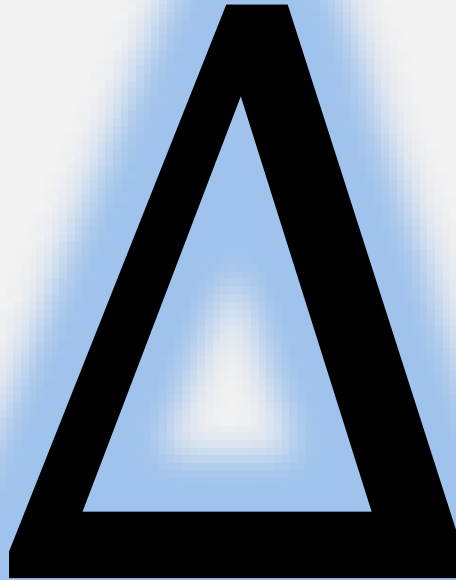


IFT 6112

08 – DISCRETE LAPLACIAN

<http://www-labs.iro.umontreal.ca/~bmpix/teaching/6112/2018/>

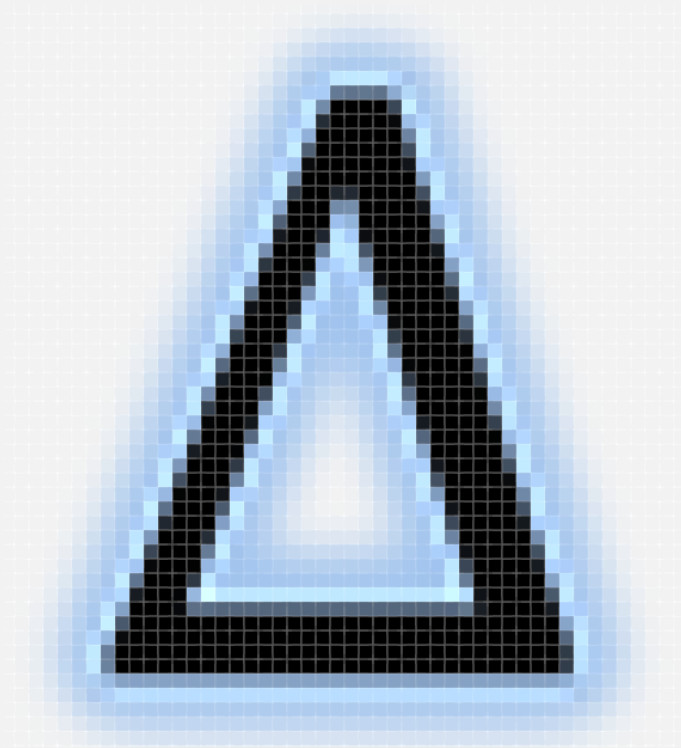


Mikhail Bessmeltsev

IFT 6112

08 – DISCRETE LAPLACIAN

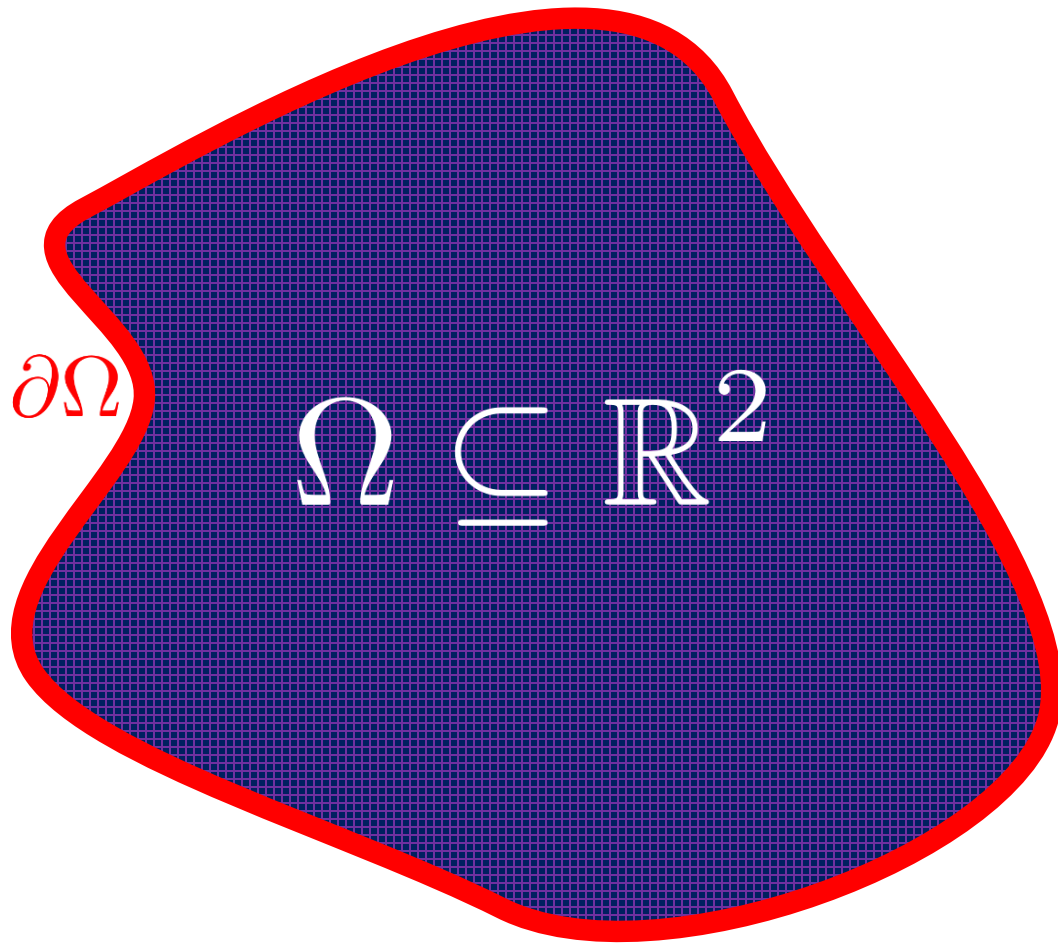
<http://www-labs.iro.umontreal.ca/~bmpix/teaching/6112/2018/>



Mikhail Bessmeltsev

Recall:

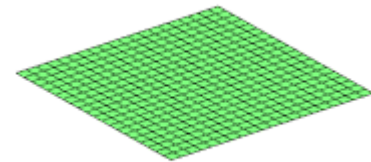
Planar Region



Wave equation:

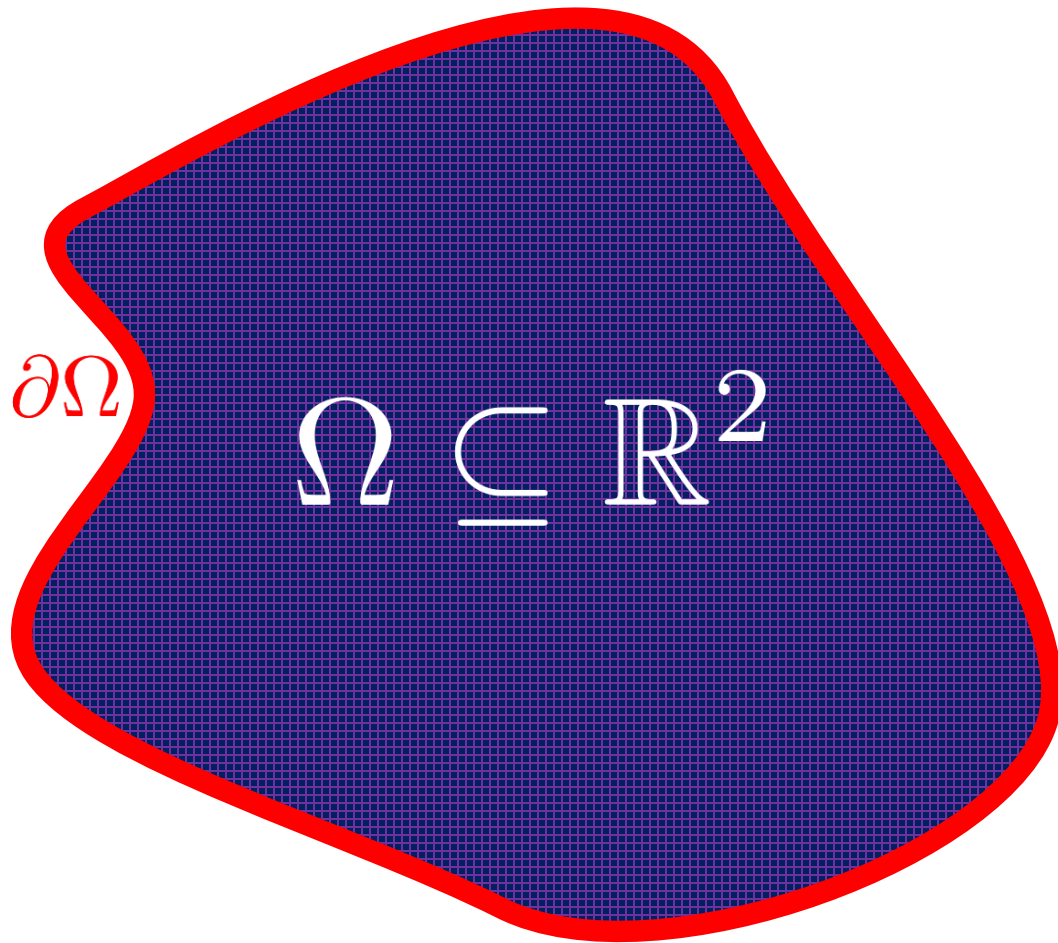
$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

$$\Delta := \sum_i \frac{\partial^2}{\partial x_i^2}$$



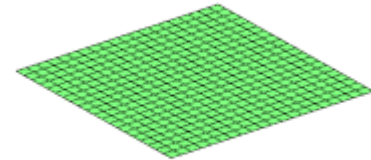
Recall:

Planar Region



How to
discretize on a
mesh?

$$\Delta := \sum_i \frac{\partial^2}{\partial x_i^2}$$



Discretizing the Laplacian

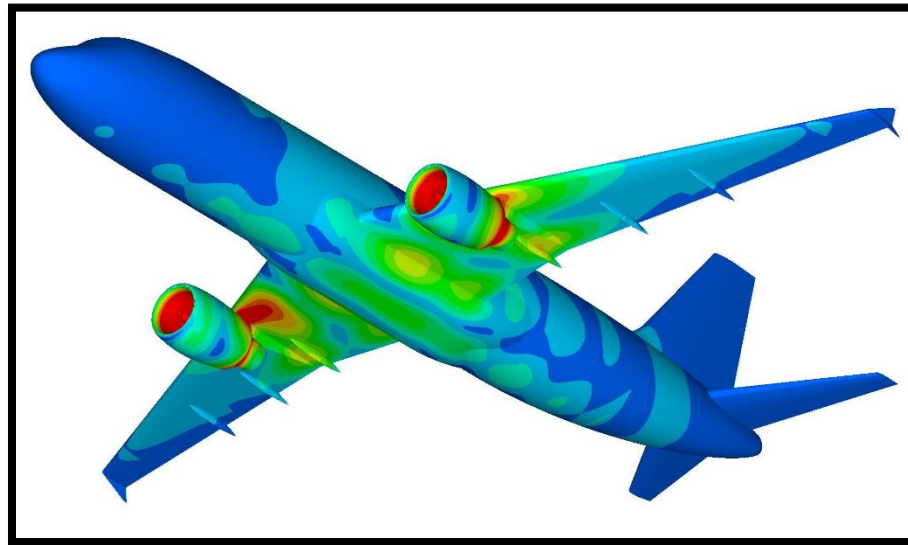
$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right)$$

?!

Today's Approach

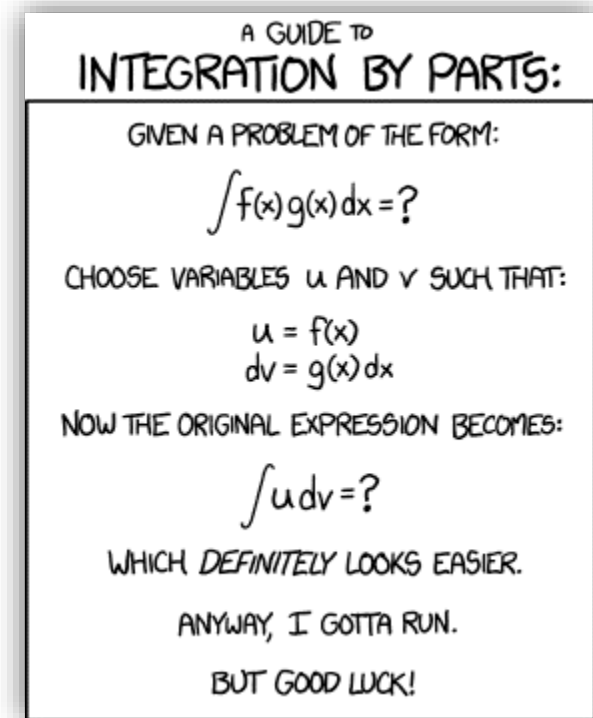
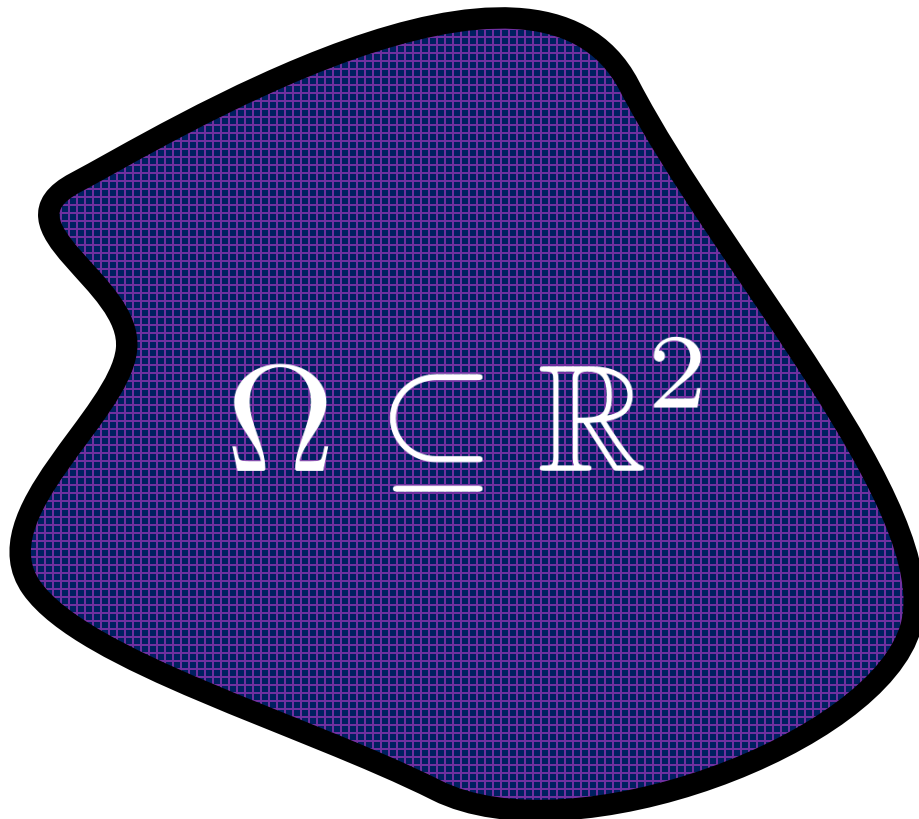
First-order Galerkin

Finite element method (FEM)



Integration by Parts to the Rescue

$$\int_{\Omega} f \Delta g \, dA = \text{boundary terms} - \int_{\Omega} \nabla f \cdot \nabla g \, dA$$



Slightly Easier?

$$\int_{\Omega} f \Delta g \, dA = \text{boundary terms} - \int_{\Omega} \nabla f \cdot \nabla g \, dA$$

Laplacian
(second derivative)

Gradient
(first
derivative)

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

Overview:

Galerkin FEM Approach

$$g = \Delta f \quad \Rightarrow \quad \langle g, \psi \rangle = \langle \Delta f, \psi \rangle$$

For any **test
function** ψ

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA$$

For any **test
function** ψ

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

For any **test
function** ψ

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

Approximate $f \approx \sum_i a_i \psi_i$ and $g \approx \sum_i b_i \psi_i$

$$\implies \text{Linear system } \sum_i b_i \langle \psi_i, \psi_j \rangle = - \sum_i a_i \langle \nabla \psi_i, \nabla \psi_j \rangle$$

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

Approximate $f \approx \sum_i a_i \psi_i$ and $g \approx \sum_i b_i \psi_i$

$$\implies \text{Linear system } \sum_i b_i \langle \psi_i, \psi_j \rangle = - \sum_i a_i \langle \nabla \psi_i, \nabla \psi_j \rangle$$

Mass matrix: $M_{ij} := \langle \psi_i, \psi_j \rangle$

Stiffness matrix: $L_{ij} := \langle \nabla \psi_i, \nabla \psi_j \rangle$

$$\implies Mb = La$$

Which
basis?

Important to Note

Not the only way

to approximate the Laplacian operator.

- Divided differences
- Higher-order elements
- Boundary element methods
- Discrete exterior calculus
- ...

L^2 Dual of a Function

Function $f : M \rightarrow \mathbb{R}$



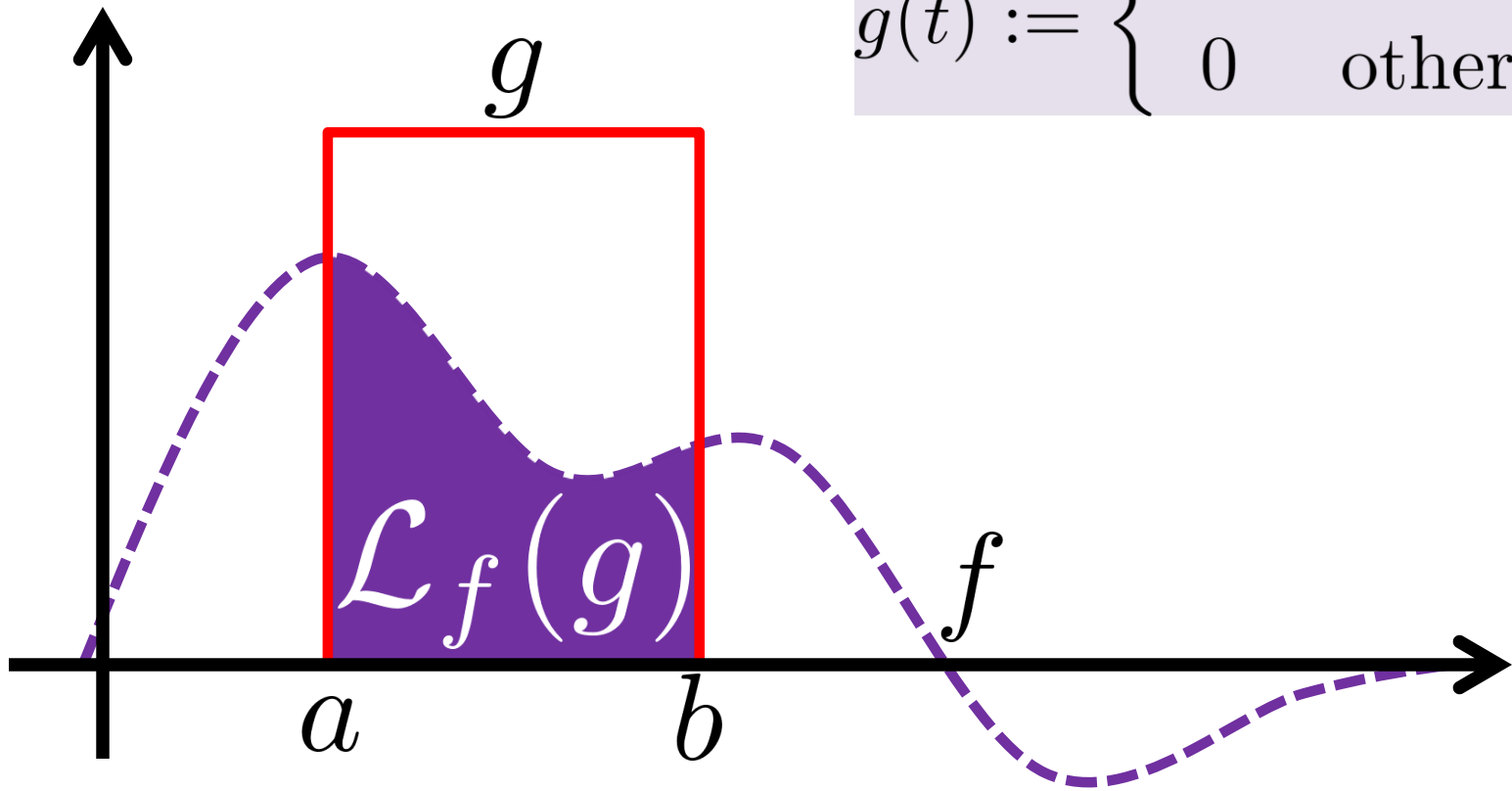
Operator $\mathcal{L}_f : L^2(M) \rightarrow \mathbb{R}$

$$\mathcal{L}_f[g] := \int_M f(x)g(x) dA$$

Test function

Observation

$$g(t) := \begin{cases} 1 & \text{if } t \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$



Can recover function from dual

Dual of Laplacian

Space of test functions (no boundary!):

$$\{g \in L^\infty(M) : g|_{\partial M} \equiv 0\}$$

$$\begin{aligned}\mathcal{L}_{\Delta f}[g] &= \int_M g \Delta f \, dA \\ &= - \int_M \nabla g \cdot \nabla f \, dA\end{aligned}$$

Use Laplacian without evaluating

Galerkin's Approach

Choose one of each:

- Function space
- Test functions

Often the same!

One Derivative is Enough

$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

First Order Finite Elements

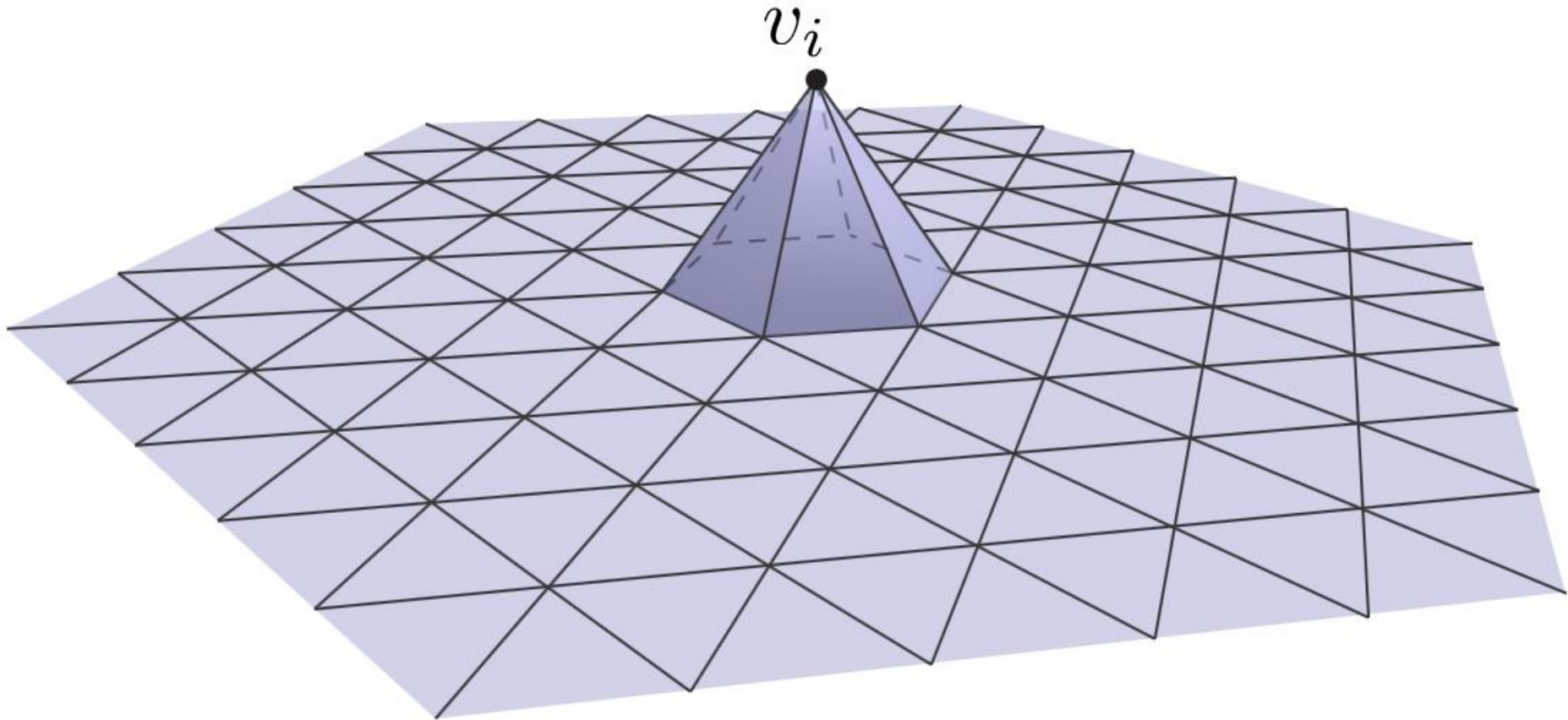
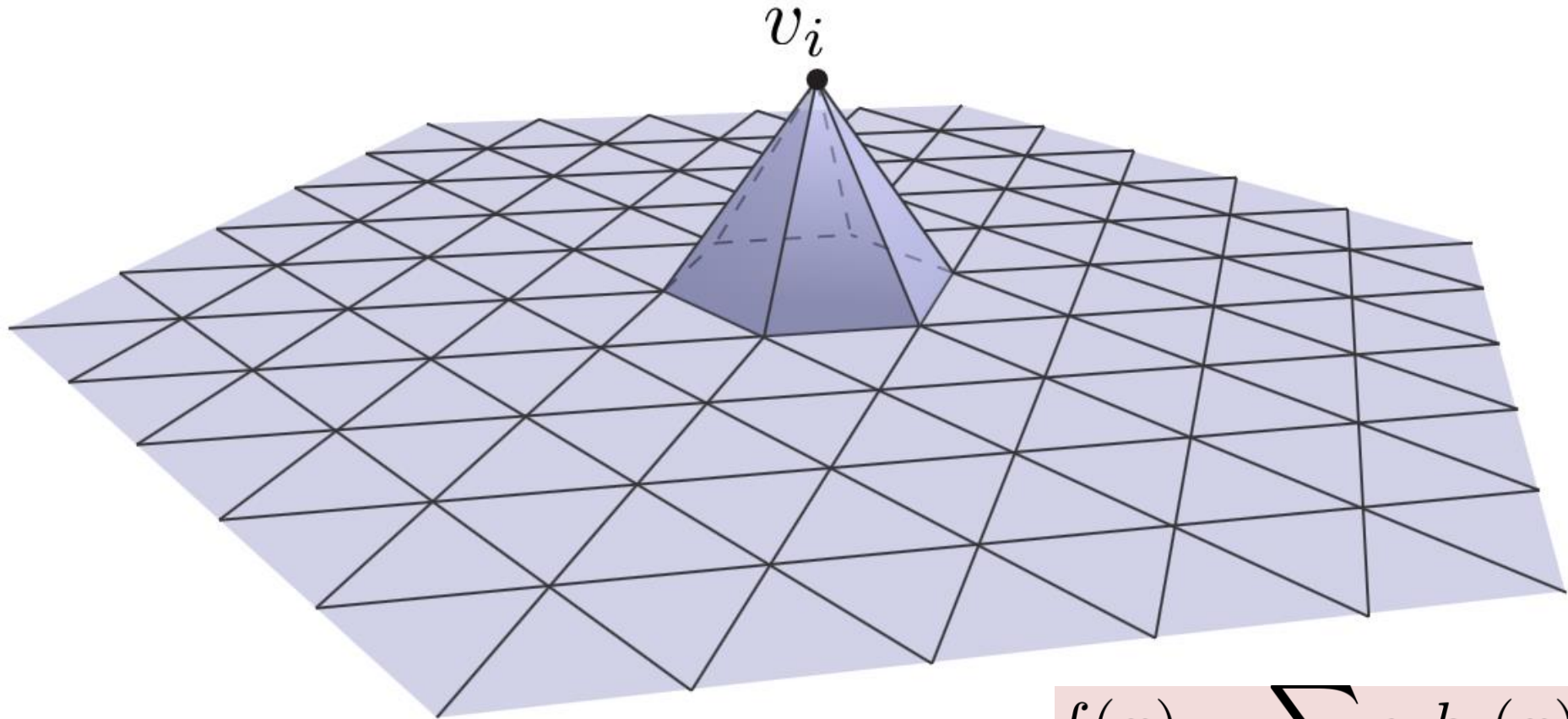


Image courtesy K. Crane, CMU

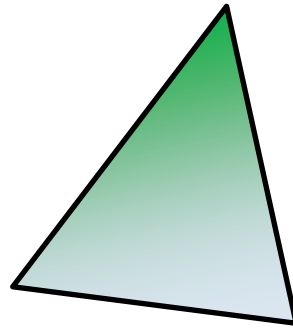
One “hat function” per vertex

Representing Functions



$$f(x) = \sum_i a_i h_i(x)$$
$$a \in \mathbb{R}^{|V|}$$

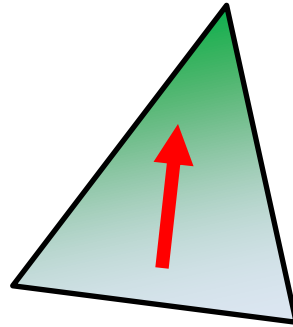
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

**Linear combination of hats
(piecewise linear)**

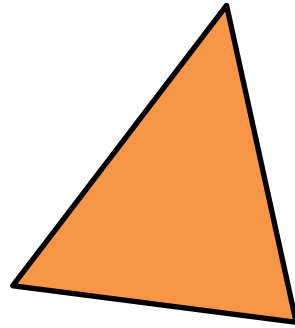
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

One vector per face

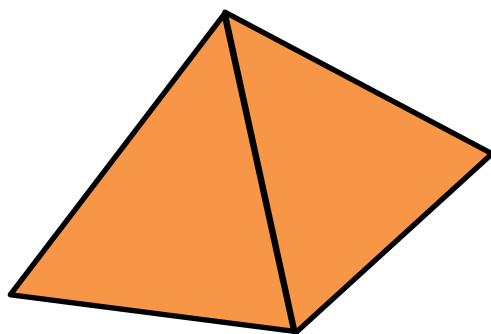
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

One scalar per face

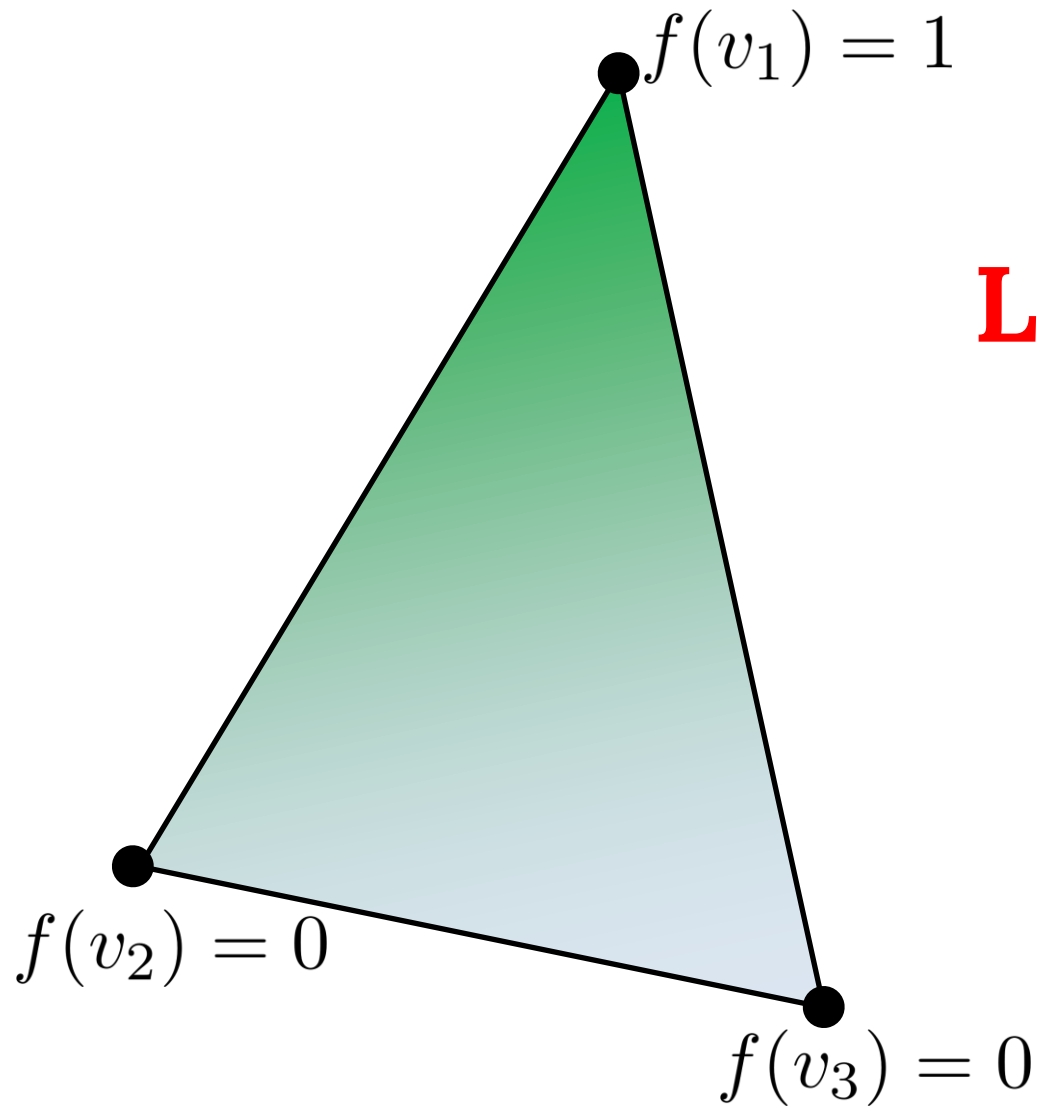
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

↑
**Sum scalars per face
multiplied by face areas**

Gradient of a Hat Function



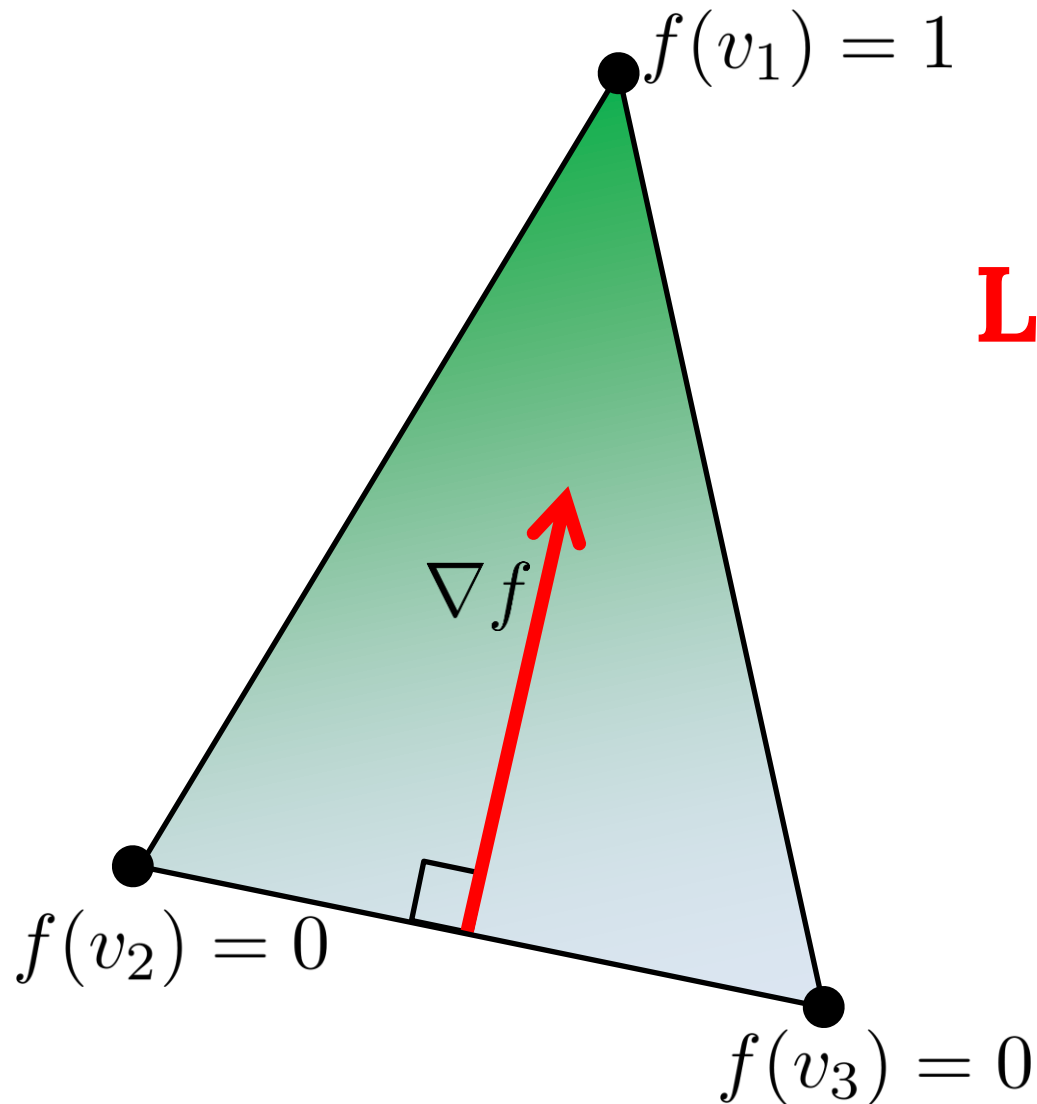
Linear along edges

$$\nabla f \cdot (v_1 - v_3) = 1$$

$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$

Gradient of a Hat Function



Linear along edges

$$\nabla f \cdot (v_1 - v_3) = 1$$

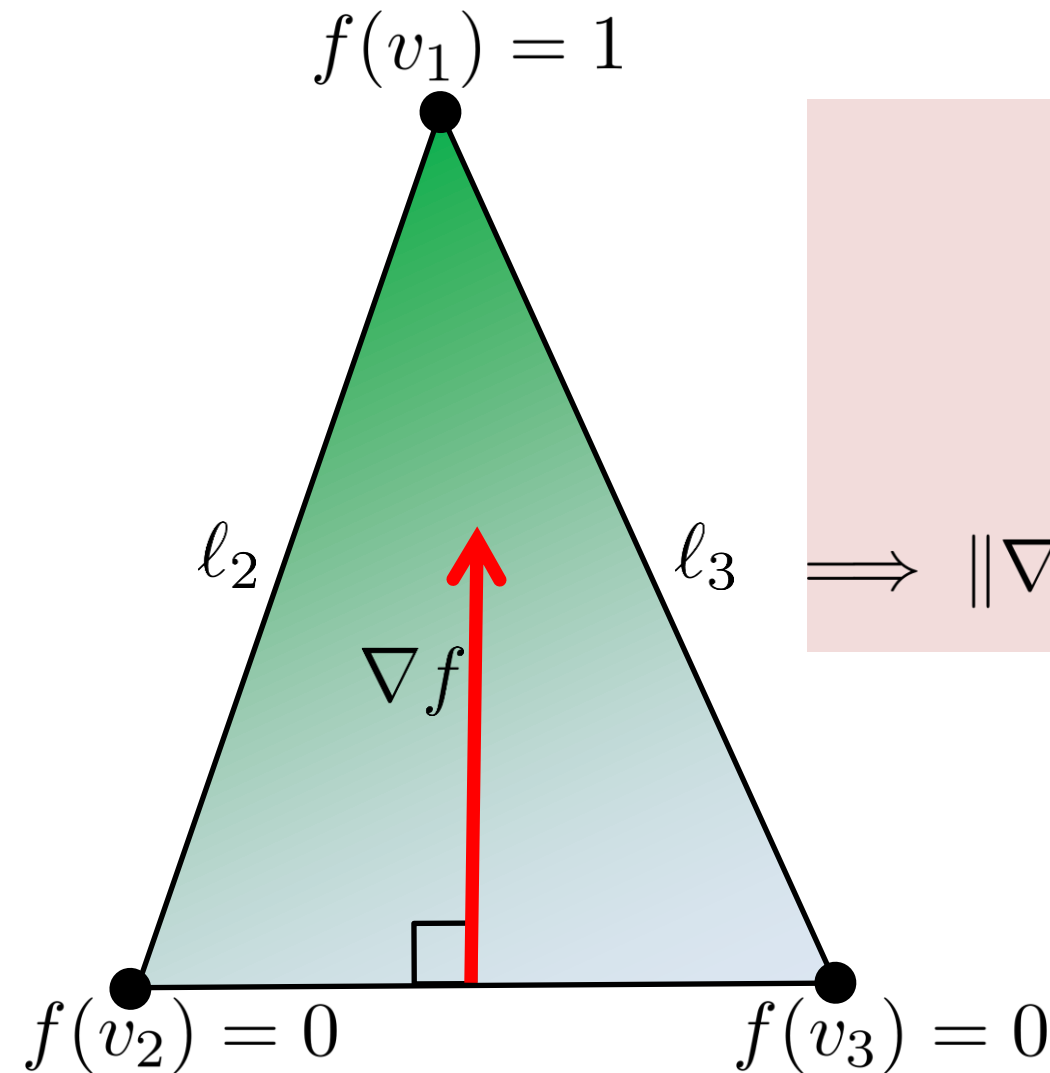
$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$



$$\nabla f \cdot (v_2 - v_3) = 0$$

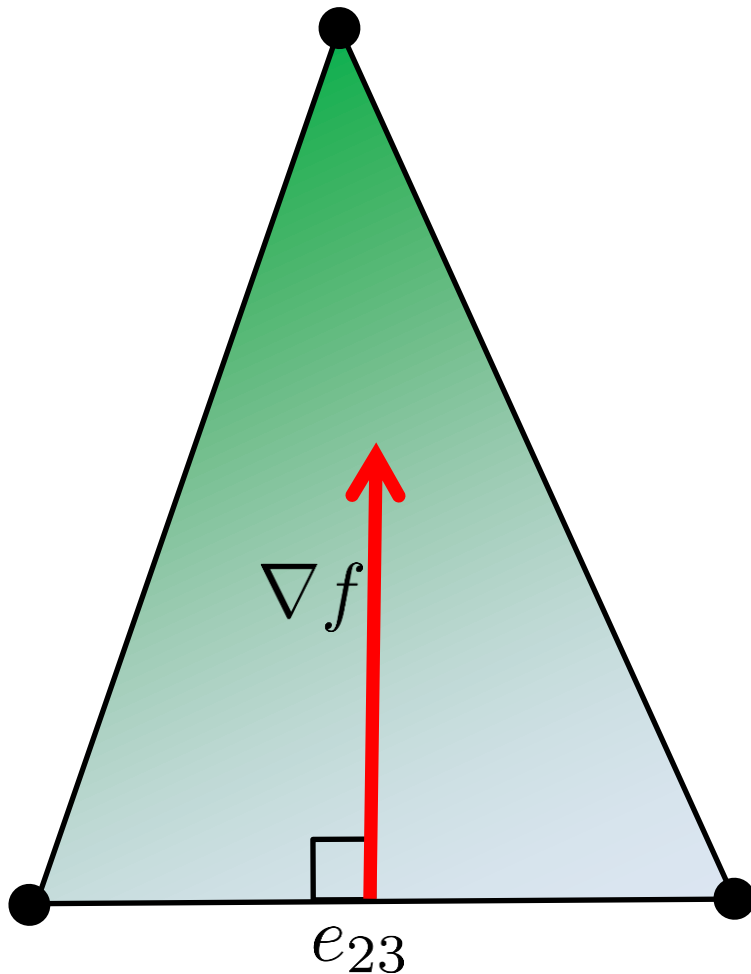
Gradient of a Hat Function



$$\begin{aligned} 1 &= \nabla f \cdot (v_1 - v_3) \\ &= \|\nabla f\| l_3 \cos\left(\frac{\pi}{2} - \theta_3\right) \\ &= \|\nabla f\| l_3 \sin \theta_3 \\ \implies \|\nabla f\| &= \frac{1}{l_3 \sin \theta_3} = \frac{1}{h} \end{aligned}$$

Gradient of a Hat Function

$$\|\nabla f\| = \frac{1}{l_3 \sin \theta_3} = \frac{1}{h}$$

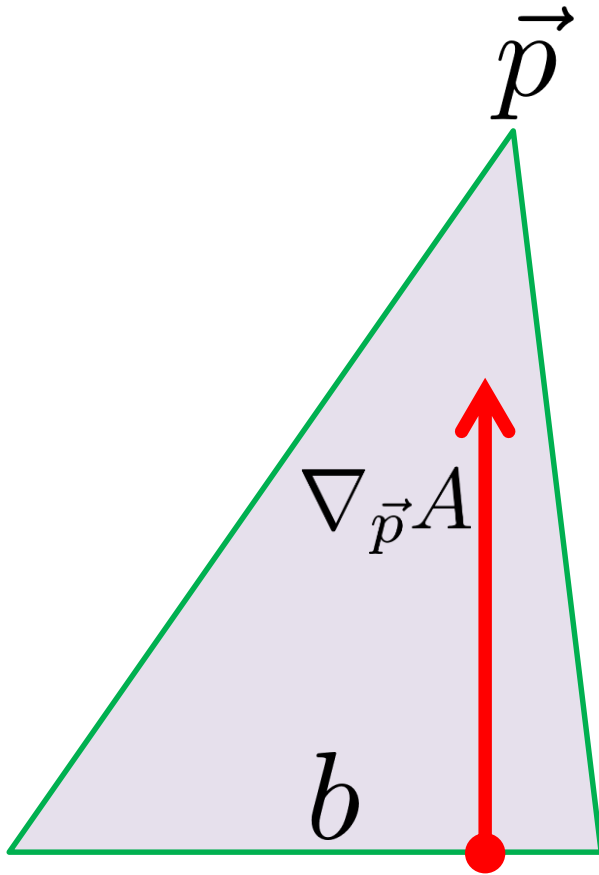


$$\nabla f = \frac{e_{23}^\perp}{2A}$$

Length of e_{23} cancels
“base” in A

Recall:

Single Triangle: Complete

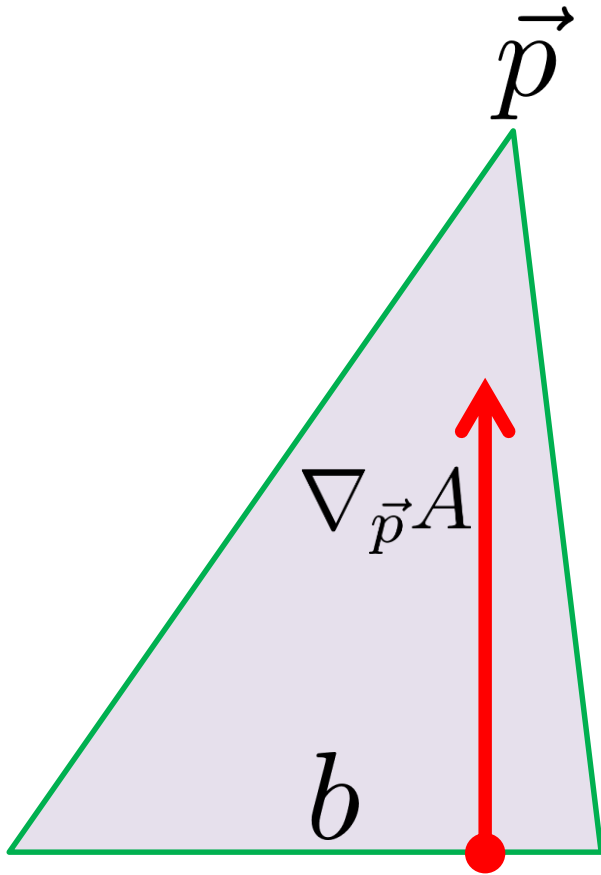


$$\vec{p} = p_n \vec{n} + p_e \vec{e} + p_{\perp} \vec{e}_{\perp}$$
$$A = \frac{1}{2} b \sqrt{p_n^2 + p_{\perp}^2}$$
$$\nabla_{\vec{p}} A = \frac{1}{2} b \vec{e}_{\perp}$$

Similar expression

Recall:

Single Triangle: Complete



$$\vec{p} = p_n \vec{n} + p_e \vec{e} + p_{\perp} \vec{e}_{\perp}$$

$$A = \frac{1}{2} b \sqrt{p_n^2 + p_{\perp}^2}$$

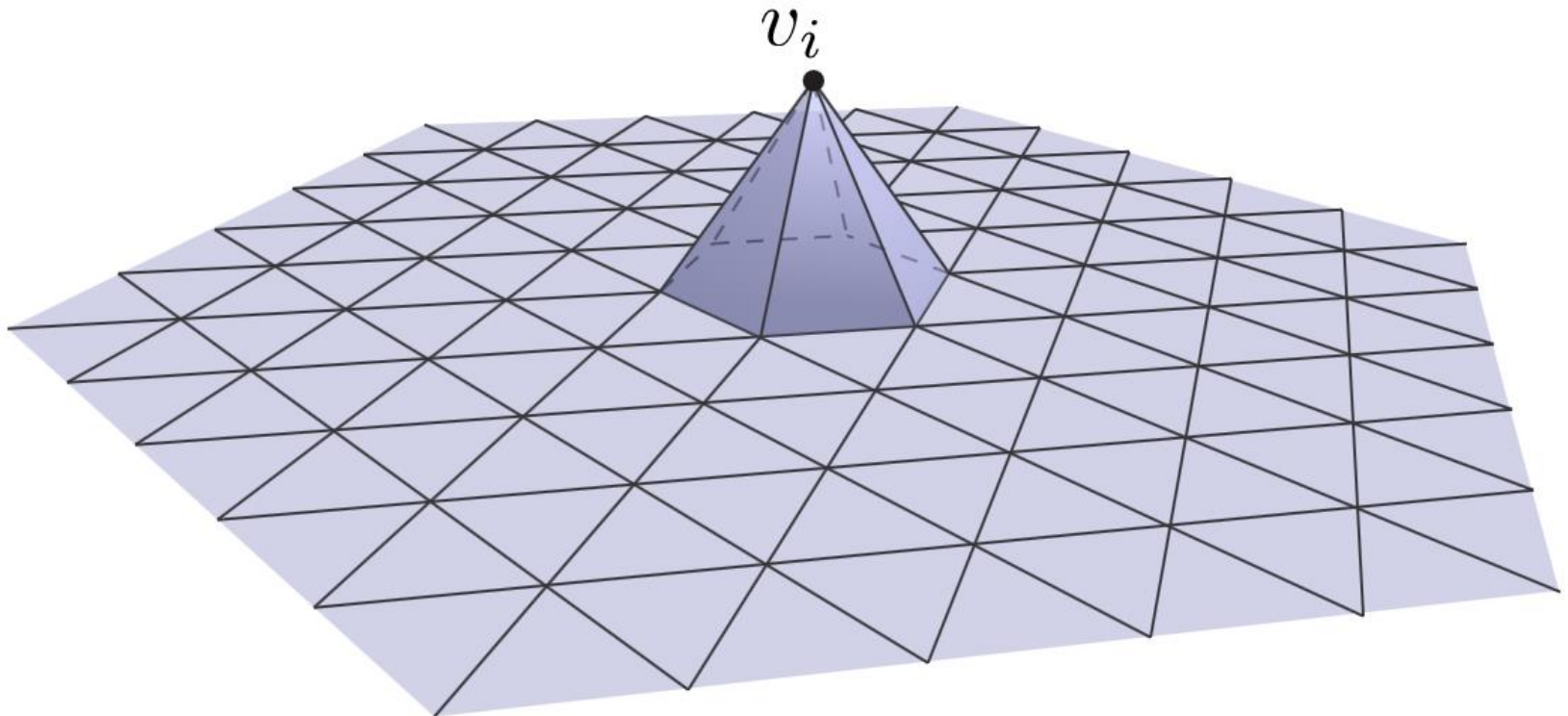
$$\nabla_{\vec{p}} A = \frac{1}{2} b \vec{e}_{\perp}$$

$$\nabla f = \frac{e_{23}^{\perp}}{2A} = \frac{\vec{e}_{\perp}}{h} = \frac{\nabla_{\vec{p}} A}{A}$$

Similar expression

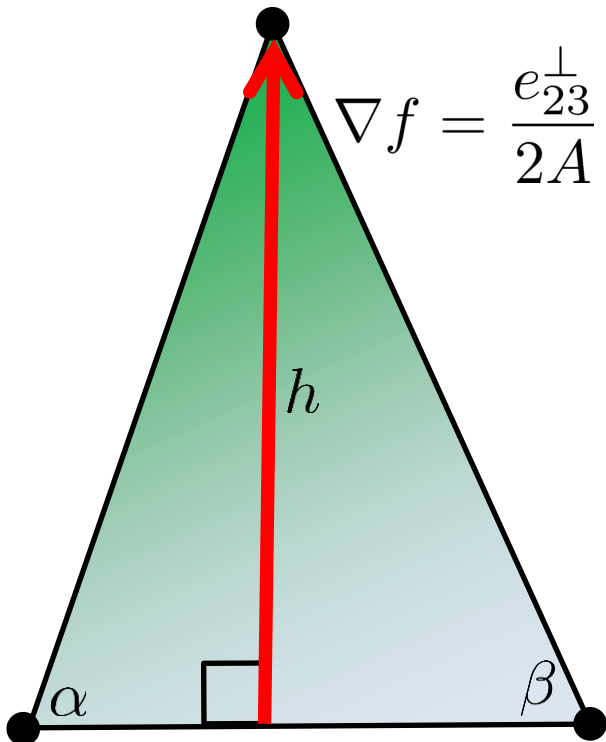
What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = - \int_M \boxed{\nabla g \cdot \nabla f} dA$$



What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = - \int_M \boxed{\nabla g \cdot \nabla f} dA$$



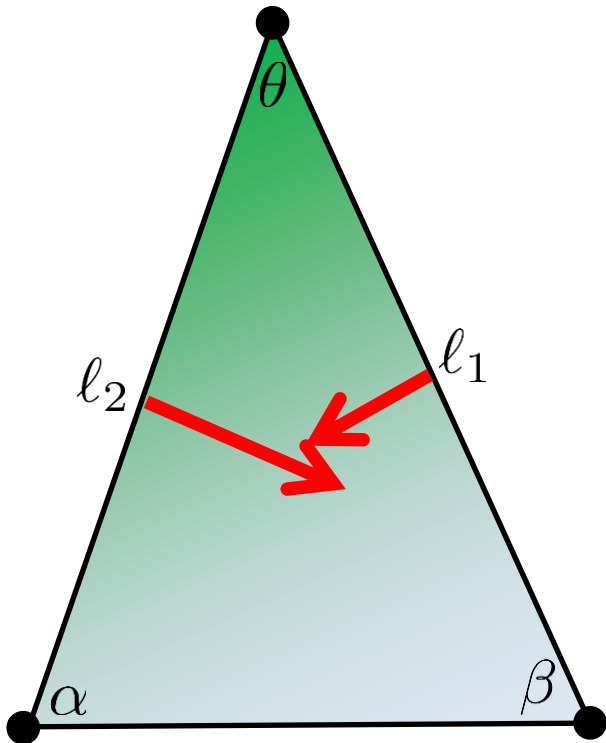
Case 1: Same vertex

$$\begin{aligned} \int_T \langle \nabla f, \nabla f \rangle dA &= A \|\nabla f\|^2 \\ &= \frac{A}{h^2} = \frac{b}{2h} \\ &= \frac{1}{2} (\cot \alpha + \cot \beta) \end{aligned}$$

What We Actually Need

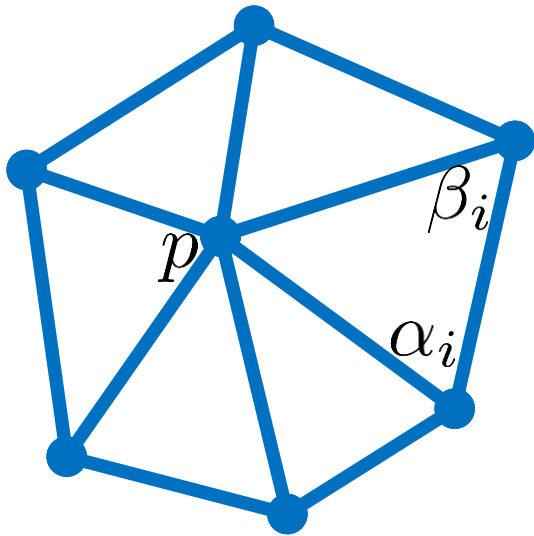
$$\mathcal{L}_{\Delta} f [g] = - \int_M \boxed{\nabla g \cdot \nabla f} dA$$

Case 2: Different vertices

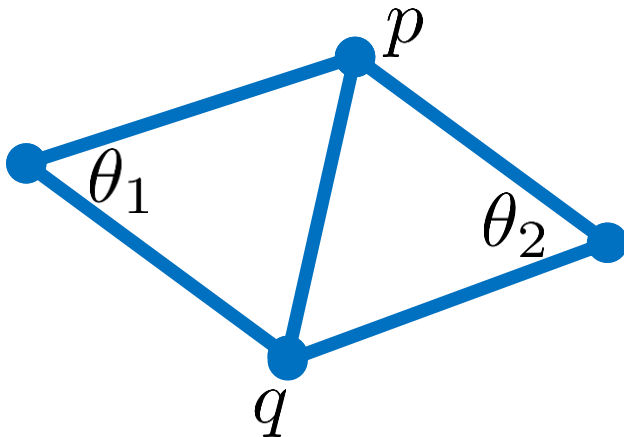


$$\begin{aligned} \int_T \langle \nabla f_{\alpha}, \nabla f_{\beta} \rangle dA &= A \langle \nabla f_{\alpha}, \nabla f_{\beta} \rangle \\ &= \frac{1}{4A} \langle e_{31}^{\perp}, e_{12}^{\perp} \rangle = -\frac{l_1 l_2 \cos \theta}{4A} \\ &= \frac{-h^2 \cos \theta}{4A \sin \alpha \sin \beta} = \frac{-h \cos \theta}{2b \sin \alpha \sin \beta} \\ &= -\frac{\cos \theta}{2 \sin(\alpha + \beta)} = -\frac{1}{2} \cot \theta \end{aligned}$$

Summing Around a Vertex



$$\langle \nabla h_p, \nabla h_p \rangle = \frac{1}{2} \sum_i (\cot \alpha_i + \cot \beta_i)$$

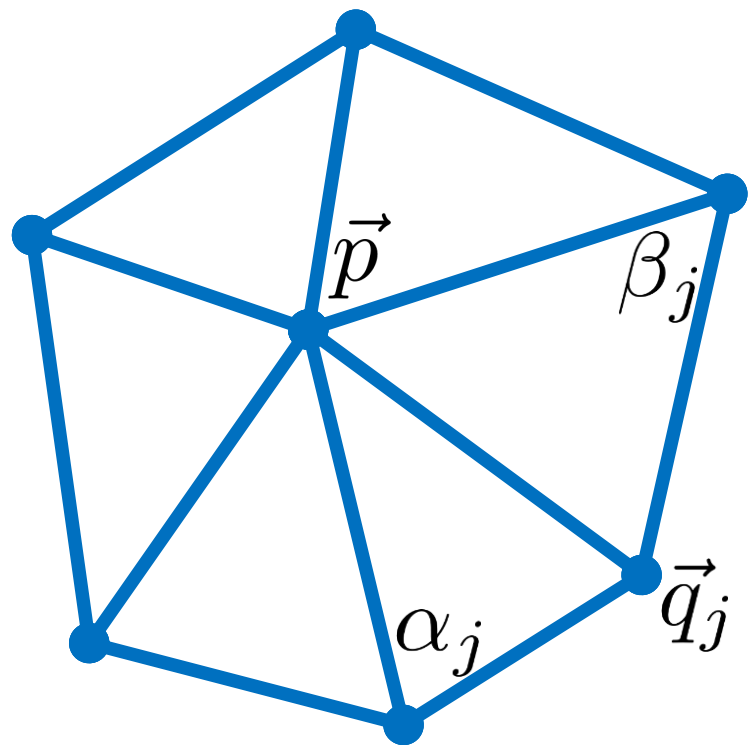


$$\langle \nabla h_p, \nabla h_q \rangle = \frac{1}{2} (\cot \theta_1 + \cot \theta_2)$$

Recall:

Summing Around a Vertex

$$\nabla_{\vec{p}} A = \frac{1}{2} \sum_j (\cot \alpha_j + \cot \beta_j) (\vec{p} - \vec{q}_j)$$

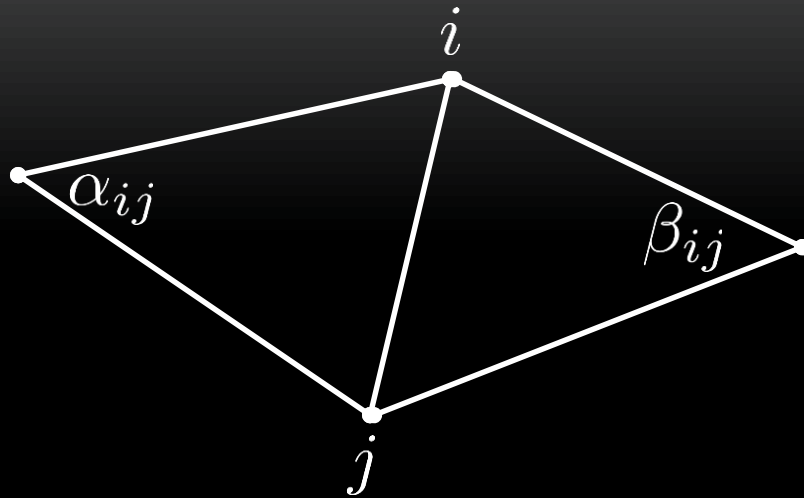


$$\nabla_{\vec{p}} A = \frac{1}{2} ((\vec{p} - \vec{r}) \cot \alpha + (\vec{p} - \vec{q}) \cot \beta)$$

Same weights up
to sign!

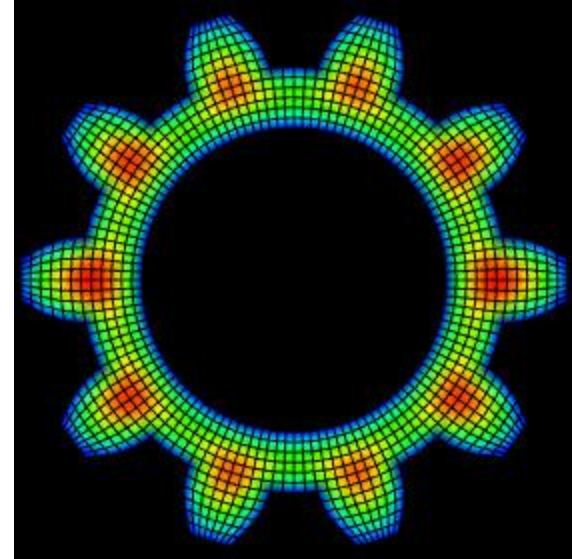
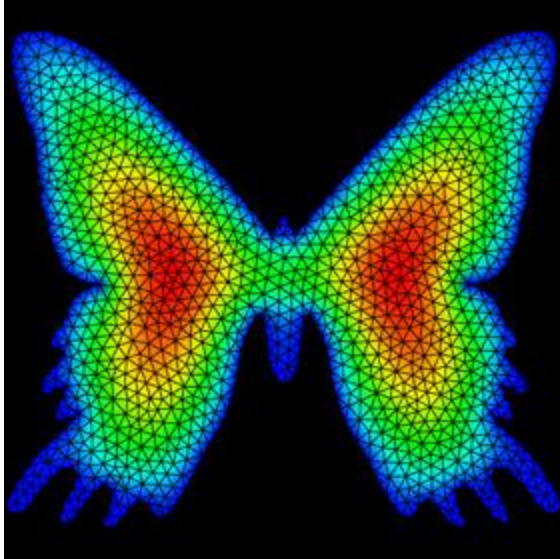
THE COTANGENT LAPLACIAN

$$L_{ij} = \begin{cases} \frac{1}{2} \sum_{i \sim k} (\cot \alpha_{ik} + \cot \beta_{ik}) & \text{if } i = j \\ -\frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}) & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$



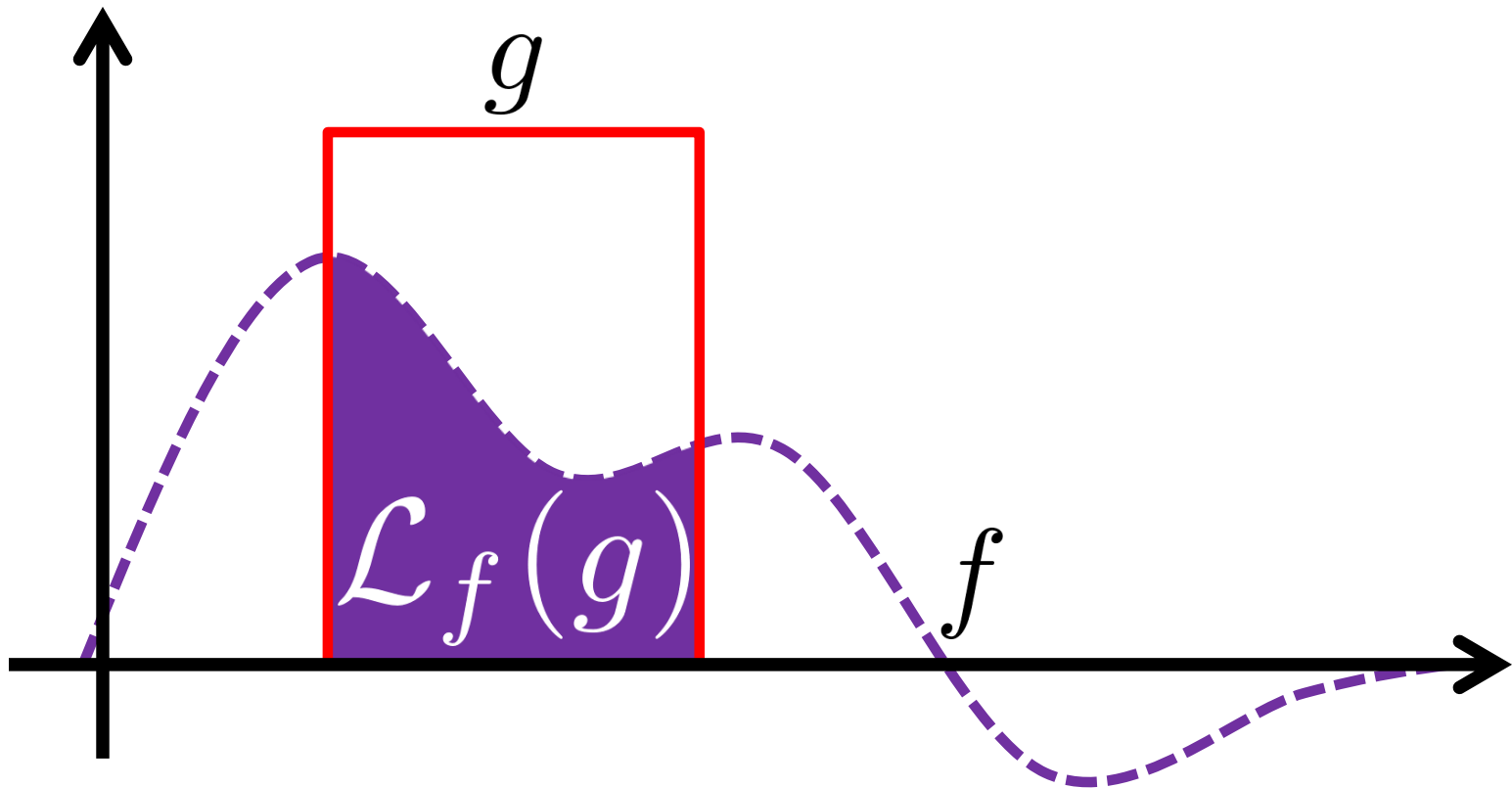
Poisson Equation

$$\Delta f = g$$



Weak Solutions

$$\int_M \phi \Delta f \, dA = \int_M \phi g \, dA \quad \forall \text{ test functions } \phi$$



FEM Hat Weak Solutions

$$\int_M h_i \Delta f \, dA = \int_M h_i g \, dA \quad \forall \text{ hat functions } h_i$$

$$\begin{aligned} \int_M h_i \Delta f \, dA &= - \int_M \nabla h_i \cdot \nabla f \, dA \\ &= - \int_M \nabla h_i \cdot \nabla \sum_j a_j h_j \, dA \\ &= - \sum_j a_j \int_M \nabla h_i \cdot \nabla h_j \, dA \\ &= \sum_j L_{ij} a_j \end{aligned}$$

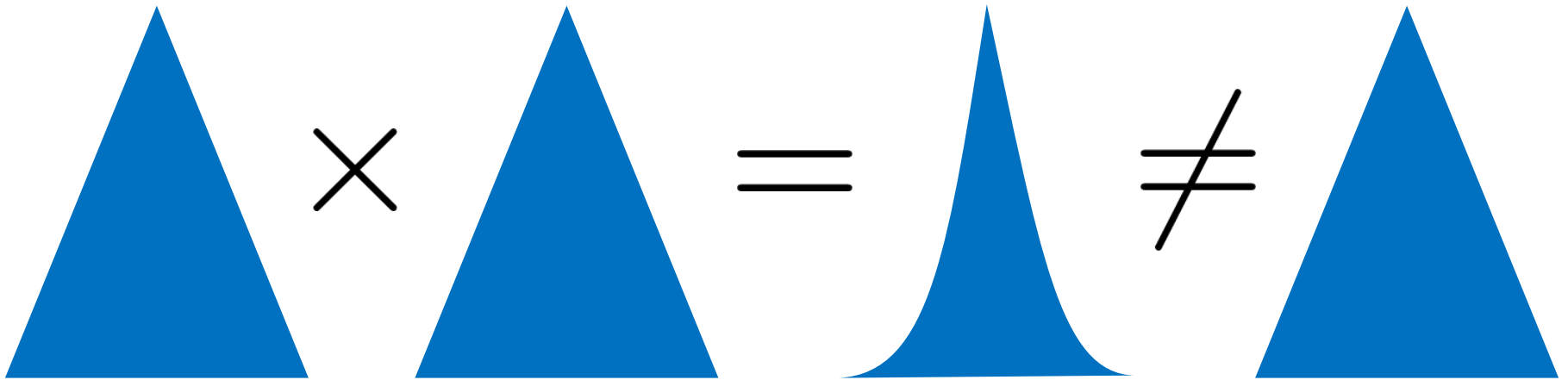
Stacking Integrated Products

$$\begin{pmatrix} \int_M h_1 \Delta f dA \\ \int_M h_2 \Delta f dA \\ \vdots \\ \int_M h_{|V|} \Delta f dA \end{pmatrix} = \begin{pmatrix} \sum_j L_{1j} a_j \\ \sum_j L_{2j} a_j \\ \vdots \\ \sum_j L_{|V|j} a_j \end{pmatrix} = L \vec{a}$$

Multiply by Laplacian matrix!

Problematic Right Hand Side

$$\int_M h_i \Delta f \, dA = \int_M h_i g \, dA \quad \forall \text{ hat functions } h_i$$



Product of hats is quadratic

A Few Ways Out

- **Just do the integral**
“Consistent” approach
- **Approximate some more**
- **Redefine g**

A Few Ways Out

- **Just do the integral**

“Consistent” approach

- **Approximate some more**

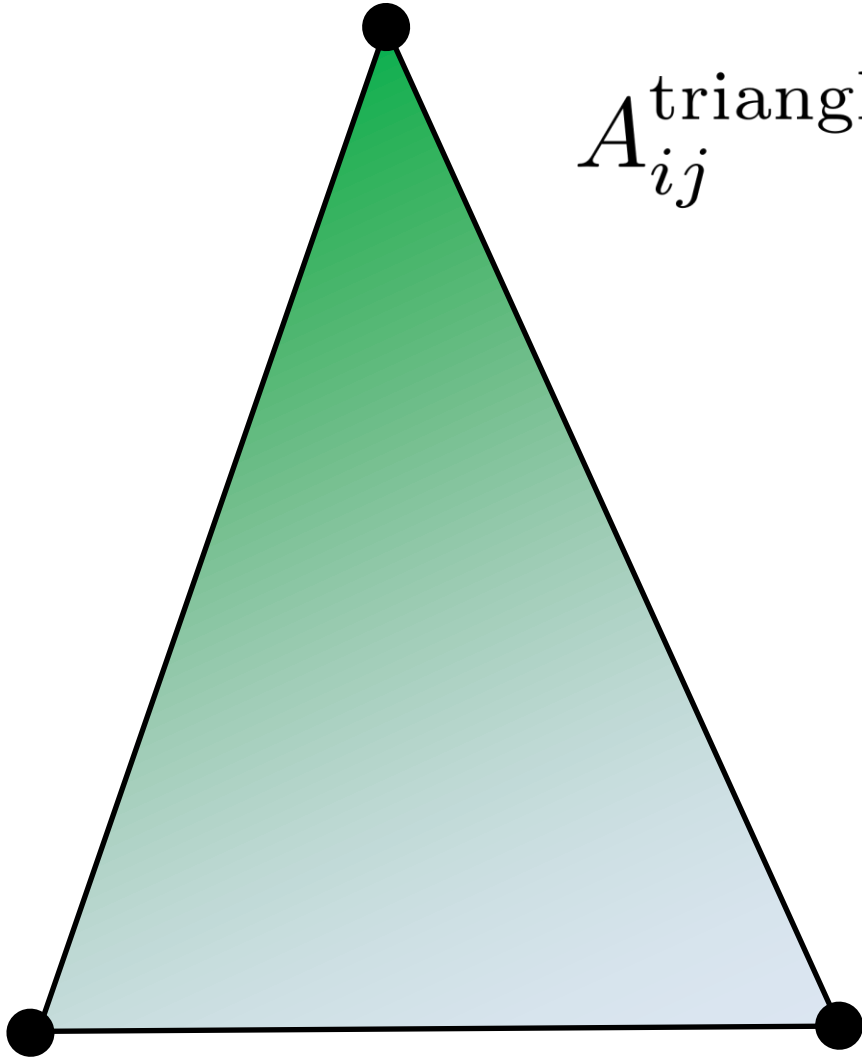
- **Redefine g**

The Mass Matrix

$$A_{ij} := \int_M h_i h_j dA$$

- **Diagonal** elements:
Norm of h_i
- **Off-diagonal** elements:
Overlap between h_i and h_j

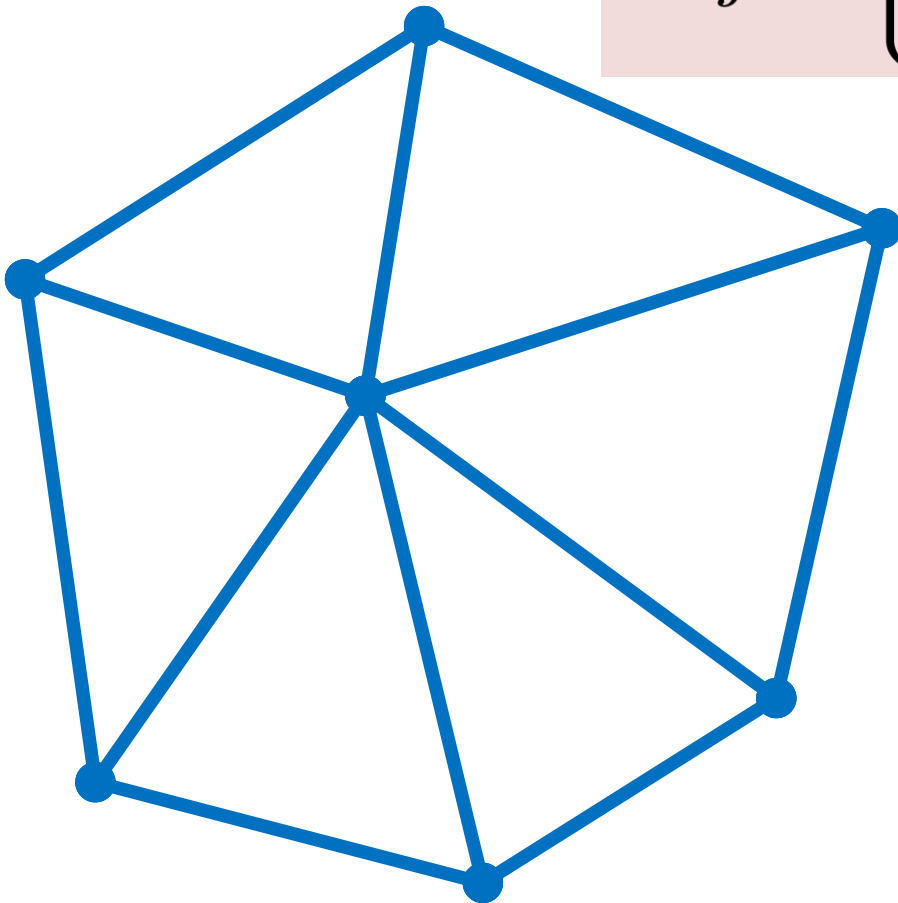
Consistent Mass Matrix



$$A_{ij}^{\text{triangle}} = \begin{cases} \frac{\text{area}}{6} & \text{if } i = j \\ \frac{\text{area}}{12} & \text{if } i \neq j \end{cases}$$

Non-Diagonal Mass Matrix

$$M_{ij} = \begin{cases} \frac{\text{one-ring area}}{6} & \text{if } i = j \\ \frac{\text{adjacent area}}{12} & \text{if } i \neq j \end{cases}$$



Properties of Mass Matrix

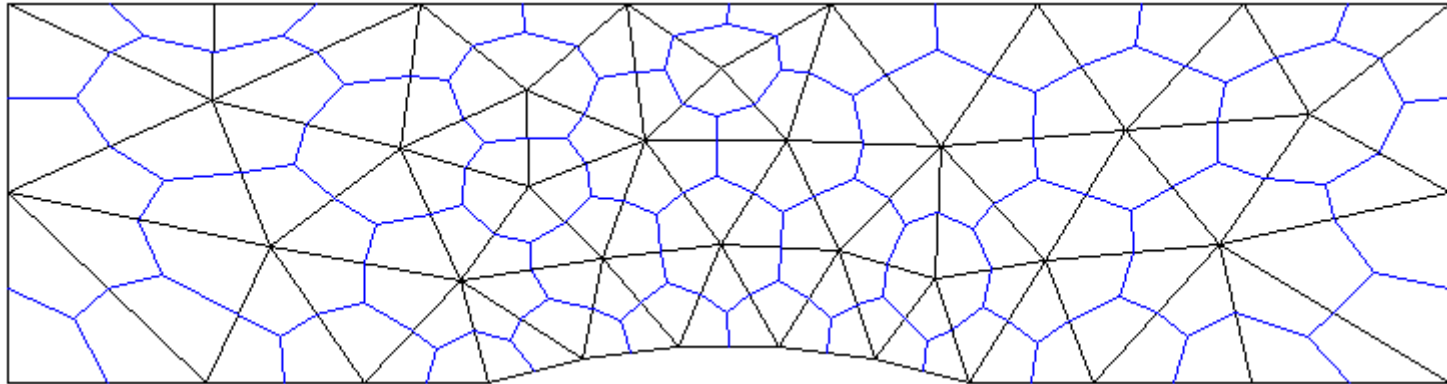
- Rows sum to one ring area / 3
- Involves only vertex and its neighbors
- Partitions surface area

Issue: Not trivial to invert!

Use for Integration

$$\begin{aligned}\int_M f &= \int_M \sum_j a_j h_j(\cdot) \mathbf{1} \\ &= \int_M \sum_j a_j h_j \sum_i h_i \\ &= \sum_{ij} A_{ij} a_j \\ &= \mathbf{1}^\top A \vec{a}\end{aligned}$$

Lumped Mass Matrix



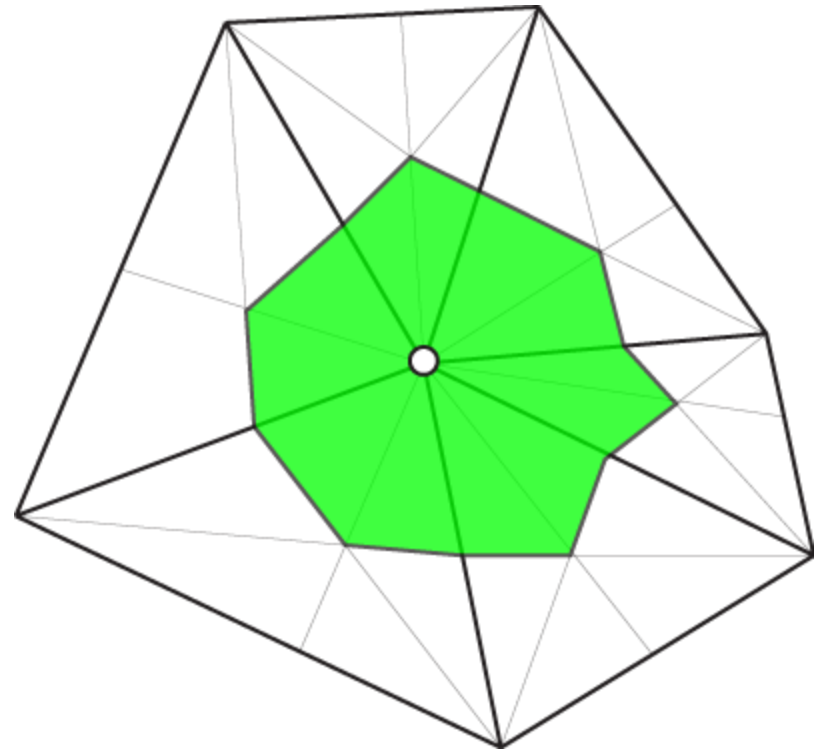
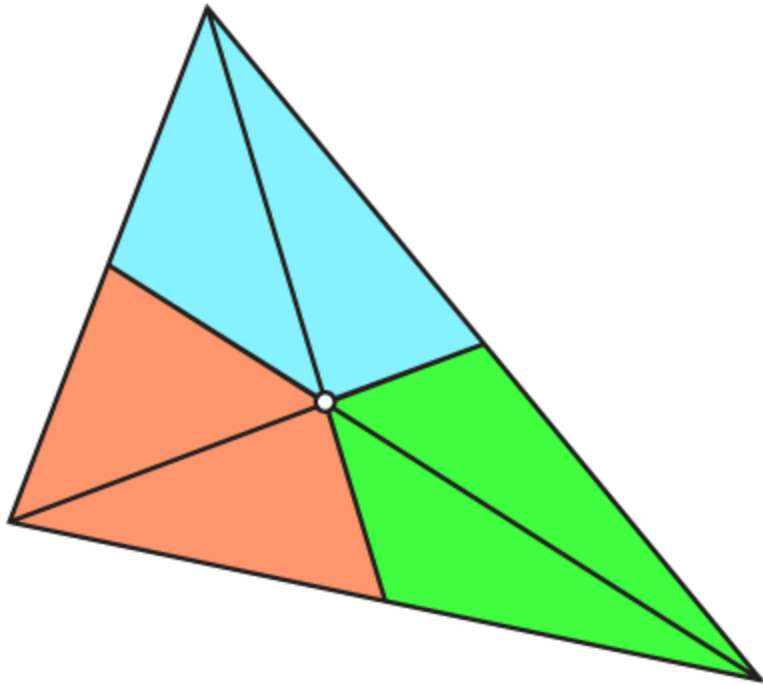
$$\tilde{a}_{ii} := \text{Area}(\text{cell } i)$$

Won't make big difference for smooth functions

<http://users.led-inc.eu/~phk/mesh-dualmesh.html>

Approximate with diagonal matrix

Simplest: Barycentric Lumped Mass



<http://www.alecjacobson.com/weblog/?p=1146>

Area/3 to each vertex

Ingredients

- **Cotangent Laplacian L**

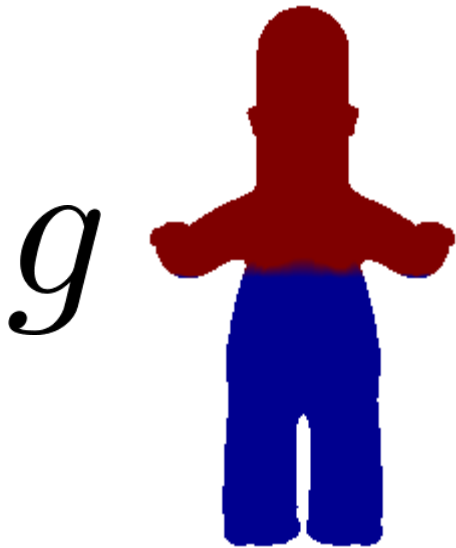
Per-vertex function to integral of its Laplacian against each hat

- **Area weights A**

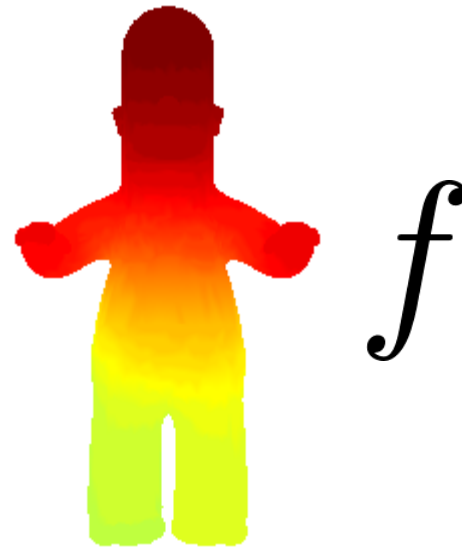
Integrals of pairwise products of hats (or approximation thereof)

Solving the Poisson Equation

$$\Delta f = g \rightarrow L \vec{f} = A \vec{g}$$



Must integrate
to zero



Determined up
to constant

Helmholtz Equation

$$\Delta f = \lambda f$$

$$\int \Delta f \cdot \psi = -\int \nabla f \cdot \nabla \psi = \int \lambda f \cdot \psi$$

$$\Rightarrow La = \lambda Ma$$

Generalized
Eigenvalue Problem

Important Detail: Boundary Conditions

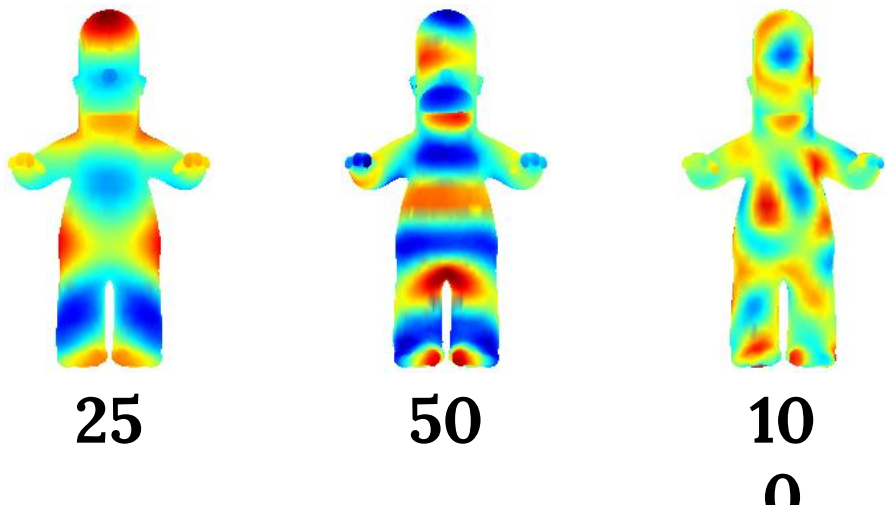
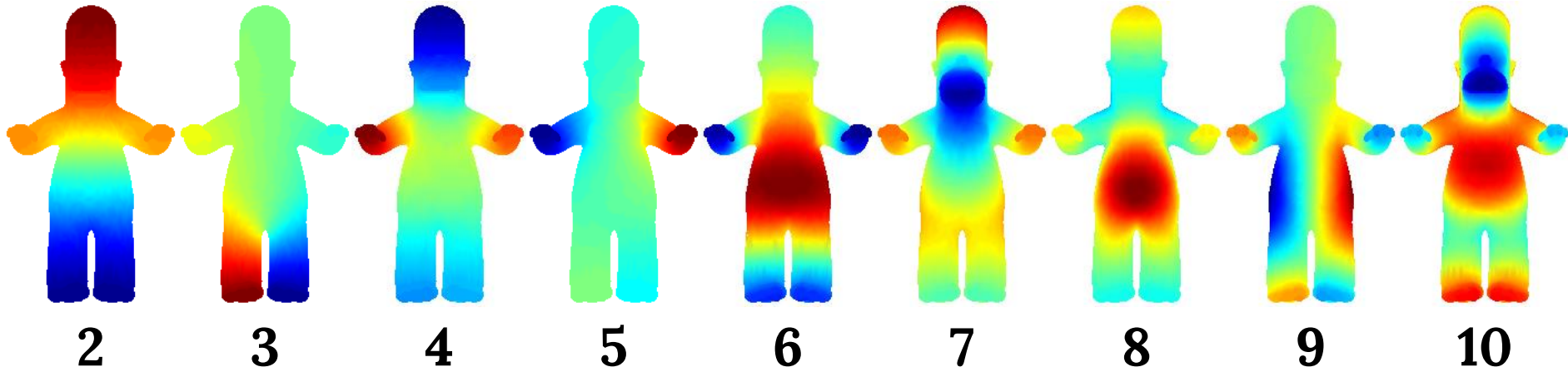
$$\begin{aligned}\Delta f(x) &= g(x) \quad \forall x \in \Omega \\ f(x) &= u(x) \quad \forall x \in \Gamma_D \\ \nabla f \cdot n &= v(x) \quad \forall x \in \Gamma_N\end{aligned}$$

Strong
form

$$\begin{aligned}\int_{\Omega} \nabla f \cdot \nabla \phi &= \int_{\Gamma_N} v(x) \phi(x) d\Gamma - \int_{\Omega} f(x) \phi(x) d\Omega \\ f(x) &= u(x) \quad \forall x \in \Gamma_D\end{aligned}$$

Weak form

Eigenhomers



What is
smallest
eigenvalue?

**On board:
FEM approach?**

Higher-Order Elements

https://www.femtable.org

finite element method

Periodic Table of the Finite Elements

The $\mathcal{P}_r^- \Lambda^k$ family

	$k=0$	$k=1$	$k=2$	$k=3$
$n=1$	P_1	dP_0		
	P_2	dP_1		
	P_3	dP_2		
$n=2$	P_1	RT_1	dP_0	
	P_2	RT_2	dP_1	
	P_3	RT_3	dP_2	
$n=3$	P_1	NI_1^s	NI_1^f	dP_0

The $\mathcal{P}_r \Lambda^k$ family

	$k=0$	$k=1$	$k=2$	$k=3$
$n=1$	P_1	dP_1		
	P_2	dP_2		
	P_3	dP_3		
$n=2$	P_1	BDM_1	dP_1	
	P_2	BDM_2	dP_2	
	P_3	BDM_3	dP_3	
$n=3$	P_1	$N2_1^s$	$N2_1^f$	dP_1

The $\mathcal{Q}_r^- \Lambda^k$ family

	$k=0$	$k=1$	$k=2$	$k=3$
$n=1$	Q_1	dQ_0		
	Q_2	dQ_1		
	Q_3	dQ_2		
$n=2$	Q_1	RT_{c1}	dQ_0	
	Q_2	RT_{c2}	dQ_1	
	Q_3	RT_{c3}	dQ_2	
$n=3$	Q_1	Nc_1^s	Nc_1^f	dQ_0

The $\mathcal{S}_r \Lambda^k$ family

	$k=0$	$k=1$	$k=2$	$k=3$
$n=1$	S_1	dP_{c1}		
	S_2	dP_{c2}		
	S_3	dP_{c3}		
$n=2$	S_1	BDM_{c1}	dP_{c1}	
	S_2	BDM_{c2}	dP_{c2}	
	S_3	BDM_{c3}	dP_{c3}	
$n=3$	S_1	AA_1^s	AA_1^f	dP_{c1}

Point Cloud Laplace: Easiest Option

$$W_{ij} = \exp\left(-\frac{\|x_i - x_j\|^2}{t}\right)$$

$$D_{ii} = \sum_j W_{ji}$$

$$L = D - W$$

$$Lf = \lambda Df$$

