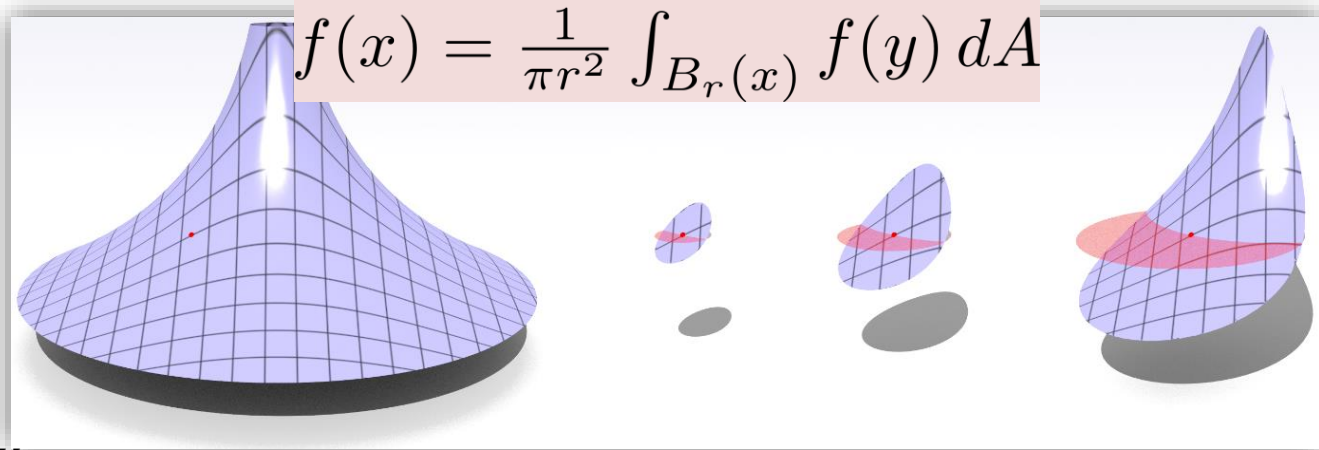
# Harmonic Functions

$$\Delta f \equiv 0$$

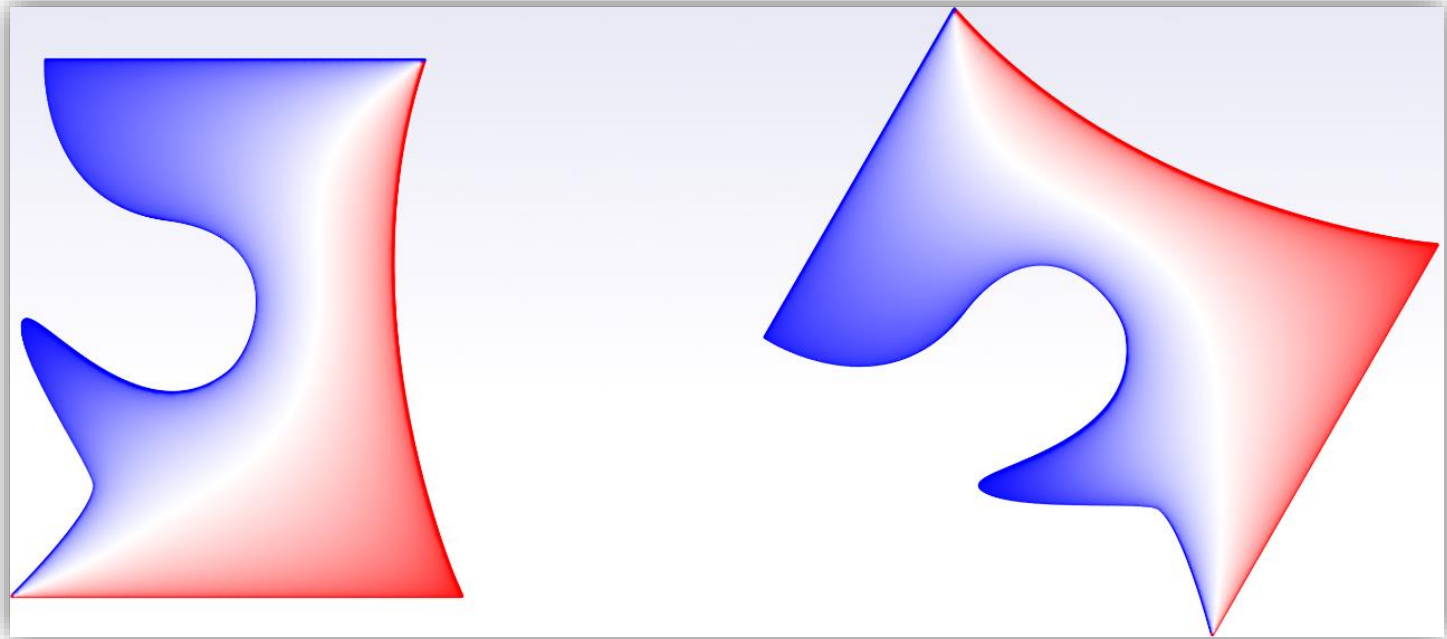


Mean value property:

$$f(x) = \frac{1}{\pi r^2} \int_{B_r(x)} f(y) dA$$



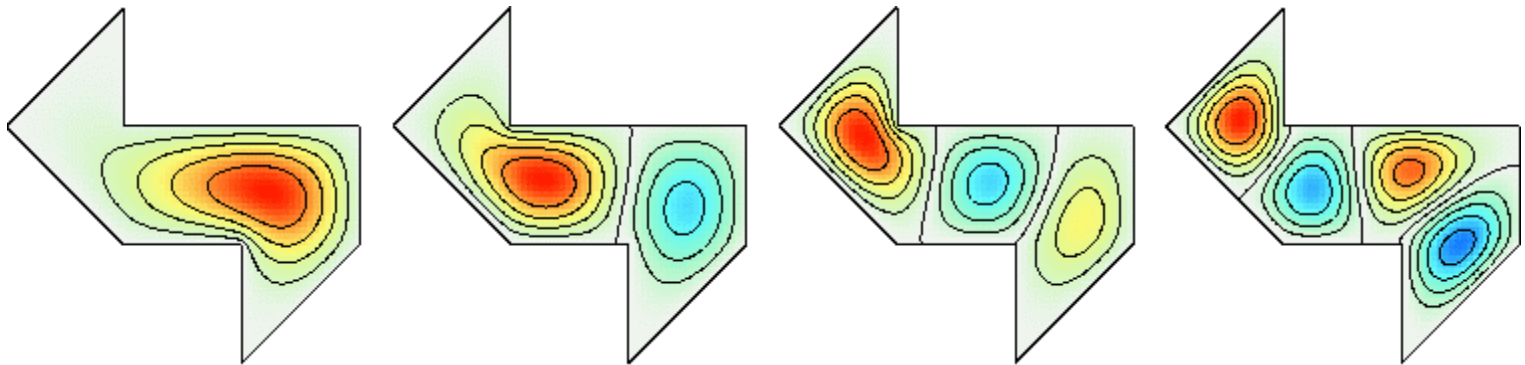
# Intrinsic Operator



Images made by E. Vouga

**Coordinate-independent**

# Another Interpretation of Eigenfunctions



Find critical points of  $E[f]$

$$\text{s.t. } \int_{\Omega} f^2 = 1$$

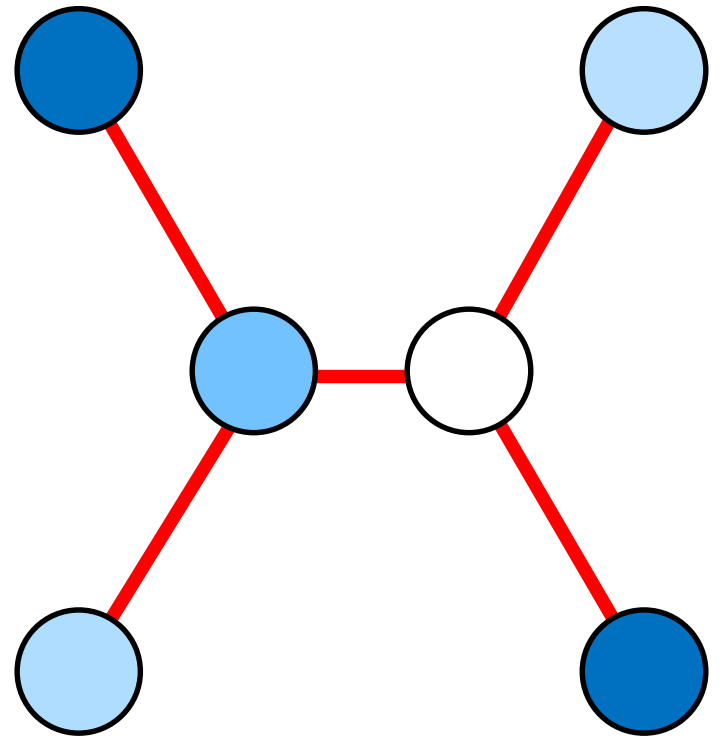
**Small eigenvalue: smooth function**

# Our Progression

- **Line segments**
- **Regions in  $\mathbb{R}^2$**
- **Graphs**
- **Surfaces/manifolds**

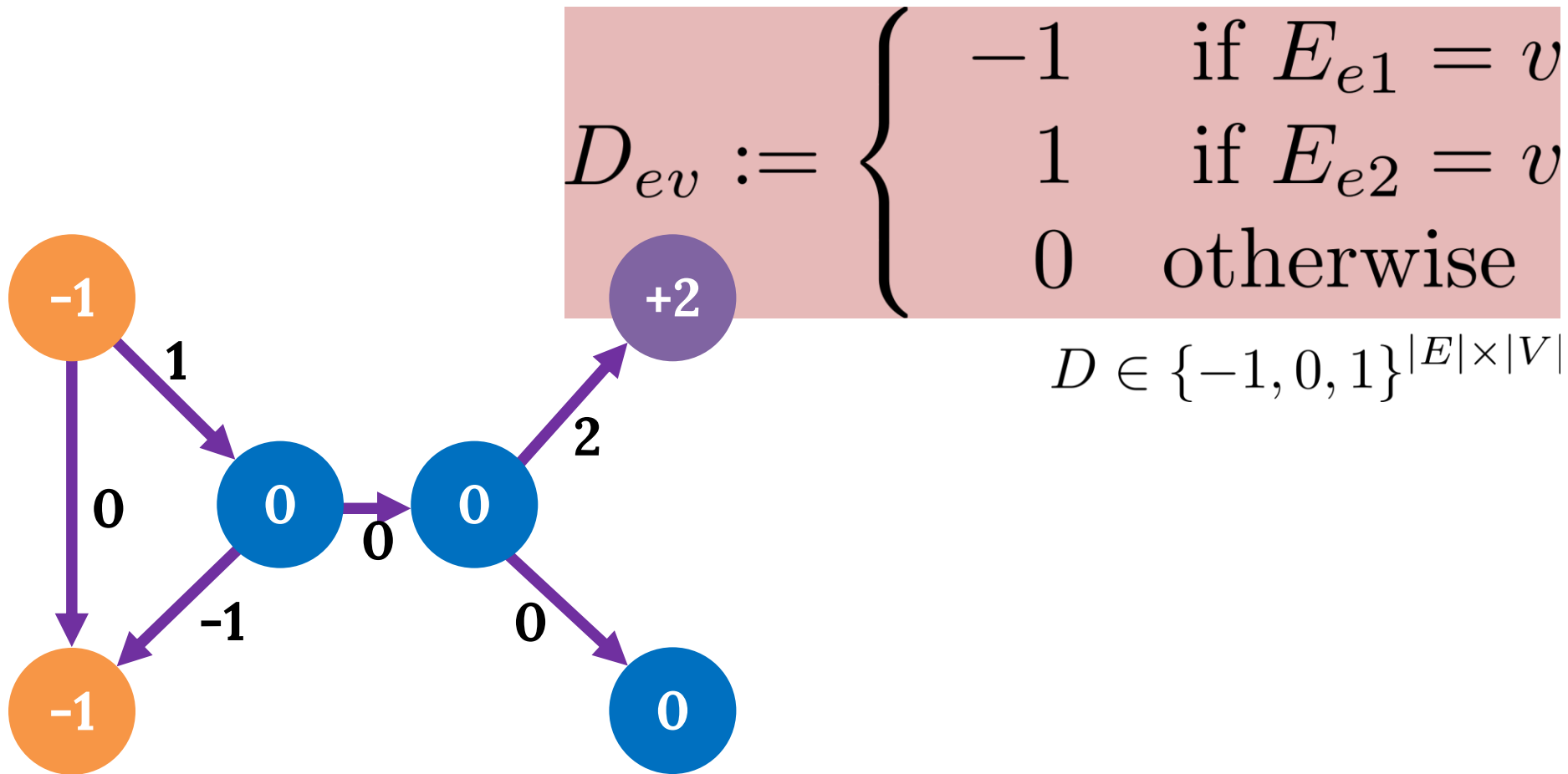
# Basic Setup

- **Function:**  
One value per vertex



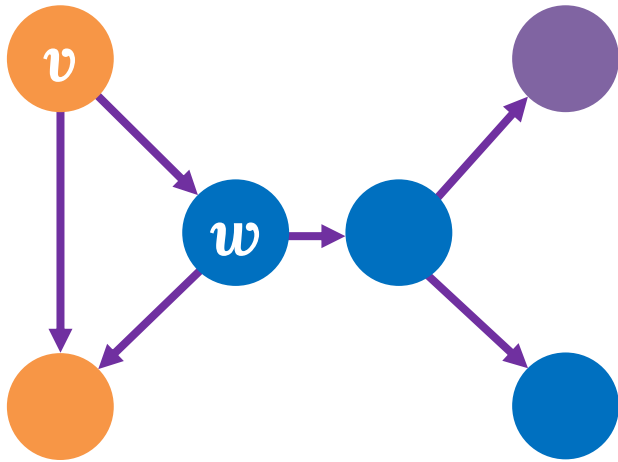
**Dirichlet energy of a  
function on a graph?**

# Differencing Operator



**Orient edges arbitrarily**

# Dirichlet Energy on a Graph



$$D_{ev} := \begin{cases} -1 & \text{if } E_{e1} = v \\ 1 & \text{if } E_{e2} = v \\ 0 & \text{otherwise} \end{cases}$$

$$E[f] := \|Df\|_2^2 = \sum_{(v,w) \in E} (f_v - f_w)^2$$



# (Unweighted) Graph Laplacian

$$E[f] = \|Df\|_2^2 = f^\top (D^\top D)f := f^\top Lf$$

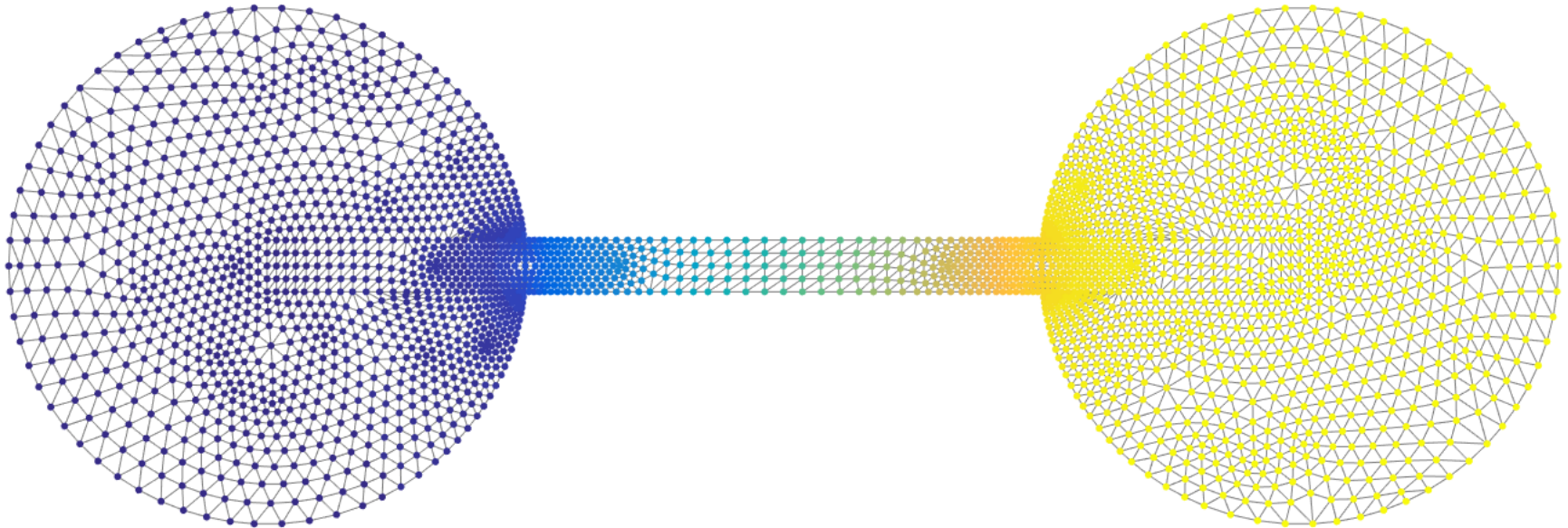
$$L_{vw} = A - D = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

Labeled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$

• **Symmetric**

• **Positive definite**

# Second-Smallest Eigenvector



$$Lx = \lambda x$$

**Used for  
graph  
partitioning**

**Fiedler vector (“algebraic connectivity”)**

# Mean Value Property

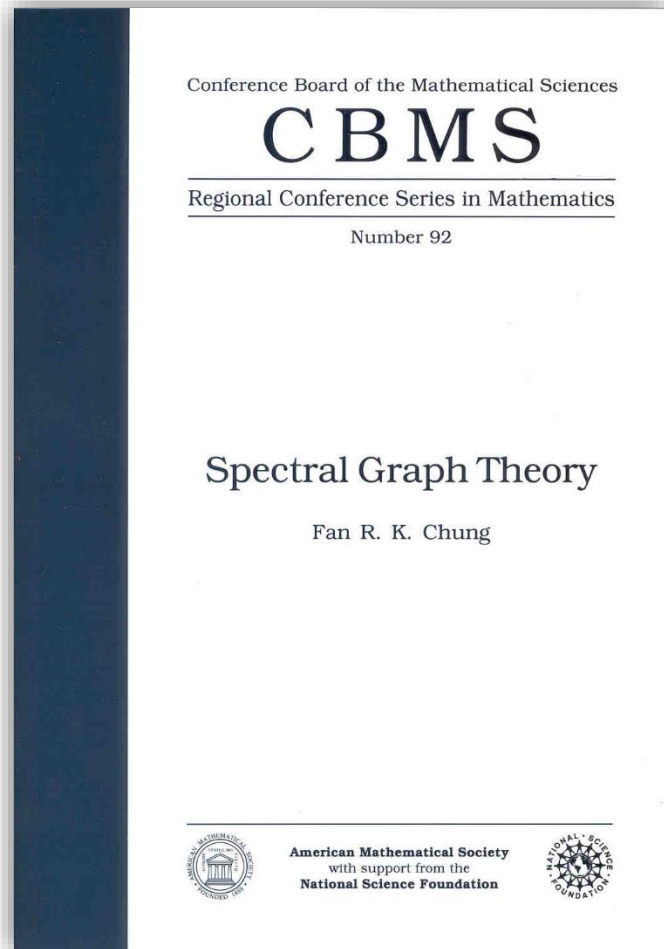
$$L_{vw} = A - D = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$$(Lx)_v = 0$$



Value at  $v$  is average of neighboring values

# For More Information...

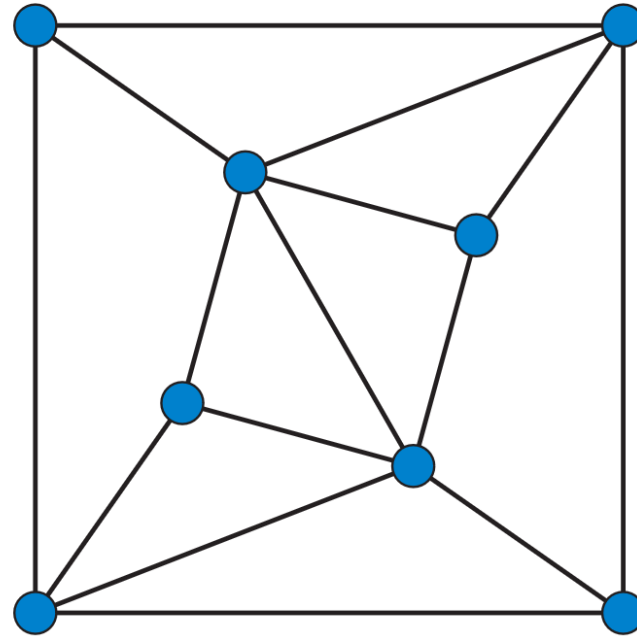
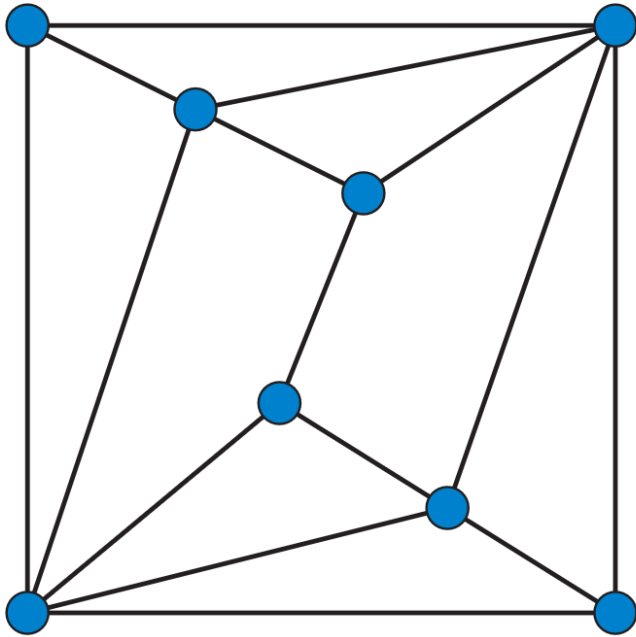


**Graph Laplacian encodes lots of information!**

**Example: Kirchoff's Theorem**  
**Number of spanning trees equals**

$$\frac{1}{n} \lambda_2 \lambda_3 \cdots \lambda_n$$

# Hear the Shape of a Graph?



*No!*

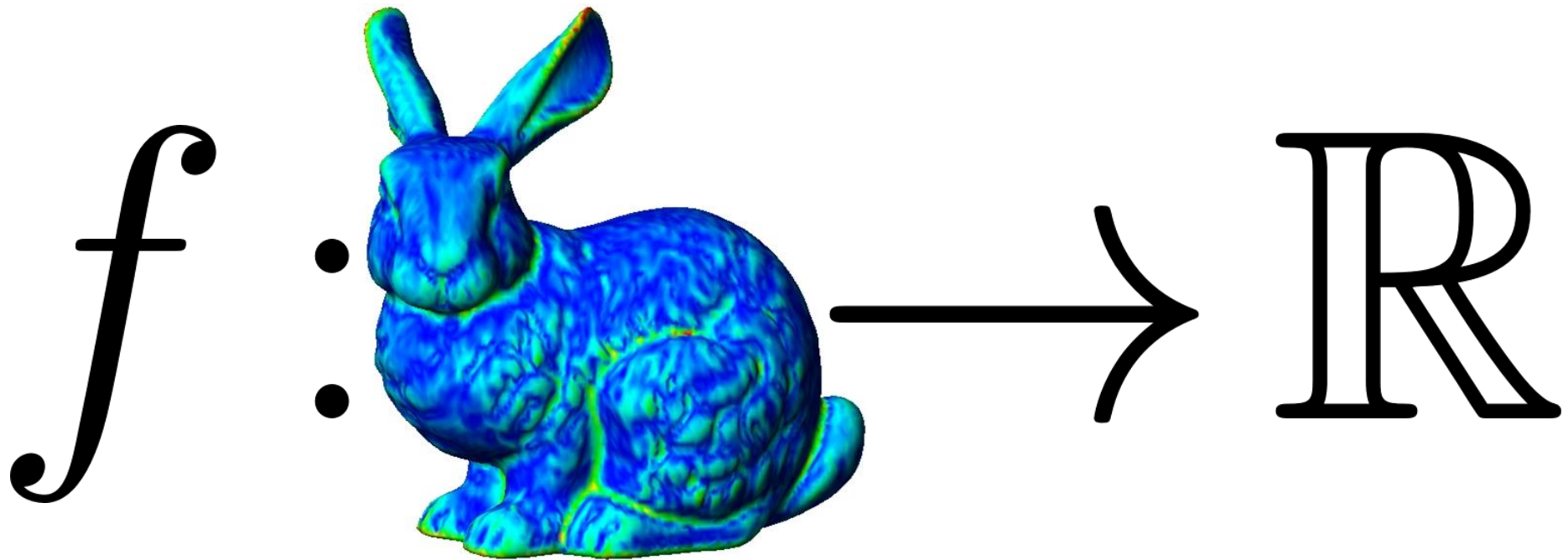
**“Enneahedra”**

# Our Progression

- **Line segments**
- **Regions in  $\mathbb{R}^2$**
- **Graphs**
- **Surfaces/manifolds**

*Recall:*

# Scalar Functions



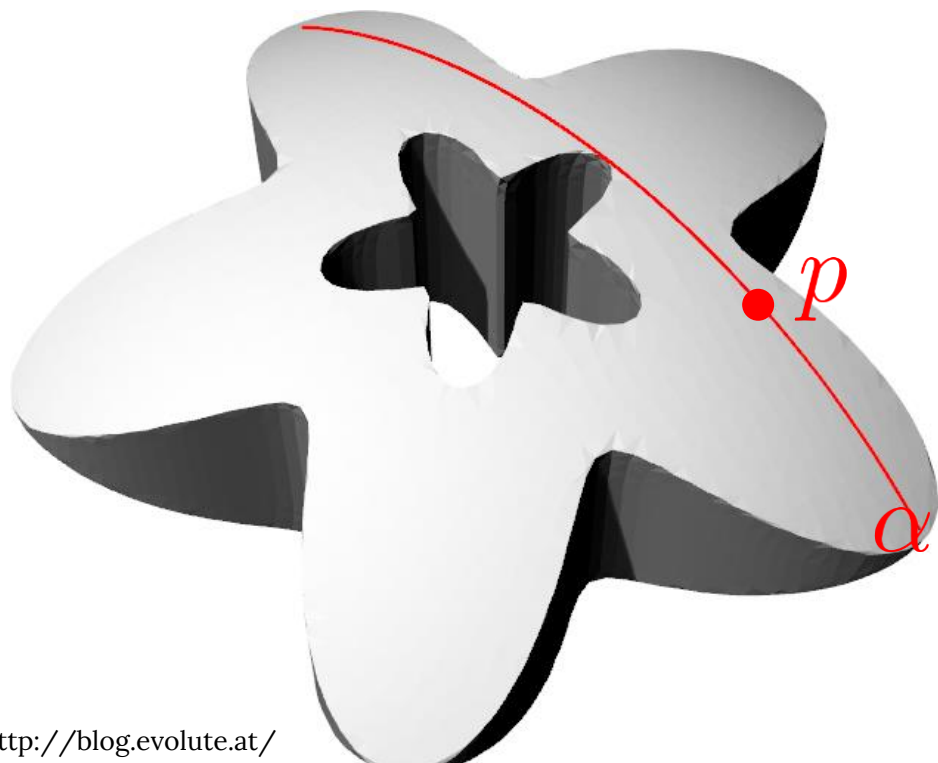
[http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo\\_OneView.jpg](http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg)

**Map points to real numbers**

# Differential of a Map

Suppose  $f: S \rightarrow \mathbb{R}$  and take  $p \in S$ . For  $v \in T_p S$ , choose a **curve**  $\alpha: (-\varepsilon, \varepsilon) \rightarrow S$  with  $\alpha(0) = p$  and  $\alpha'(0) = v$ . Then the differential of  $f$  is  $df: T_p S \rightarrow \mathbb{R}$  with

$$(df)_p(v) := \left. \frac{d}{dt} \right|_{t=0} (f \circ \alpha)(t) = (f \circ \alpha)'(0).$$



**On the board (time-permitting):**

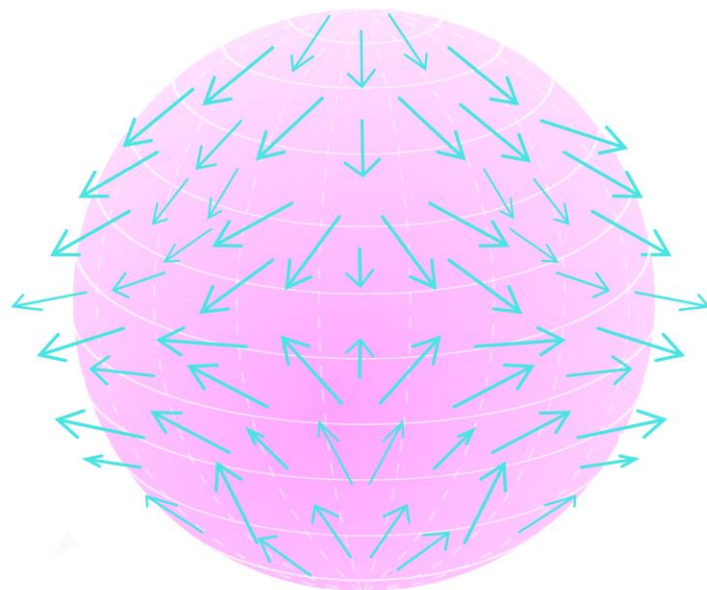
- Does not depend on choice of  $\alpha$
- Linear map



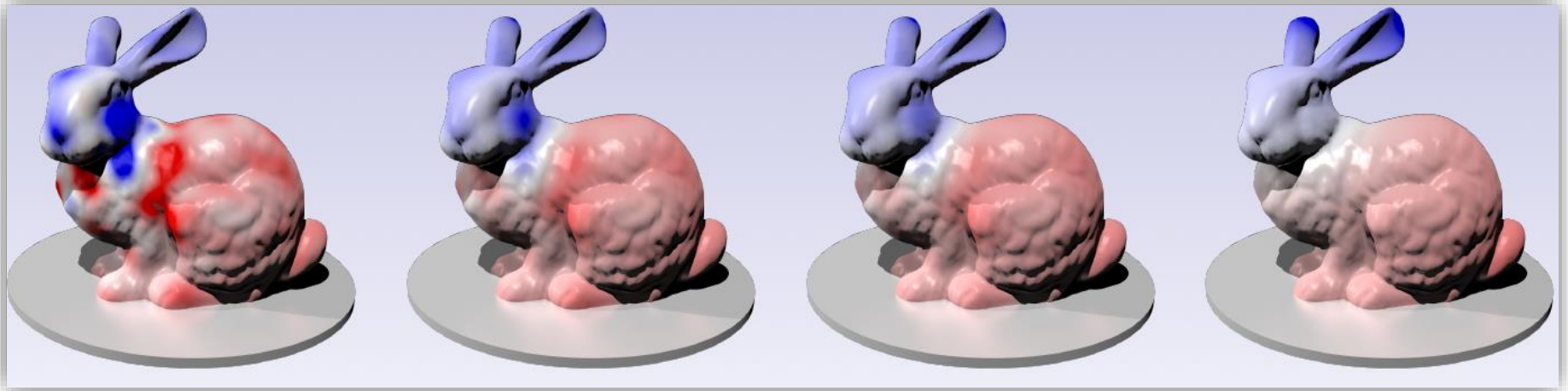
# Gradient Vector Field

$\nabla f : S \rightarrow \mathbb{R}^3$  with

$$\begin{cases} \langle (\nabla f)(p), v \rangle = (df)_p(v), v \in T_p S \\ \langle (\nabla f)(p), N(p) \rangle = 0 \end{cases}$$



# Dirichlet Energy



Decreasing  $E$

$$E[f] := \int_S \|\nabla f\|_2^2 dA$$

# From Inner Product to Operator

$$\langle f, g \rangle_{\Delta} := \int_S \nabla f(x) \cdot \nabla g(x) dA$$

$$:= \langle f, \Delta g \rangle$$

**Implies**

$$\langle f, f \rangle \geq 0$$

**On the board:**

**“Motivation” from finite-dimensional linear algebra.**

**Laplace-Beltrami operator**

# What is Divergence?

$V : S \rightarrow \mathbb{R}^3$  where  $V(p) \in T_p S$

$dV_p : T_p S \rightarrow \mathbb{R}^3$

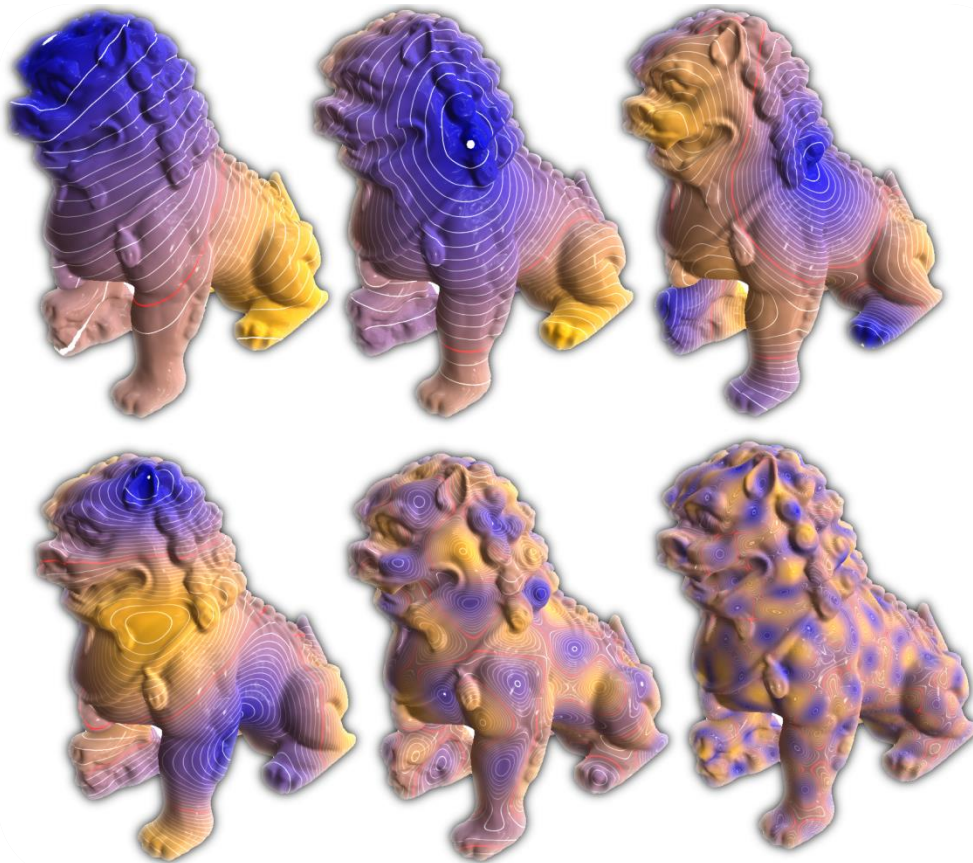
$\{e_1, e_2\} \subset T_p S$  orthonormal basis

$$(\nabla \cdot V)_p := \sum_{i=1}^2 \langle e_i, dV(e_i) \rangle_p$$

Things we **should check** (but probably won't):

- Independent of choice of basis
  - $\Delta = \nabla \cdot \nabla$

# Eigenfunctions



$$\Delta\psi_i = \lambda_i\psi_i$$

**Vibration modes of  
surface (not  
volume!)**

# Chladni Plates



<https://www.youtube.com/watch?v=CGiSlMFFlI>

# Performance Art?



[https://www.youtube.com/watch?v=Fyzqd2\\_T09Q](https://www.youtube.com/watch?v=Fyzqd2_T09Q)

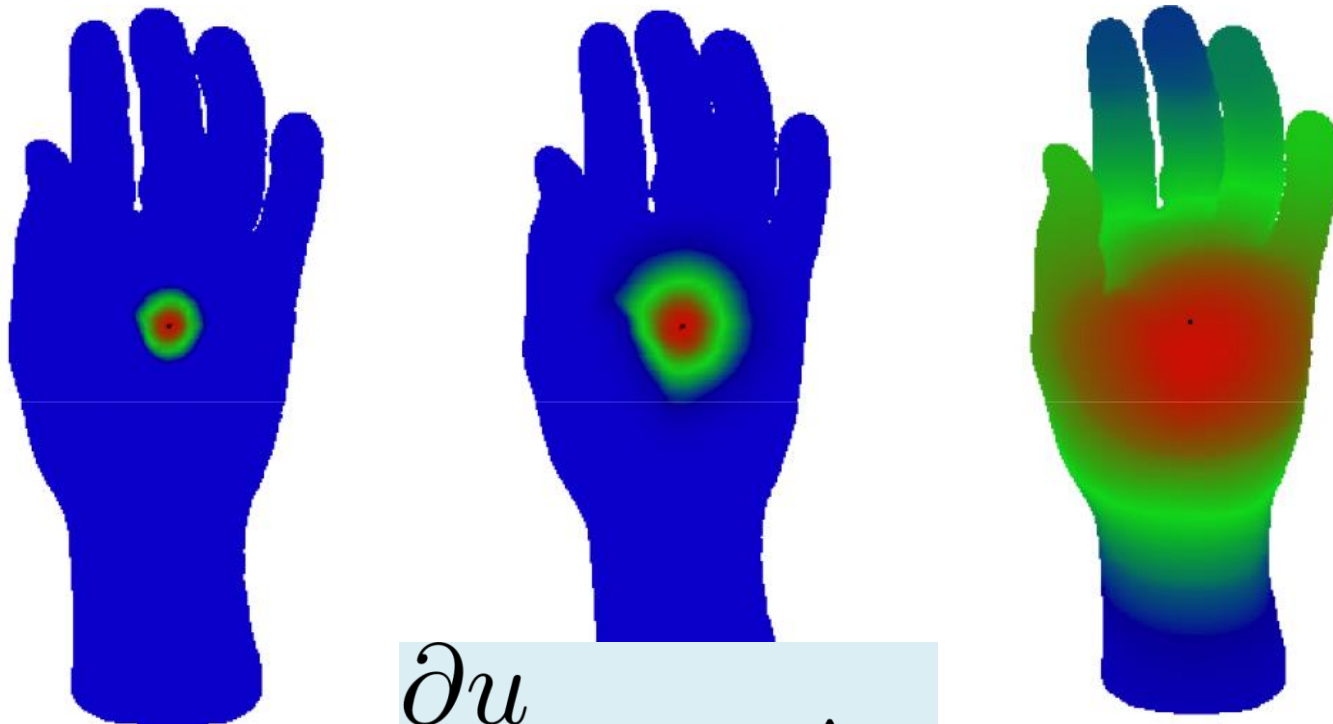
# Practical Application



<https://www.youtube.com/watch?v=3uMZzVvnSiU>



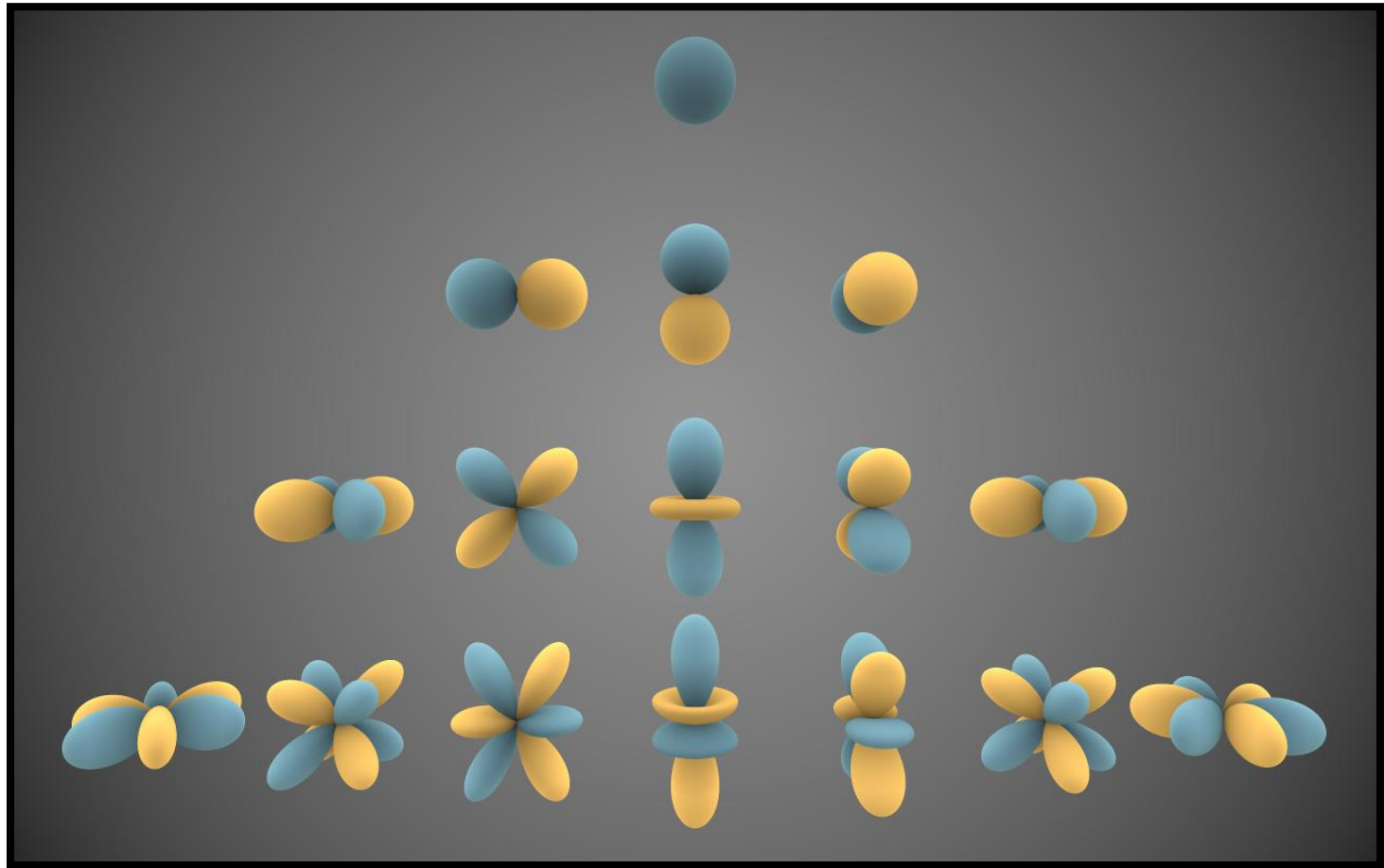
# Additional Connection to Physics



$$\frac{\partial u}{\partial t} = -\Delta u$$

**Heat equation**

# Spherical Harmonics



# Weyl's Law

$N(\lambda) := \#$  eigenfunctions  $\leq \lambda$

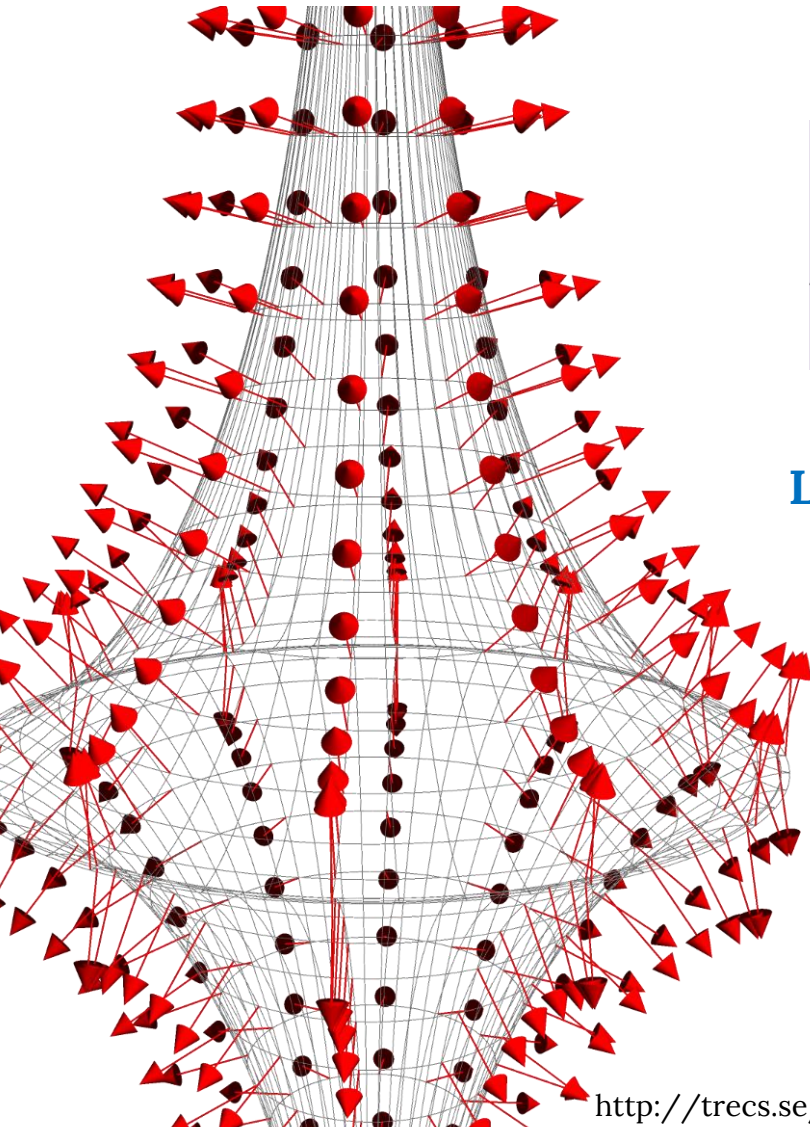
$\omega_d :=$  volume of unit ball in  $\mathbb{R}^d$

$$\lim_{\lambda \rightarrow \infty} \frac{N(\lambda)}{\lambda^{d/2}} = (2\pi)^{-d} \omega_d \text{vol}(\Omega)$$

Corollary:  $\text{vol}(\Omega) = (2\pi)^d \lim_{R \rightarrow \infty} \frac{N(R)}{R^{d/2}}$

For surfaces:  $\lambda_n \sim \frac{4\pi}{\text{vol}(\Omega)} n$

# Laplacian of xyz function



$$\Delta \vec{x} = \frac{1}{2} (\kappa_1 + \kappa_2) N$$

*Intuition:*

**Laplacian measures difference with neighbors.**