

IFT 6112

09 – LAPLACIAN APPLICATIONS

<http://www-labs.iro.umontreal.ca/~bmpix/teaching/6112/2018/>



Mikhail Bessmeltsev

Review:

Rough Intuition

http://pngimg.com/upload/hammer_PNG3886.png



**You can learn a lot
about a shape by
hitting it (lightly)
with a hammer!**

Review:

Spectral Geometry

What can you learn about its shape
from

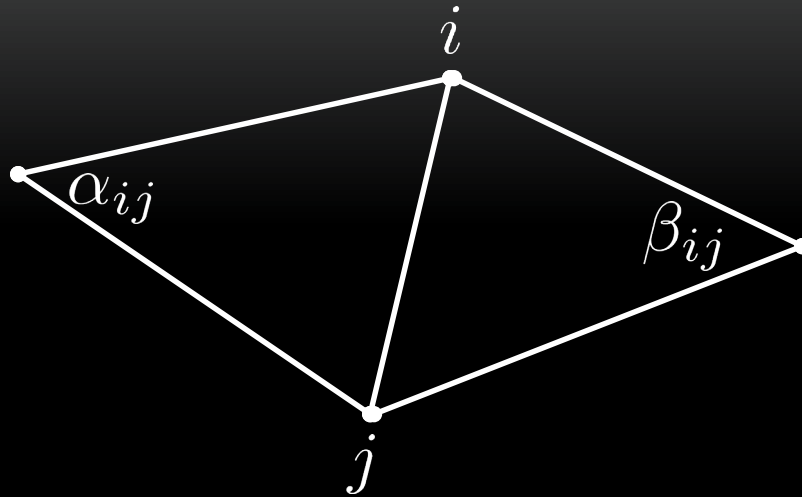
vibration frequencies and
oscillation patterns?

$$\Delta f = \lambda f$$

Review:

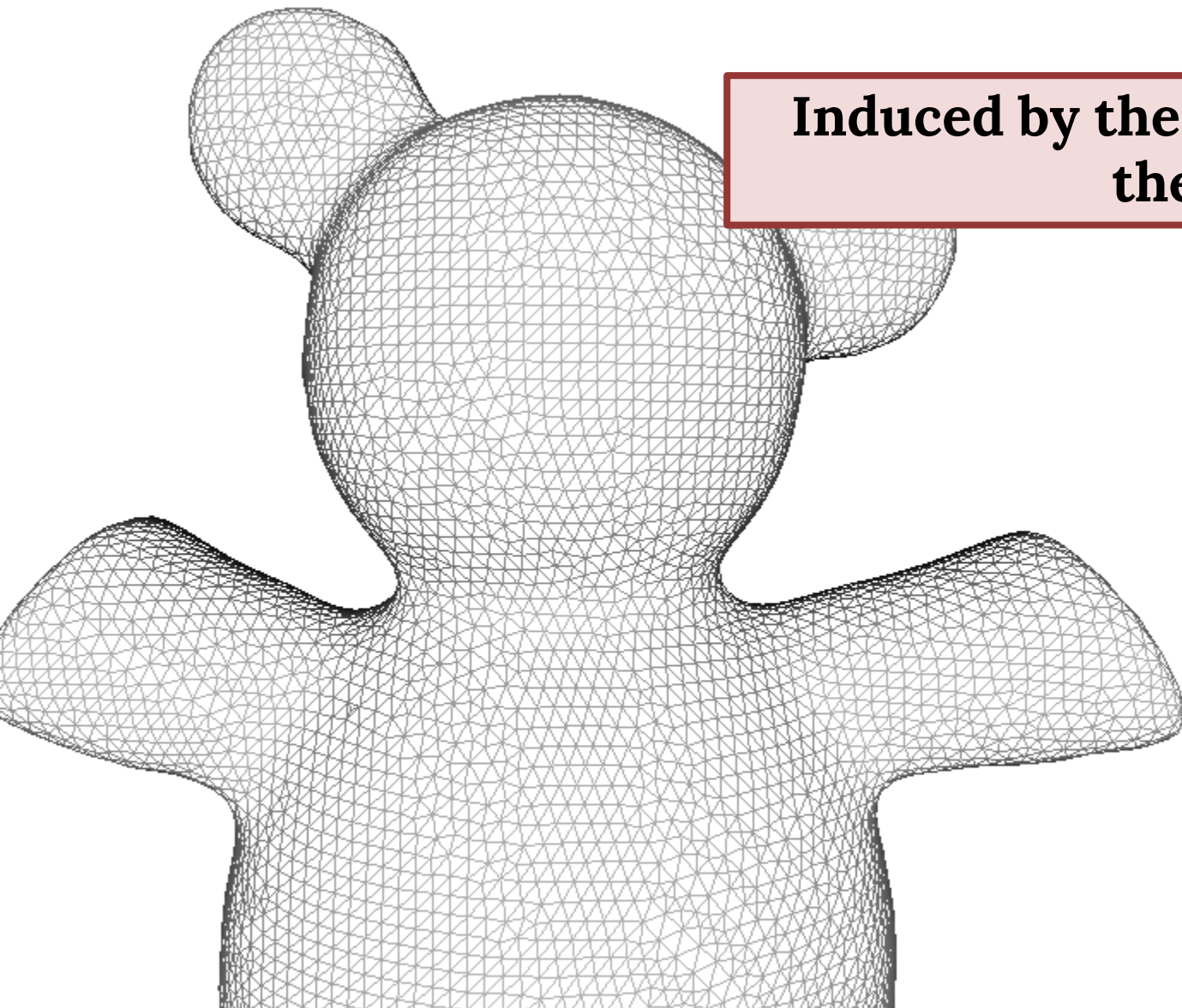
THE COTANGENT LAPLACIAN

$$L_{ij} = \begin{cases} \frac{1}{2} \sum_{i \sim k} (\cot \alpha_{ik} + \cot \beta_{ik}) & \text{if } i = j \\ -\frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}) & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$



Laplacian is sparse!

Induced by the **connectivity** of
the triangle mesh.



How can we use L ?

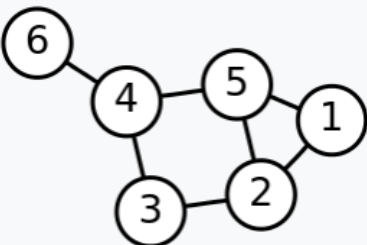
- (useful properties of the Laplacian)
- In Computer Graphics and
Geometry Modeling/Processing
- In Machine Learning

How can we use L ?

- (useful properties of the Laplacian)
- In Computer Graphics and
Geometry Modeling/Processing
- In Machine Learning

One Object, Many Interpretations

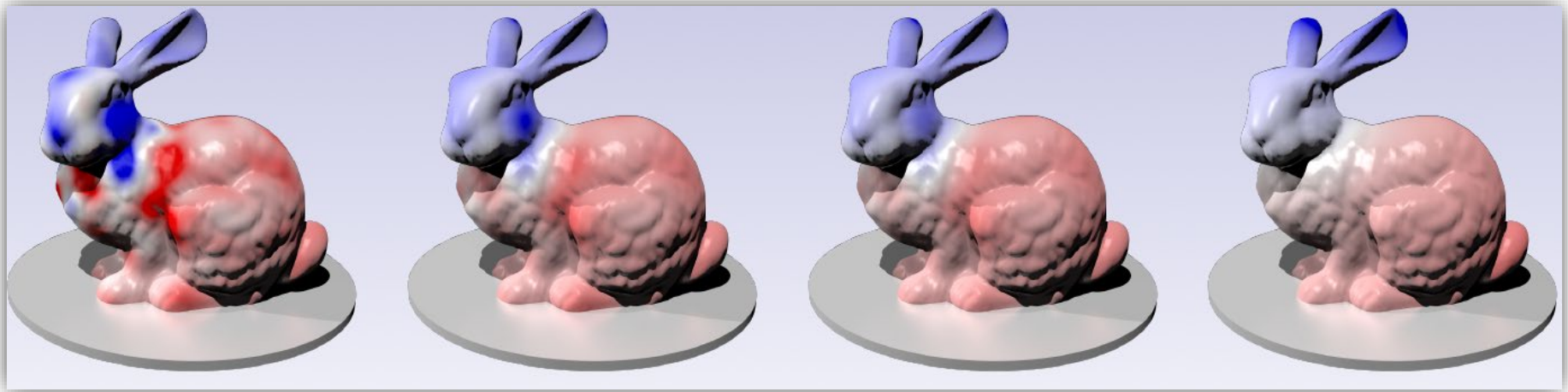
$$L_{vw} = A - D = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

Labeled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$

https://en.wikipedia.org/wiki/Laplacian_matrix

Deviation from neighbors

One Object, Many Interpretations



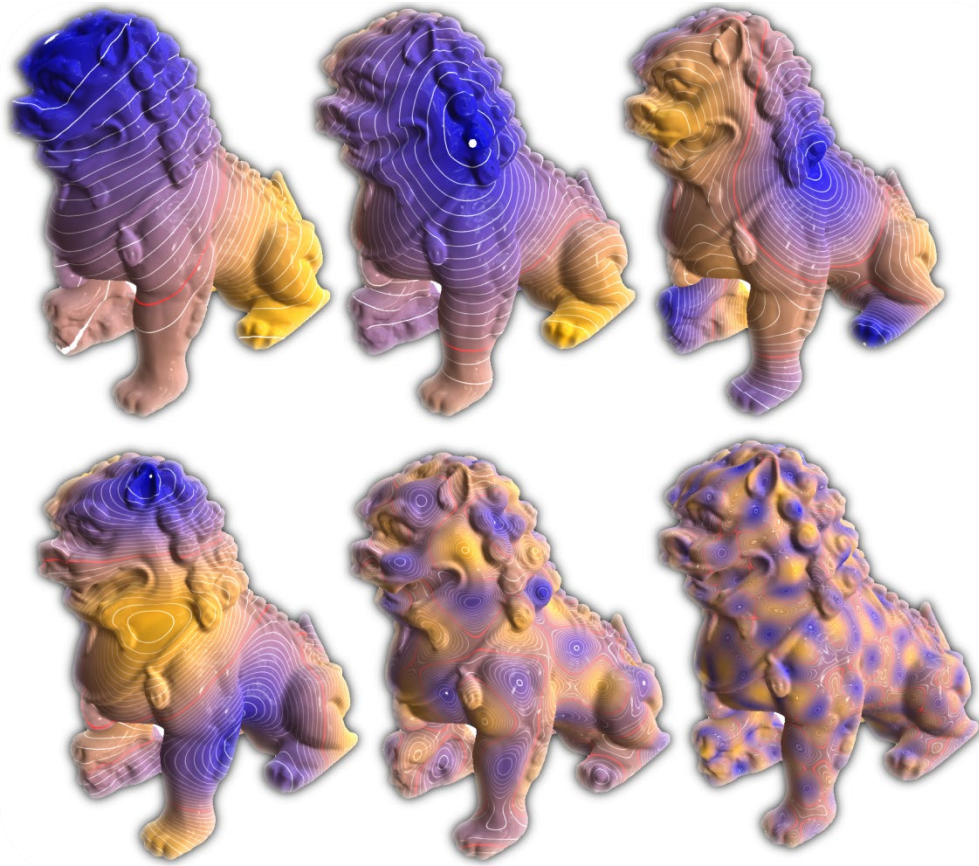
Decreasing E

$$E[f] := \int_S \|\nabla f\|_2^2 dA = - \int_S f(x) \Delta f(x) dA(x)$$

Images made by E. Vouga

Dirichlet energy: Measures smoothness

One Object, Many Interpretations



$$\Delta\psi_i = \lambda_i\psi_i$$

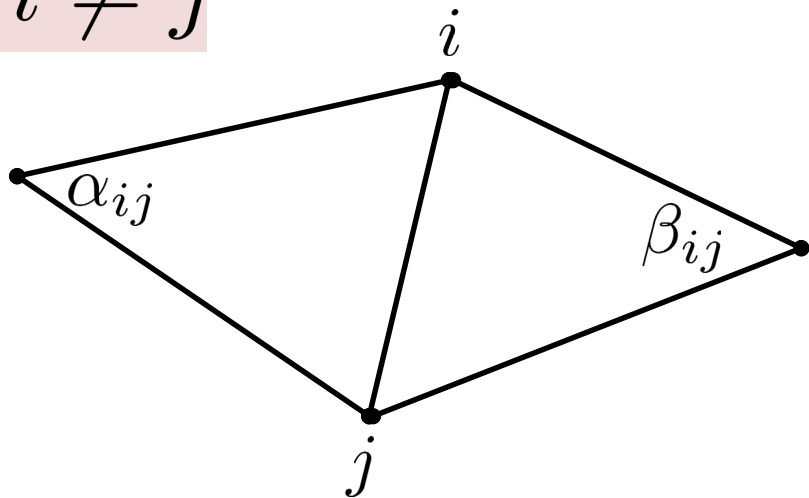
**Vibration modes of
surface (not volume!)**

Key Observation (in discrete case)

$$L_{ij} = \begin{cases} \frac{1}{2} \sum_{i \sim k} (\cot \alpha_{ik} + \cot \beta_{ik}) & \text{if } i = j \\ -\frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}) & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

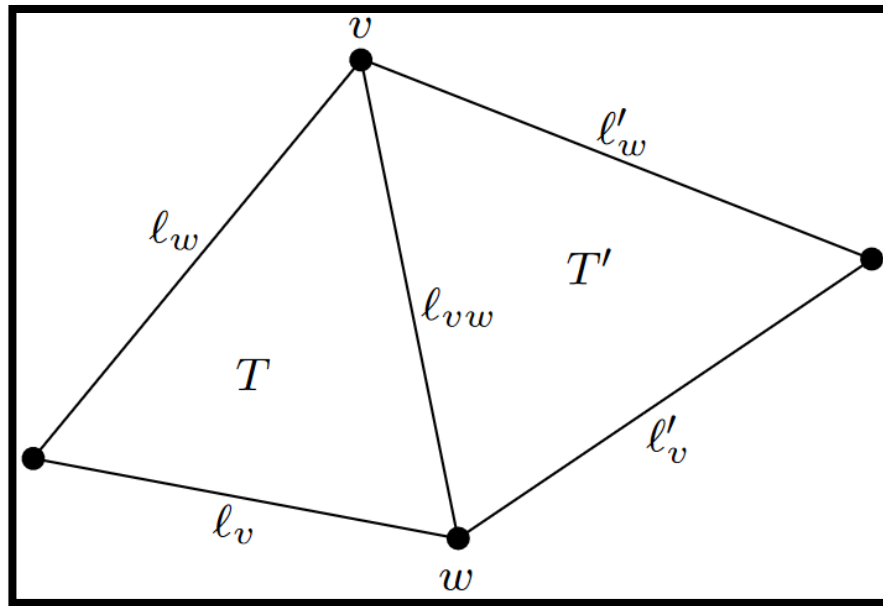
$$M_{ij} = \begin{cases} \frac{\text{one-ring area}}{6} & \text{if } i = j \\ \frac{\text{adjacent area}}{12} & \text{if } i \neq j \end{cases}$$

Can be written in terms of angles and areas!



After (More) Trigonometry

$$L_{vw} = \frac{1}{8} \begin{cases} -\sum_{u \sim v} L_{uv} & \text{when } v = w \\ \mu(T)^{-1}(\ell_{vw}^2 - \ell_v^2 - \ell_w^2) & \text{when } v \sim w \\ +\mu(T')^{-1}(\ell_{vw}^2 - \ell'_v{}^2 - \ell'_w{}^2) & \\ 0 & \text{otherwise} \end{cases}$$



Image/formula in “Functional Characterization of Intrinsic and Extrinsic Geometry,” TOG 2017 (Corman et al.)

Laplacian only depends on edge lengths

Isometry

Bending without stretching.

Lots of Interpretations

Global isometry

$$d_1(x, y) = d_2(f(x), f(y))$$

Local isometry

$$g_1 = f^* g_2$$
$$g_1(v, w) = g_2(f_* v, f_* w)$$

Intrinsic Techniques



<http://www.revedreams.com/crochet/yarncrochet/nonorientable-crochet/>

Isometry invariant

Isometry Invariance: Hope



Isometry Invariance: Reality

“Rigidity”



<http://www.4tnz.com/content/got-toilet-paper>

Few shapes *can* deform isometrically

Isometry Invariance: Reality

“Rigidity”



\approx isometries?

<http://www.4tnz.com/content/got-toilet-paper>

Few shapes can deform isometrically

Useful Fact

Graphical Models 74 (2012) 121–129



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Graphical Models

journal homepage: www.elsevier.com/locate/gmod



Discrete heat kernel determines discrete Riemannian metric

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ABSTRACT

The Laplace–Beltrami operator of a smooth Riemannian manifold is determined by the Riemannian metric. Conversely, the heat kernel constructed from the eigenvalues and eigenfunctions of the Laplace–Beltrami operator determines the Riemannian metric. This work proves the analogy on Euclidean polyhedral surfaces (triangle meshes), that the discrete heat kernel and the discrete Riemannian metric (unique up to a scaling) are mutually determined by each other. Given a Euclidean polyhedral surface, its Riemannian metric is represented as edge lengths, satisfying triangle inequalities on all faces. The Laplace–Beltrami operator is formulated using the cotangent formula, where the edge weight is defined as the sum of the cotangent of angles against the edge. We prove that the edge

Beware

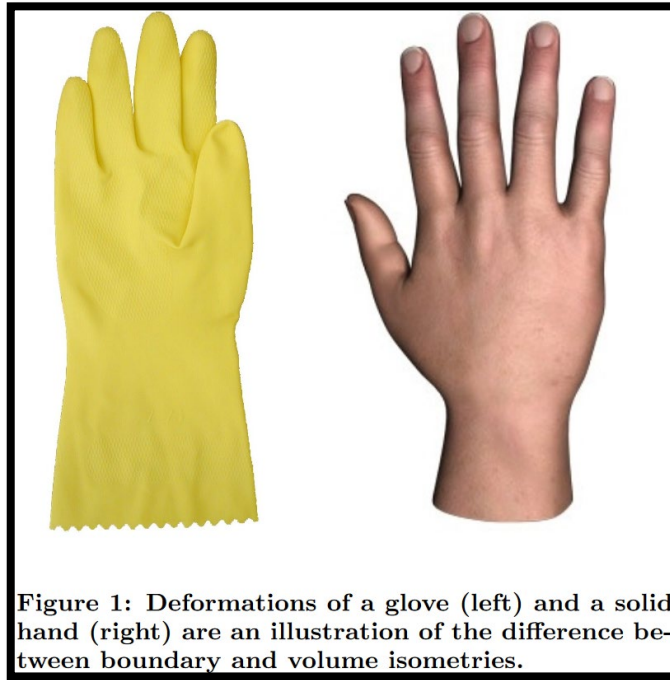


Figure 1: Deformations of a glove (left) and a solid hand (right) are an illustration of the difference between boundary and volume isometries.

But
calculations on
a volume are
expensive!

Image from: Raviv et al. "Volumetric Heat Kernel Signatures." 3DOR 2010.

Not the same.

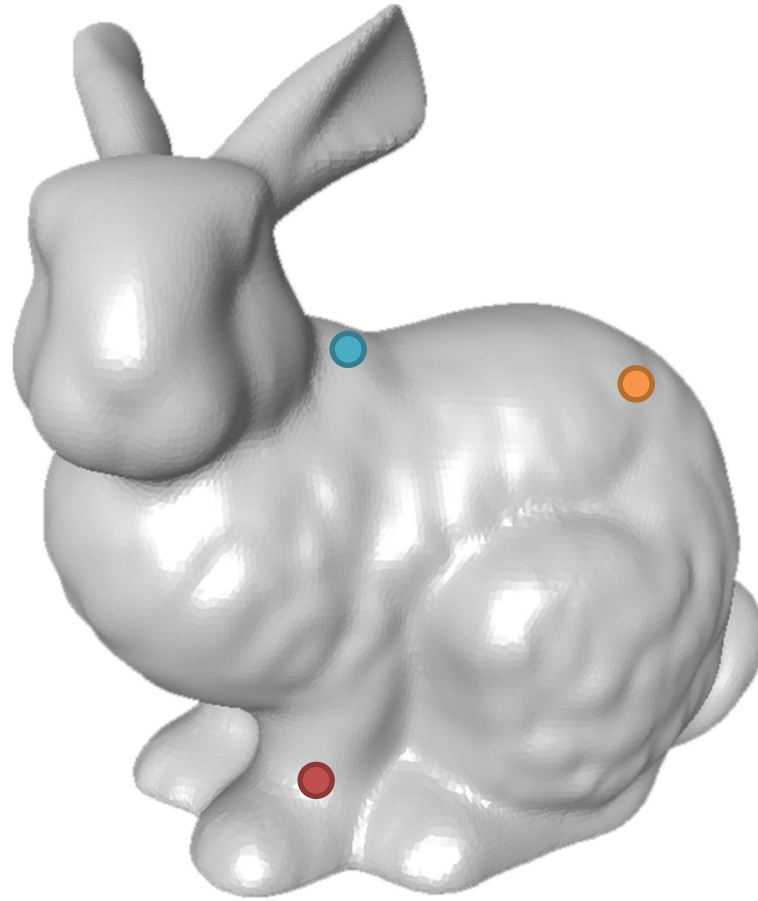
Why Study the Laplacian?

- **Encodes intrinsic geometry**
Edge lengths on triangle mesh, Riemannian metric on manifold
- **Multi-scale**
Filter based on frequency
- **Geometry through linear algebra**
Linear/eigenvalue problems, sparse positive definite matrices
- **Connection to physics**
Heat equation, wave equation, vibration, ...

How can we use L ?

- (useful properties of the Laplacian)
- In Computer Graphics and
Geometry Modeling/Processing
- In Machine Learning

Example Task: Shape Descriptors



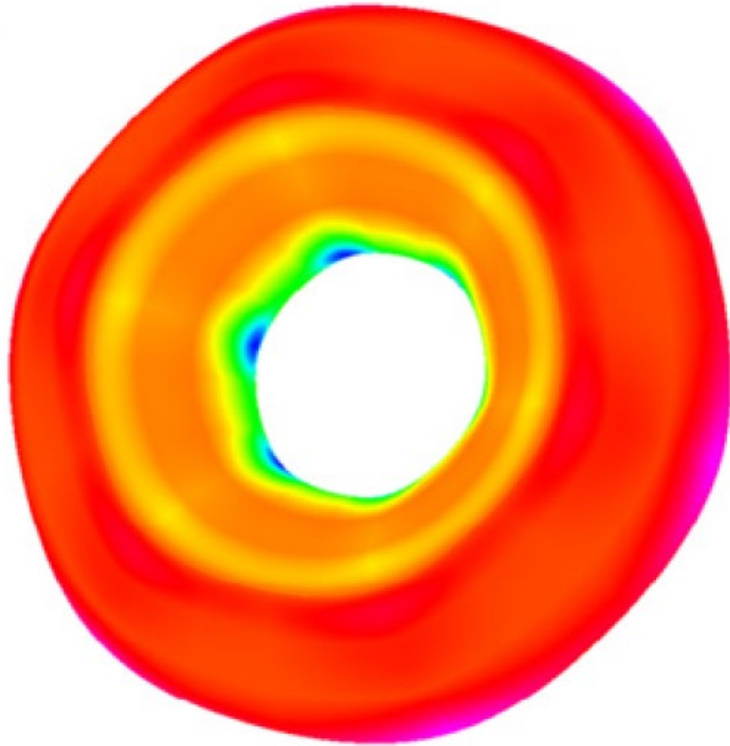
http://liris.cnrs.fr/meshbenchmark/images/fig_attacks.jpg

Pointwise quantity

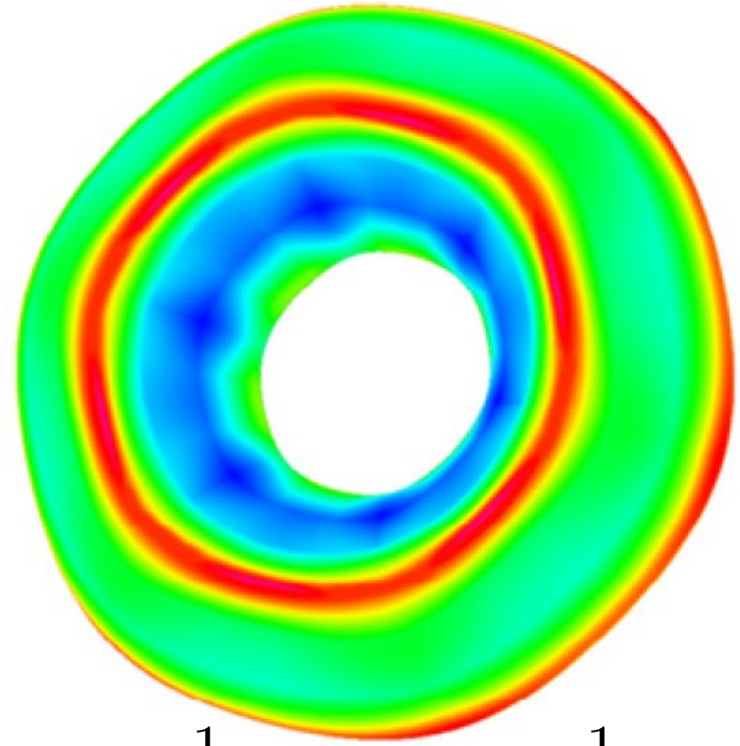
Descriptor Tasks

- **Characterize local geometry**
Feature/anomaly detection
- **Describe point's role on surface**
Symmetry detection, correspondence

Descriptors We've Seen Before



$$K := \kappa_1 \kappa_2 = \det \mathbb{II}$$



$$H := \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2} \operatorname{tr} \mathbb{II}$$

<http://www.sciencedirect.com/science/article/pii/S0010448510001983>

Gaussian and mean curvature

Desirable Properties

- **Distinguishing**

Provides useful information about a point

- **Stable**

Numerically and geometrically

- **Intrinsic**

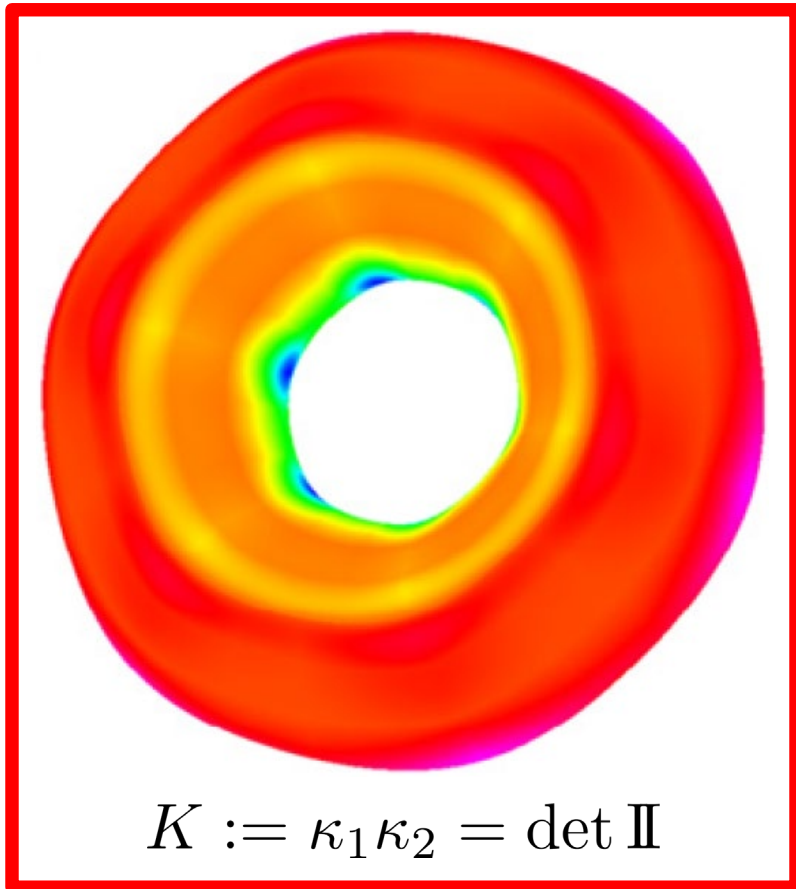
No dependence on embedding *Sometimes undesirable!*

Intrinsic Descriptors

Invariant under

- **Rigid motion**
- **Bending without stretching**

Intrinsic Descriptor



Theorema Egregium
 (“Totally Awesome
Theorem”):
Gaussian curvature is
intrinsic.

End of the Story?

Noisy!



$$K = \kappa_1 \kappa_2$$

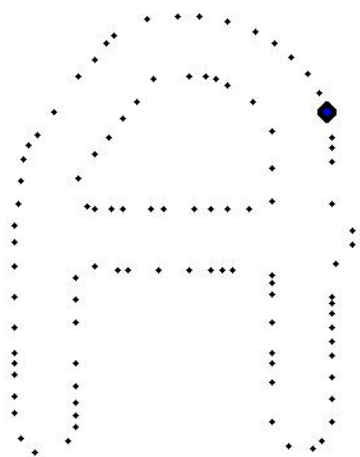
Second derivative quantity

Desirable Properties

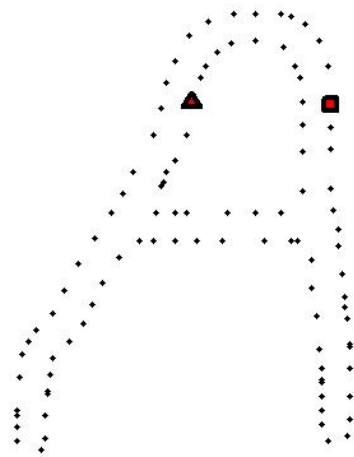
Incorporates neighborhood information in an intrinsic fashion

Stable under small deformation

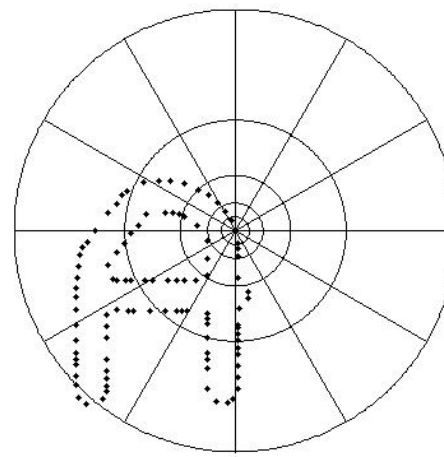
Shape Context



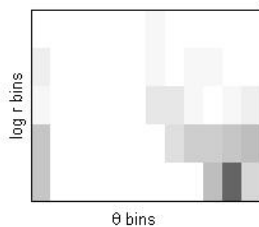
(a)



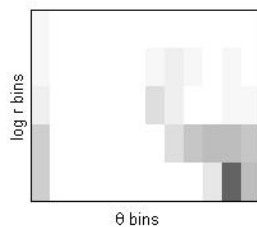
(b)



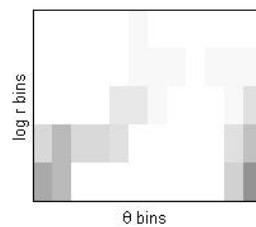
(c)



(d)



(e)



(f)

Shape Context

- + Translational invariance
- + Scale invariance
- Rotational invariance

Shape Context

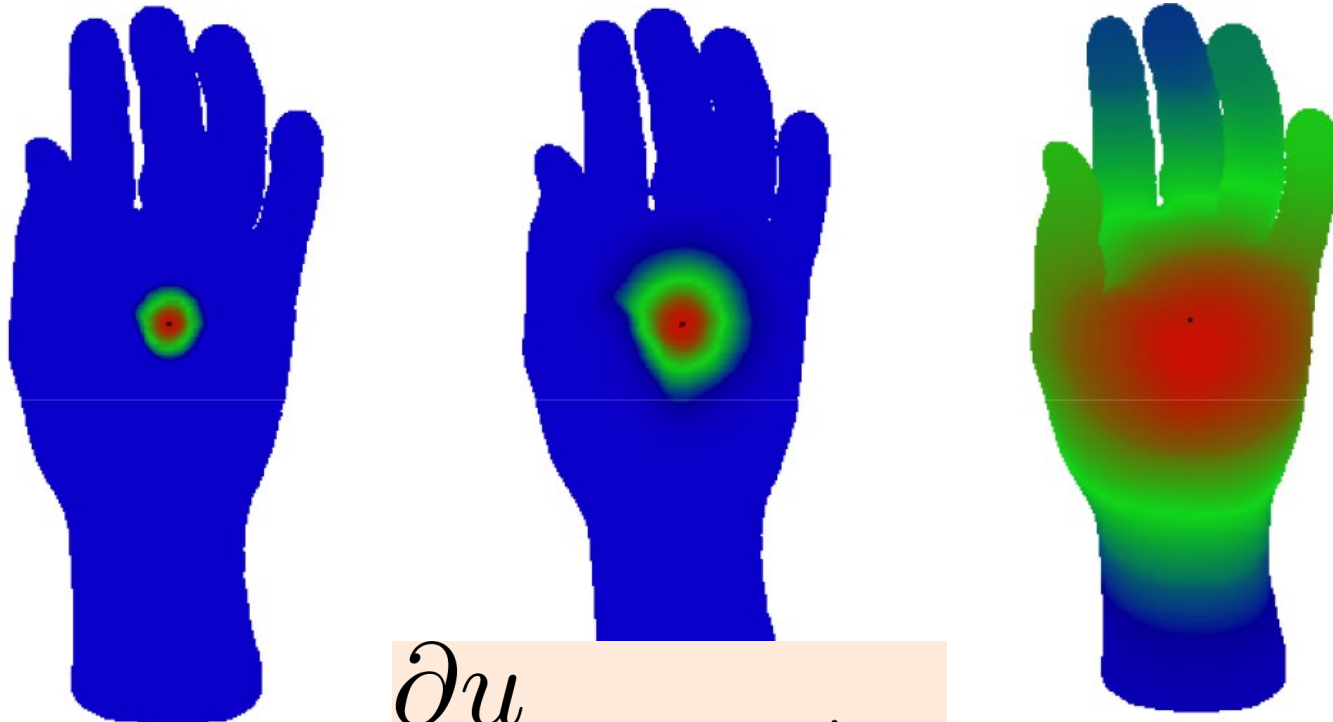
Idea!

Compute angles relative to the tangent

- + Translational invariance
- + Scale invariance
- + Rotational invariance

Recall:

Connection to Physics



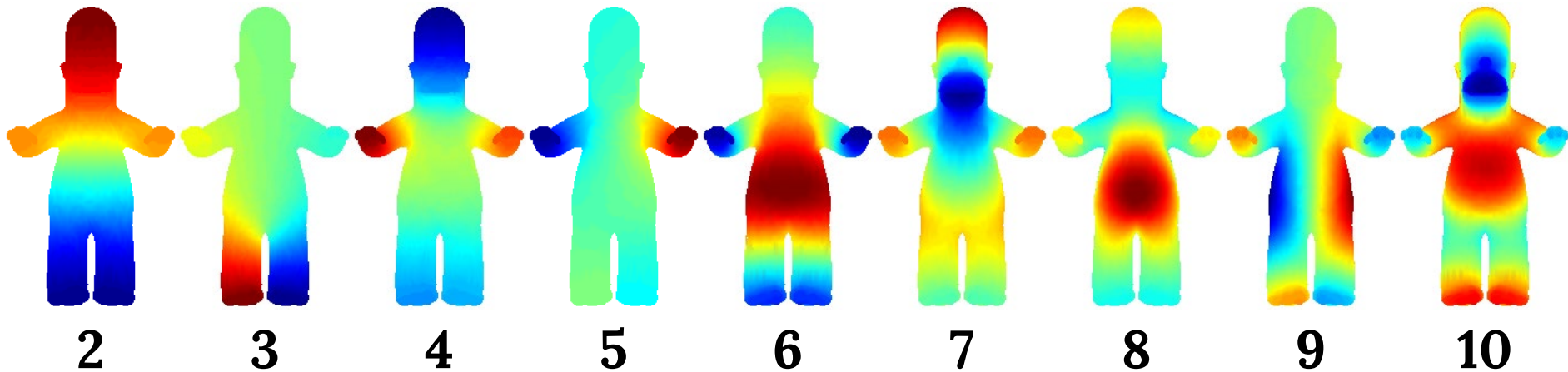
$$\frac{\partial u}{\partial t} = -\Delta u$$

Heat equation

Intrinsic Observation

Heat diffusion patterns are not affected if you bend a surface.

Global Point Signature

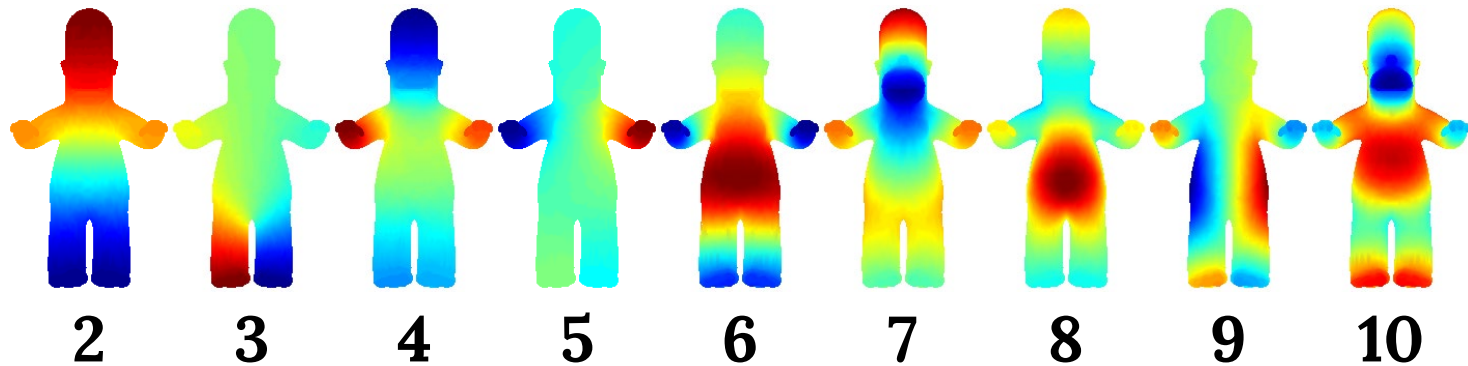


$$\text{GPS}(p) := \left(-\frac{1}{\sqrt{\lambda_1}} \phi_1(p), -\frac{1}{\sqrt{\lambda_2}} \phi_2(p), -\frac{1}{\sqrt{\lambda_3}} \phi_3(p), \dots \right)$$

“Laplace-Beltrami Eigenfunctions for Deformation Invariant Shape Representation”

Rustamov, SGP 2007

Global Point Signature

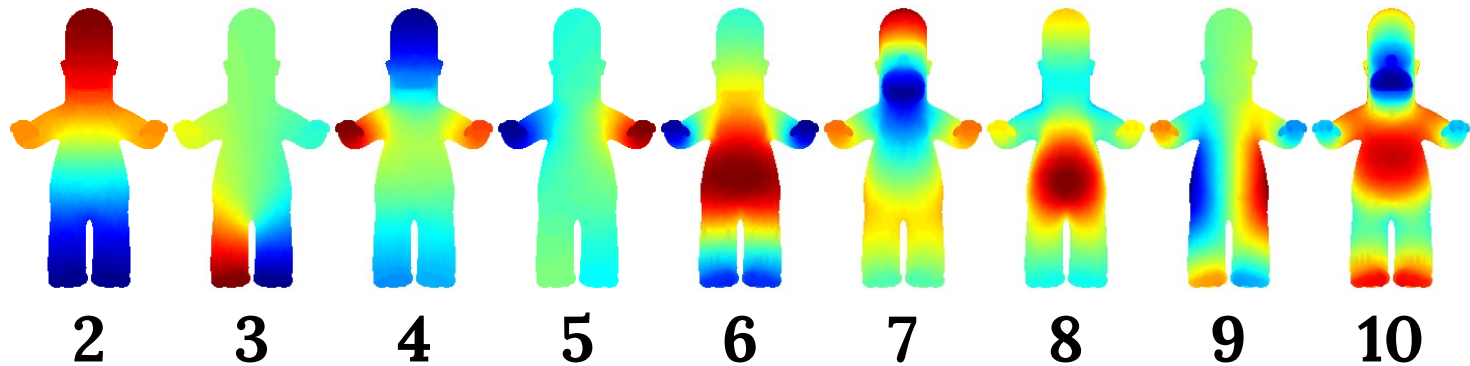


$$\text{GPS}(p) := \left(-\frac{1}{\sqrt{\lambda_1}} \phi_1(p), -\frac{1}{\sqrt{\lambda_2}} \phi_2(p), -\frac{1}{\sqrt{\lambda_3}} \phi_3(p), \dots \right)$$

If surface does not **self-intersect,
neither does the GPS embedding.**

Proof: Laplacian eigenfunctions span $L^2(\Sigma)$; if $\text{GPS}(p) = \text{GPS}(q)$, then all functions on Σ would be equal at p and q .

Global Point Signature



$$\text{GPS}(p) := \left(-\frac{1}{\sqrt{\lambda_1}} \phi_1(p), -\frac{1}{\sqrt{\lambda_2}} \phi_2(p), -\frac{1}{\sqrt{\lambda_3}} \phi_3(p), \dots \right)$$

GPS is isometry-invariant.

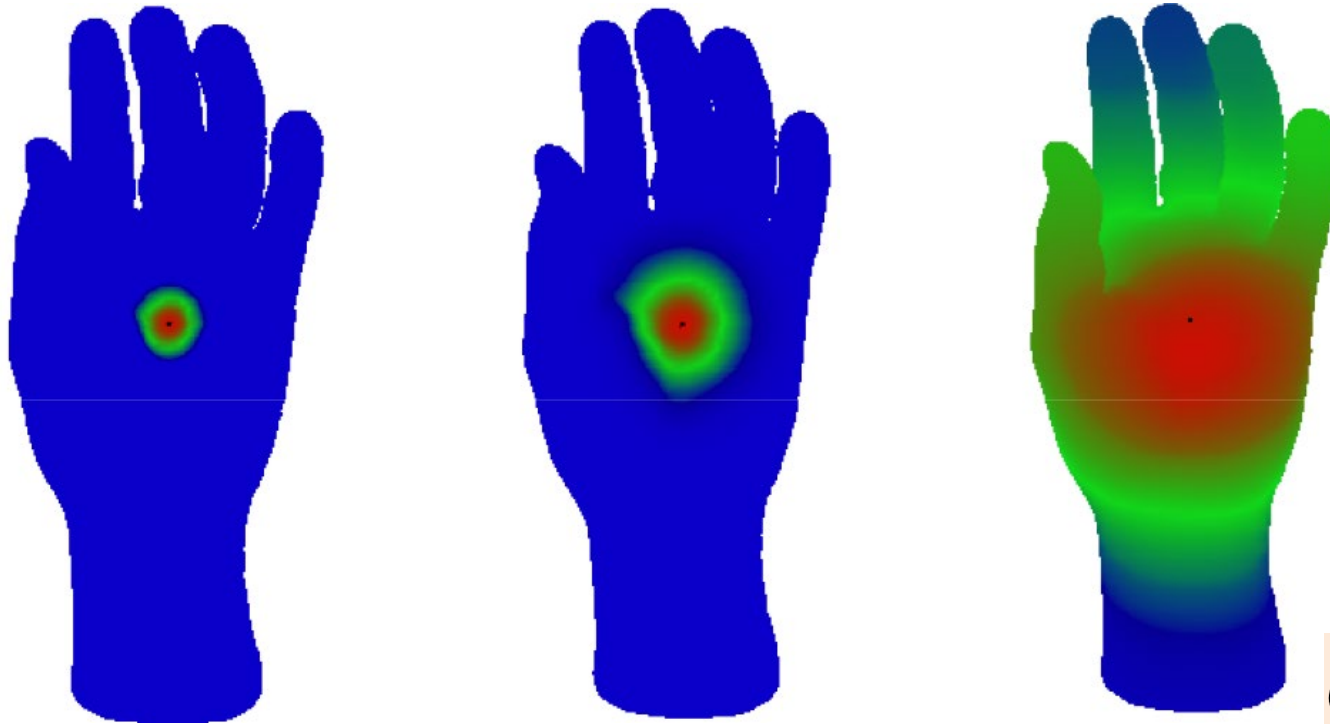
Proof: Comes from the Laplacian.

Drawbacks of GPS

- Assumes **unique** λ 's
- Potential for eigenfunction
“**switching**”
- **Nonlocal** feature

New idea:

PDE Applications of the Laplacian



$$\frac{\partial u}{\partial t} = -\Delta u$$

http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

PDE Applications of the Laplacian



$$\frac{\partial^2 u}{\partial t^2} = -i\Delta u$$

Image courtesy G. Peyré

Wave equation

PDE Applications of the Laplacian



Use this behavior to characterize shape.

$$\frac{\partial^2}{\partial t^2} = -i\Delta u$$

Image courtesy G. Peyré

Wave equation

Solutions in the LB Basis

$$\frac{\partial u}{\partial t} = -\Delta u$$

Heat equation

$$u = \sum_{n=0}^{\infty} a_n e^{-\lambda_n t} \phi_n(x)$$

$$a_n = \int_{\Sigma} u_0(x) \cdot \phi_n(x) dA$$

Heat Kernel Signature (HKS)

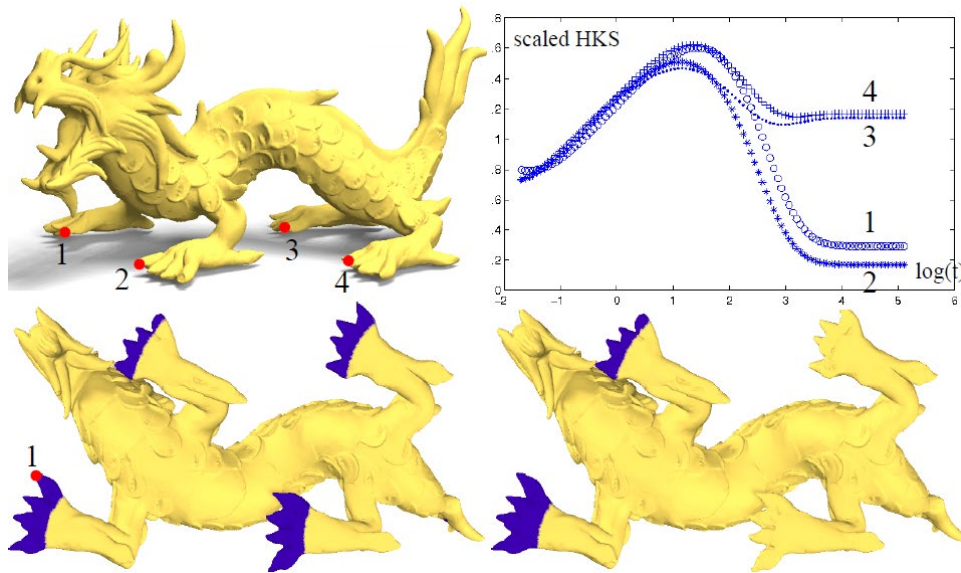
$$k_t(x, x) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \phi_n(x)^2$$

Continuous function of $t \in [0, \infty)$

**How much heat
diffuses from x to
itself in time t ?**

Heat Kernel Signature (HKS)

$$k_t(x, x) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \phi_n(x)^2$$



**“A concise and provably informative multi-scale signature based on heat diffusion”
Sun, Ovsjanikov, and Guibas; SGP 2009**

Heat Kernel Signature (HKS)

$$k_t(x, x) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \phi_n(x)^2$$

Good properties:

- Isometry-invariant
- Multiscale
- Not subject to switching
- Easy to compute
- Related to curvature at small scales

Heat Kernel Signature (HKS)

$$k_t(x, x) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \phi_n(x)^2$$

Bad properties:

- Issues remain with repeated eigenvalues
- Theoretical guarantees require (near-)isometry

Wave Kernel Signature (WKS)

$$\text{WKS}(E, x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\psi_E(x, t)|^2 dt = \sum_{n=0}^{\infty} \phi_n(x)^2 f_E(\lambda_n)^2$$

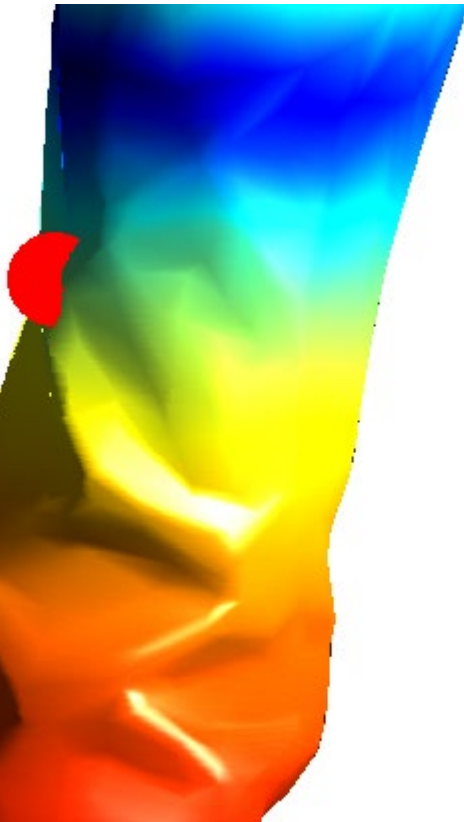
Average probability
over time that particle
is at x .

Initial energy
distribution

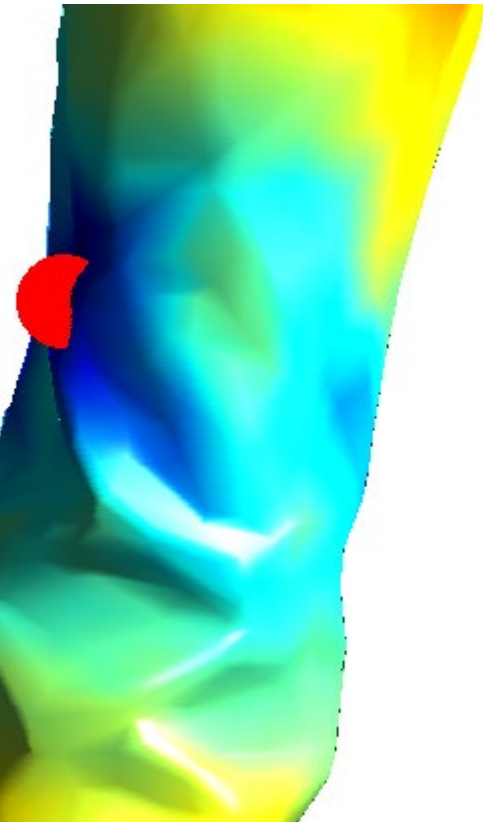
**“The Wave Kernel Signature: A Quantum Mechanical Approach to Shape Analysis”
Aubry, Schlickewei, and Cremers; ICCV Workshops 2012**

Wave Kernel Signature (WKS)

$$\text{WKS}(E, x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\psi_E(x, t)|^2 dt = \sum_{n=0}^{\infty} \phi_n(x)^2 f_E(\lambda_n)^2$$



HKS



WKS

Wave Kernel Signature (WKS)

$$\text{WKS}(E, x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\psi_E(x, t)|^2 dt = \sum_{n=0}^{\infty} \phi_n(x)^2 f_E(\lambda_n)^2$$

Good properties:

- [Similar to HKS]
- Localized in frequency
- Stable under some non-isometric deformation
- Some multi-scale properties

Wave Kernel Signature (WKS)

$$\text{WKS}(E, x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\psi_E(x, t)|^2 dt = \sum_{n=0}^{\infty} \phi_n(x)^2 f_E(\lambda_n)^2$$

Bad properties:

- [Similar to HKS]
- Can filter out *large-scale* features

Many Others

**Lots of spectral descriptors
in terms of Laplacian
eigenstructure.**

Combination with Machine Learning

$$p(x) = \sum_k f(\lambda_k) \phi_k^2(x)$$

Learn f rather than defining it

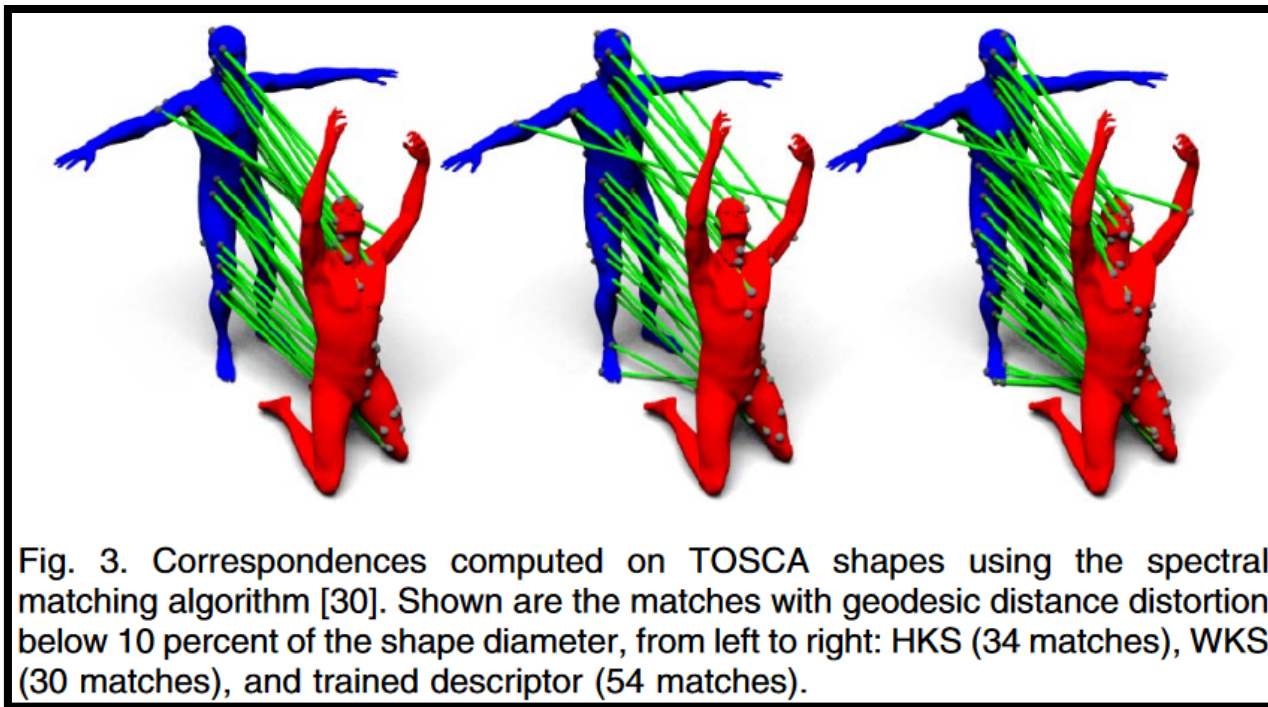
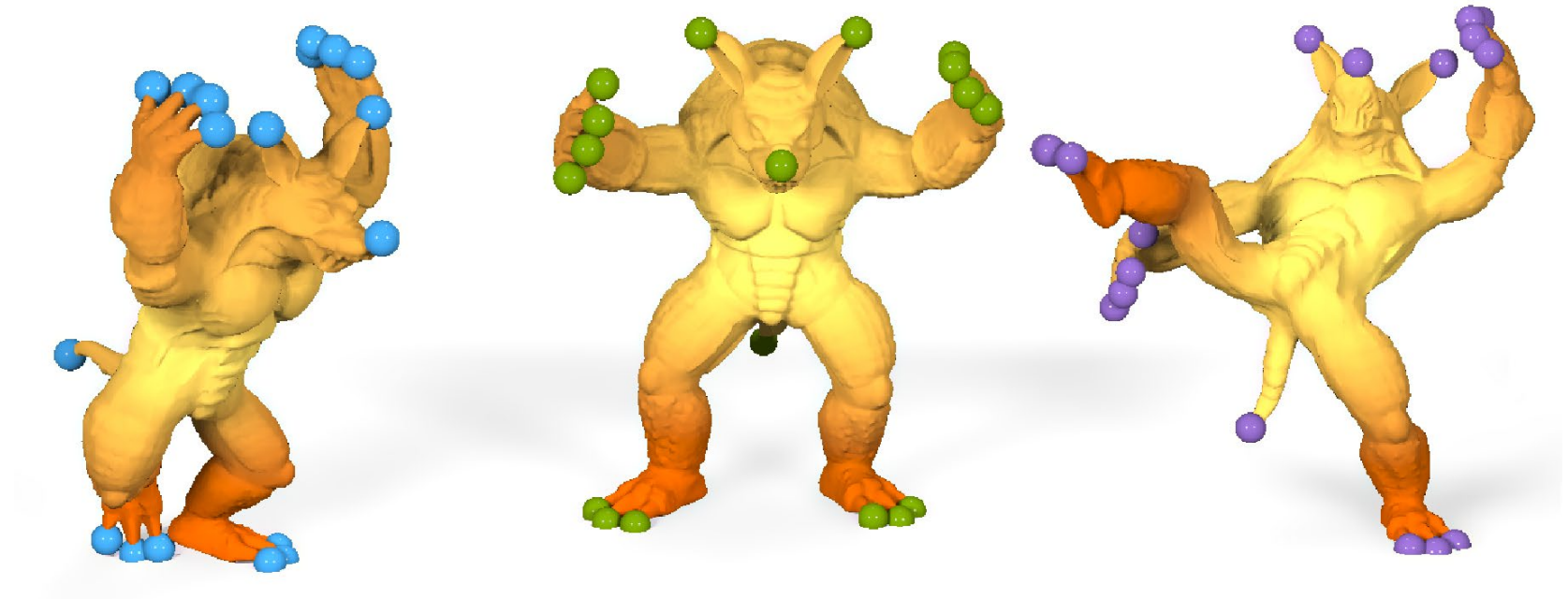


Fig. 3. Correspondences computed on TOSCA shapes using the spectral matching algorithm [30]. Shown are the matches with geodesic distance distortion below 10 percent of the shape diameter, from left to right: HKS (34 matches), WKS (30 matches), and trained descriptor (54 matches).

Application: Feature Extraction

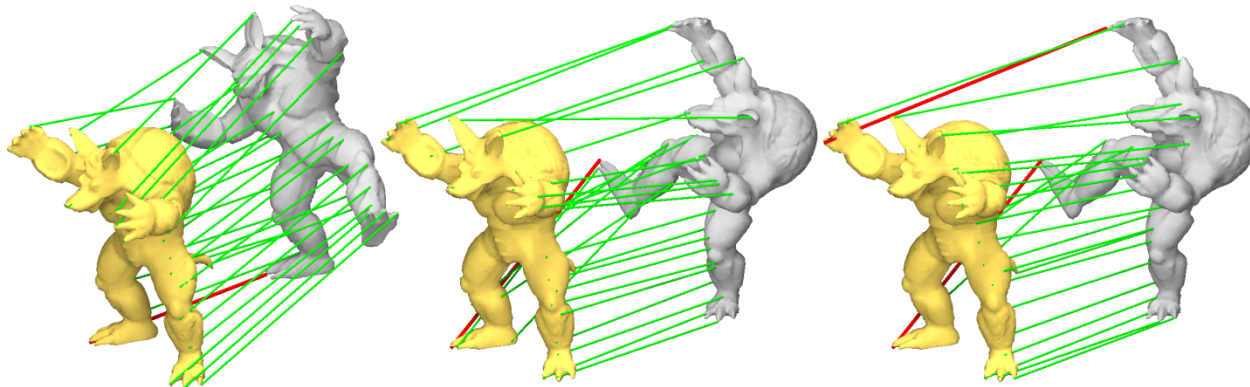
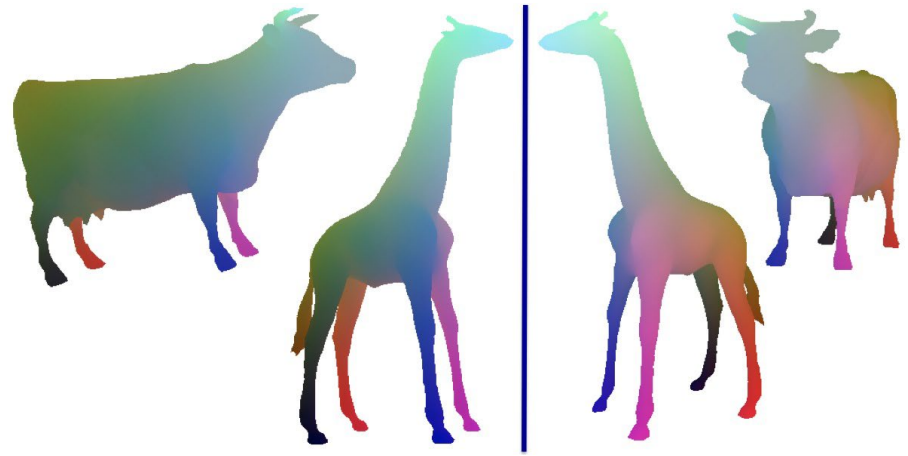
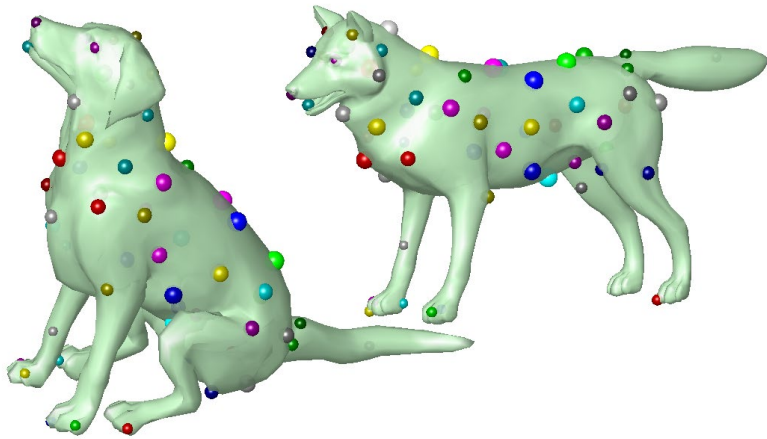


Maxima of $k_t(x, x)$ over x for large t .

A Concise and Provably Informative Multi-Scale Signature Based on Heat Diffusion
Sun, Ovsjanikov, and Guibas; SGP 2009

Feature points

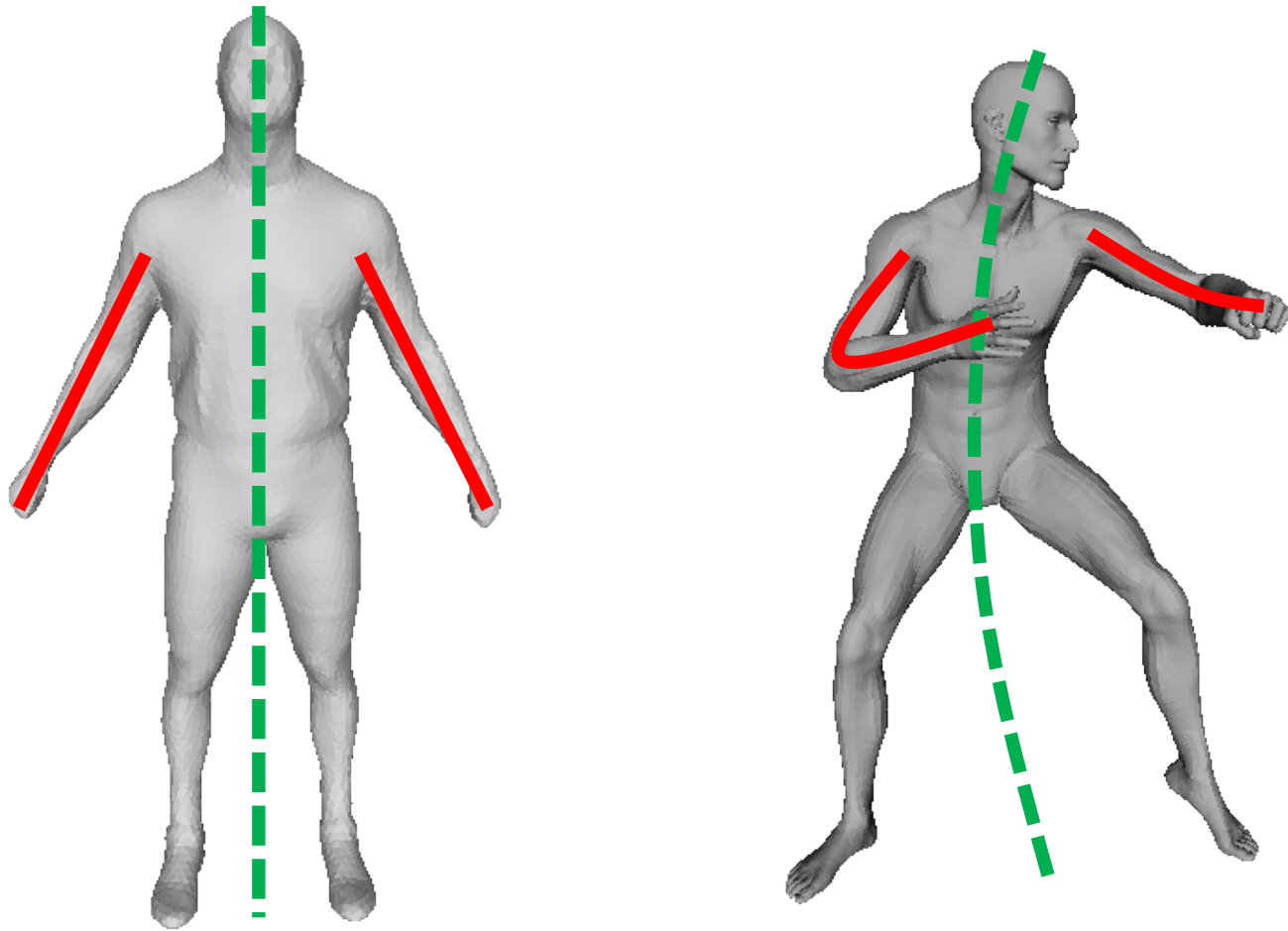
Preview: Correspondence



Descriptor Matching

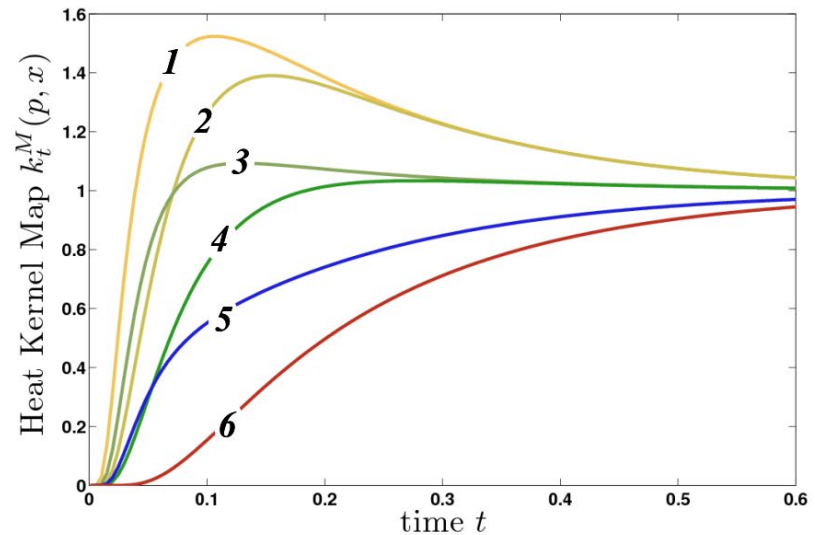
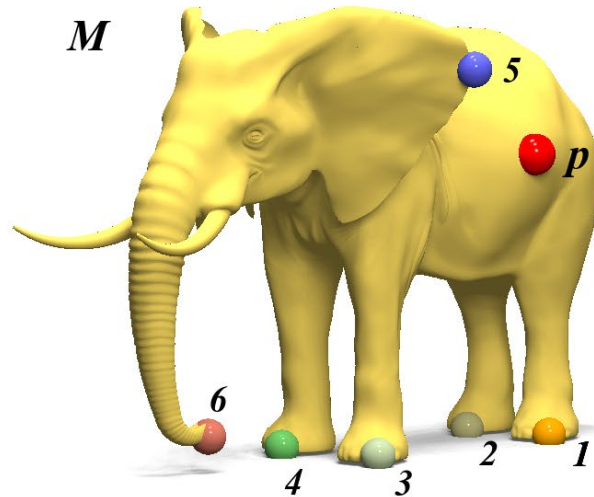
Simply match **closest points
in descriptor space.**

Descriptor Matching Problem



Symmetry

Heat Kernel Map



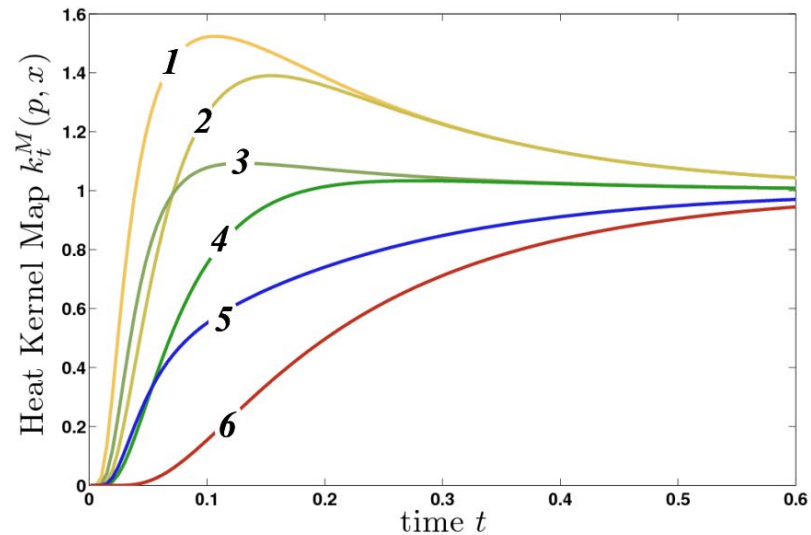
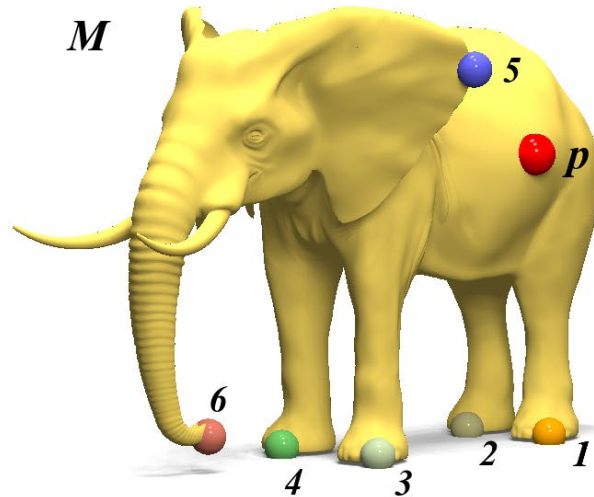
$$\text{HKM}_p(x, t) := k_t(p, x)$$

How much heat diffuses from p to x in time t ?

One Point Isometric Matching with the Heat Kernel

Ovsjanikov et al. 2010

Heat Kernel Map



$$\text{HKM}_p(x, t) := k_t(p, x)$$

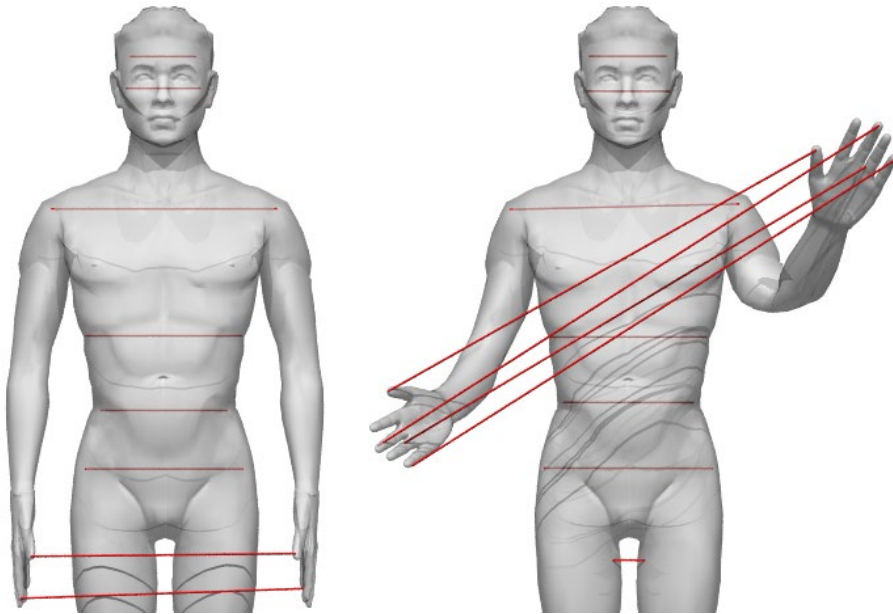
Theorem: Only have to match **one** point!

One Point Isometric Matching with the Heat Kernel

Ovsjanikov et al. 2010

KNN

Self-Map: Symmetry



**Intrinsic
symmetries become
extrinsic in GPS
space!**

Global Intrinsic Symmetries of Shapes
Ovsjanikov, Sun, and Guibas 2008

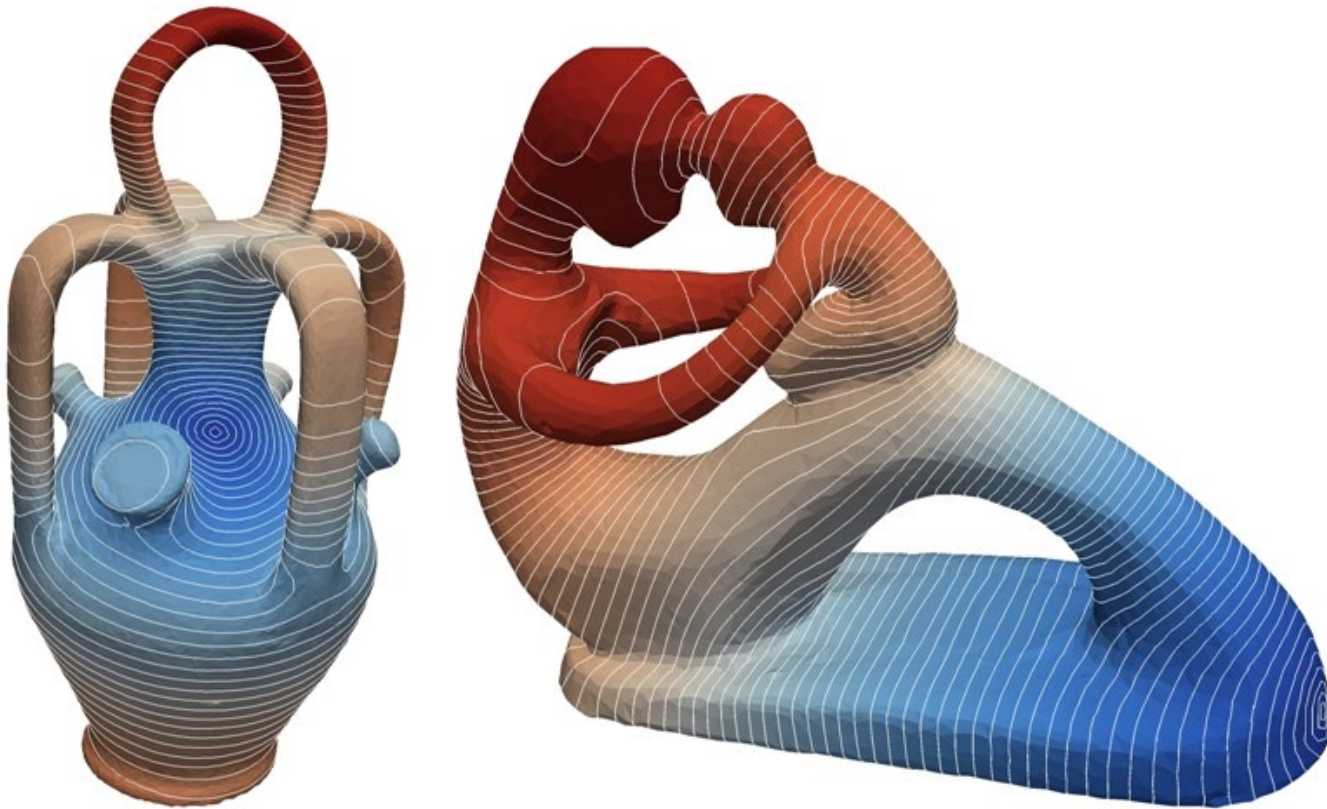
“Discrete intrinsic” symmetries

All Over the Place

Laplacians appear
everywhere in shape
analysis and geometry
processing.

Biharmonic Distances

$$d_b(p, q) := \|g_p - g_q\|_2, \text{ where } \Delta g_p = \delta_p$$

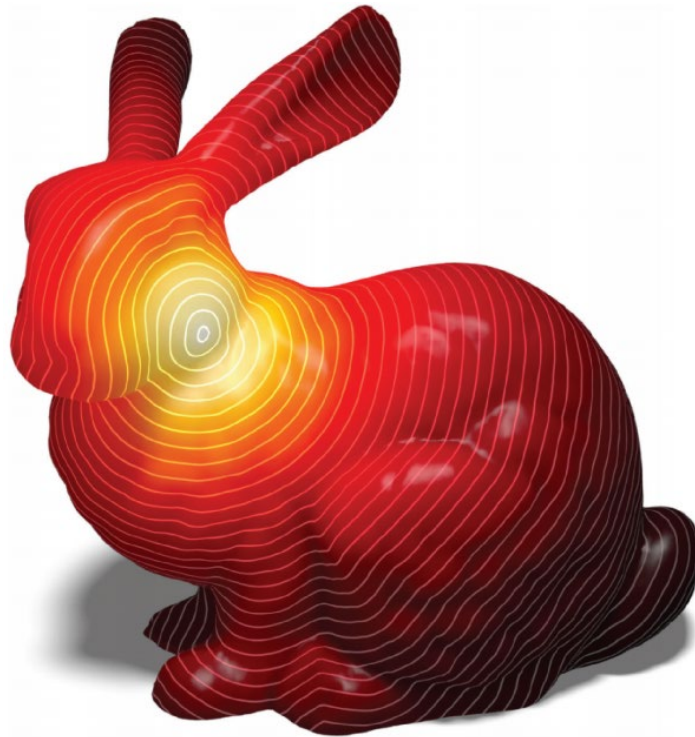


“Biharmonic distance”
Lipman, Rustamov & Funkhouser, 2010

Geodesic Distances

$$d_g(p, q) = \lim_{t \rightarrow 0} \sqrt{-4t \log k_{t,p}(q)}$$

**“Varadhan’s
Theorem”**



“Geodesics in heat”

Crane, Weischedel, and Wardetzky; TOG 2013

Finding geodesics

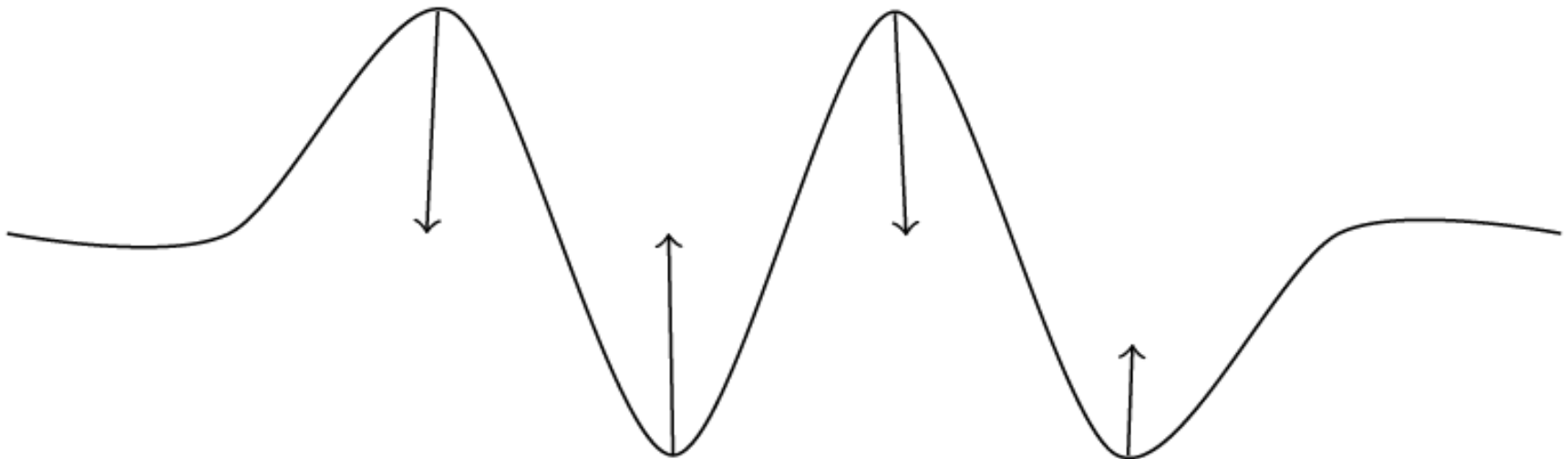
Algorithm 1 The Heat Method

- I. Integrate the heat flow $\dot{u} = \Delta u$ for time t .
 - II. Evaluate the vector field $X = -\nabla u / |\nabla u|$.
 - III. Solve the Poisson equation $\Delta \phi = \nabla \cdot X$.
-

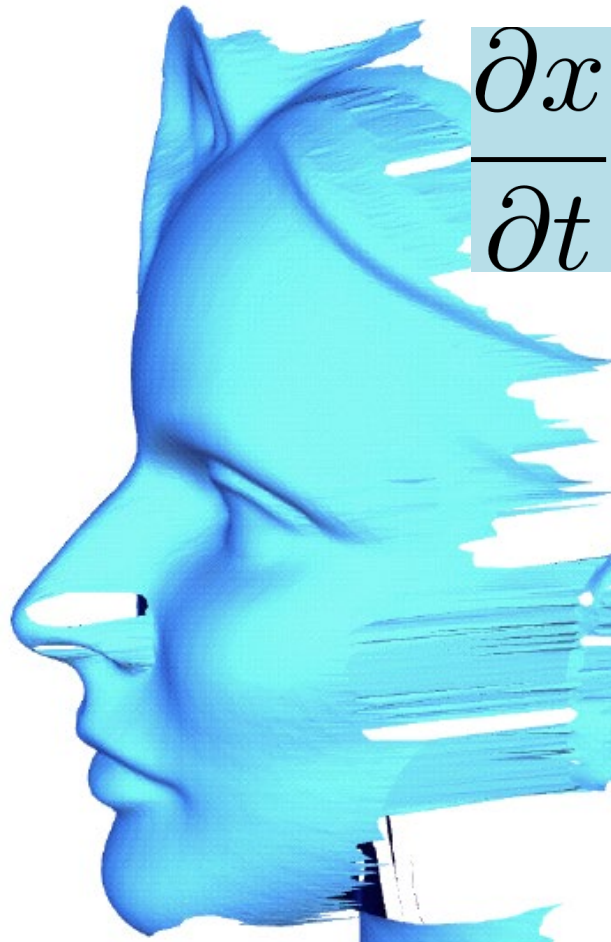
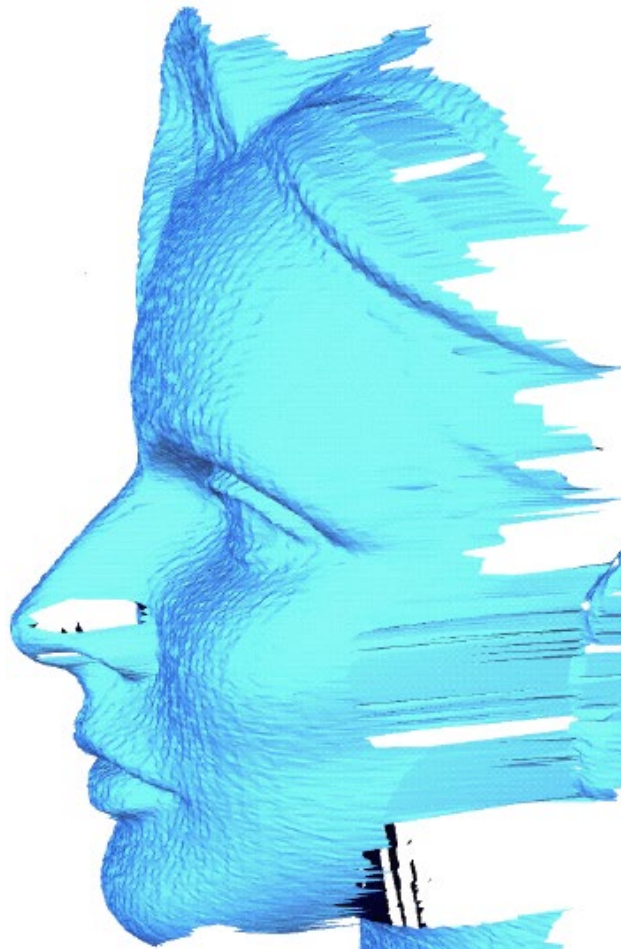


Mean Curvature Flow

$$\frac{\partial x}{\partial t} = \Delta(x) \cdot x$$



Mean Curvature Flow

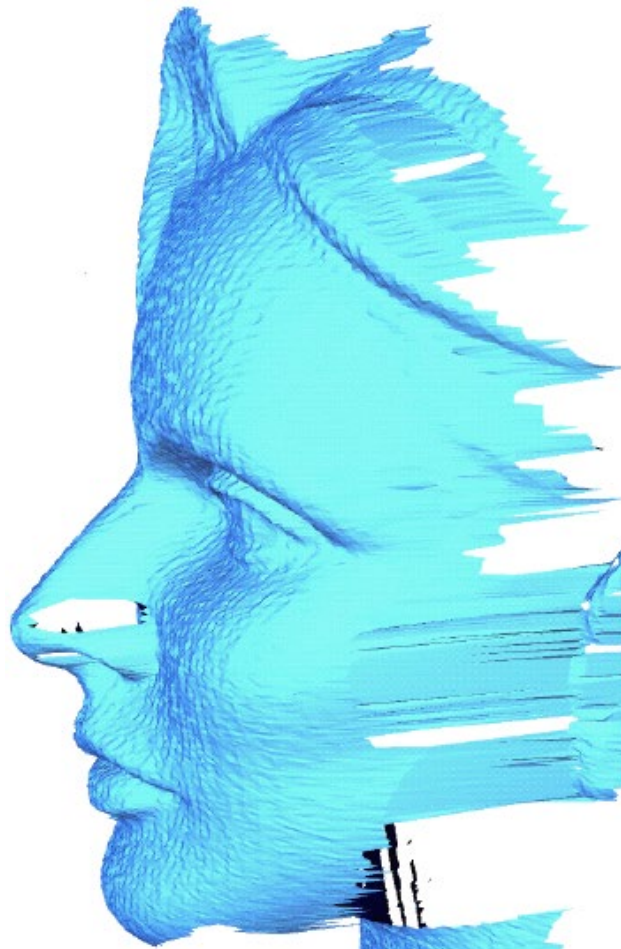


$$\frac{\partial x}{\partial t} = \Delta(x) \cdot x$$

“Implicit fairing of irregular meshes using diffusion and curvature flow”

Desbrun et al., 1999

Mean Curvature Flow



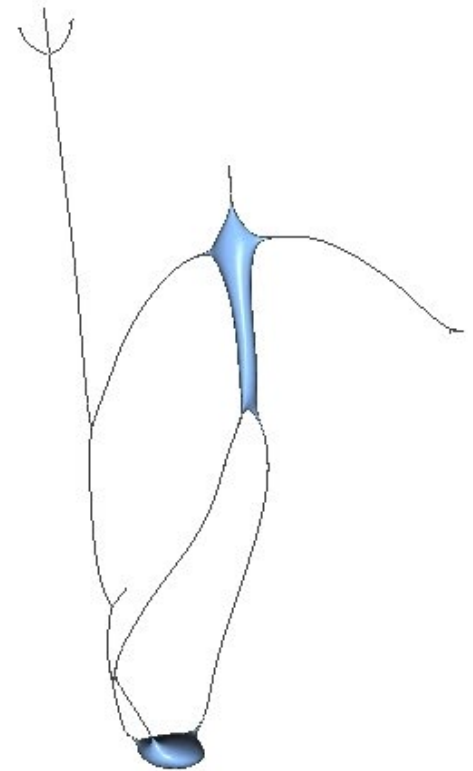
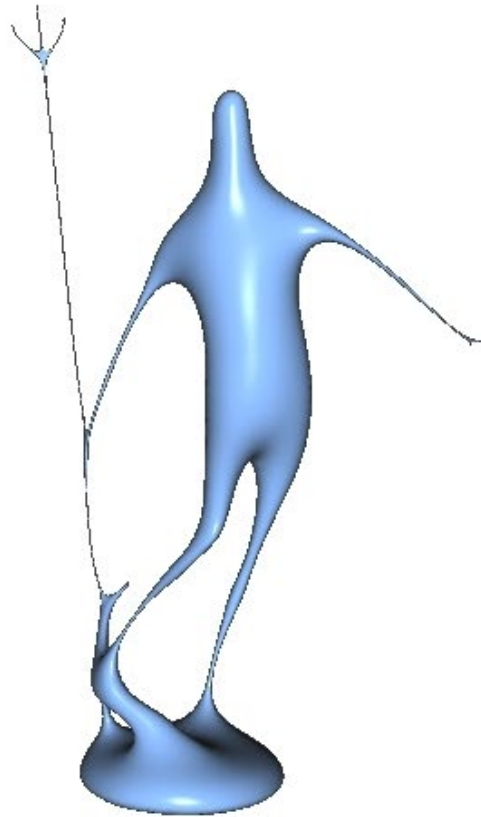
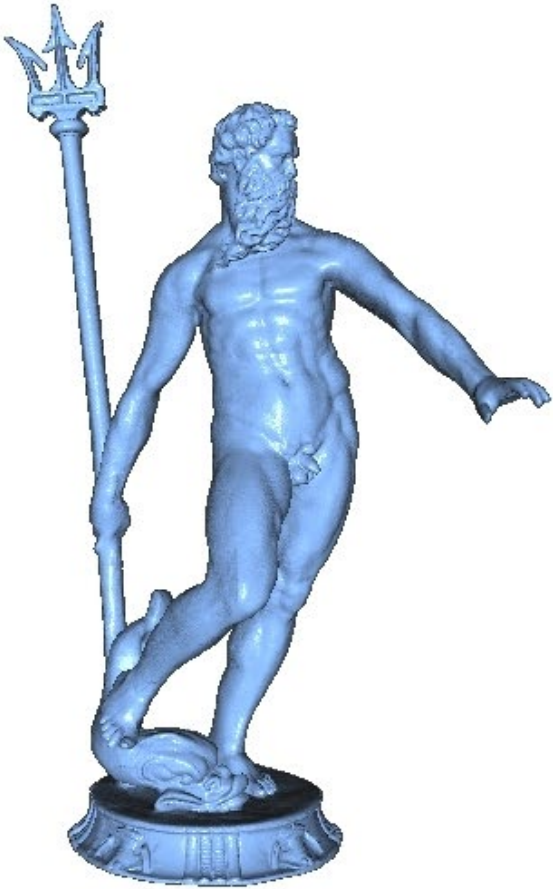
$$\frac{\partial x}{\partial t} = \Delta(x) \cdot x$$

Changes at each time step

“Implicit fairing of irregular meshes using diffusion and curvature flow”

Desbrun et al., 1999

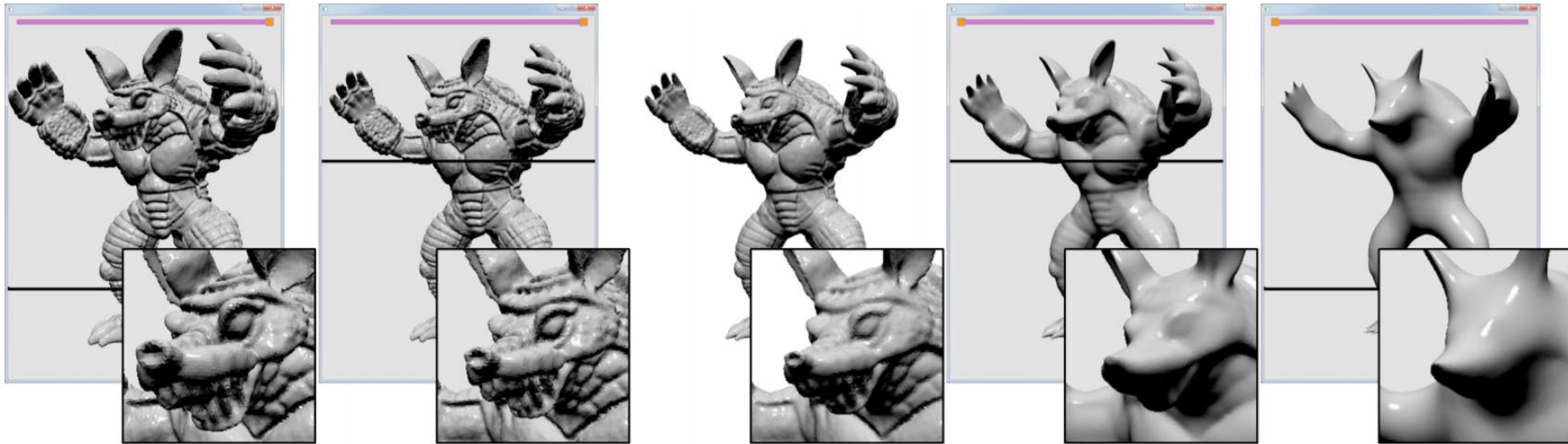
Recall:



Another fairing

Screened Poisson Equation

$$E(G) = \alpha^2 \|G - F\|^2 + \|\nabla_M G - \beta \nabla_M F\|^2$$



Useful Technique

$$\frac{\partial f}{\partial t} = -\Delta f \text{ (heat equation)}$$

$$\rightarrow M \frac{\partial f}{\partial t} = Lf \text{ after discretization in space}$$

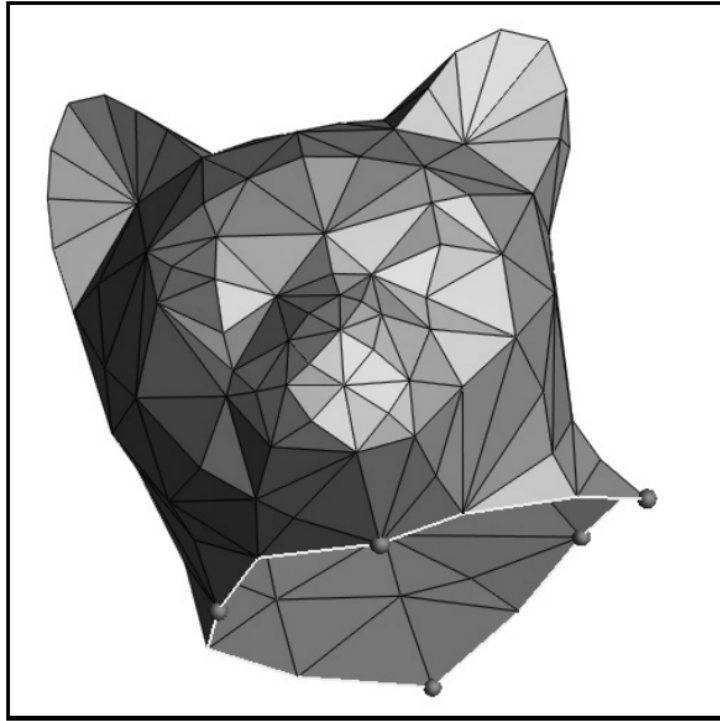
$$\rightarrow M \frac{f_T - f_0}{T} = Lf_T \text{ after time discretization}$$

Choice: Evaluate at time T

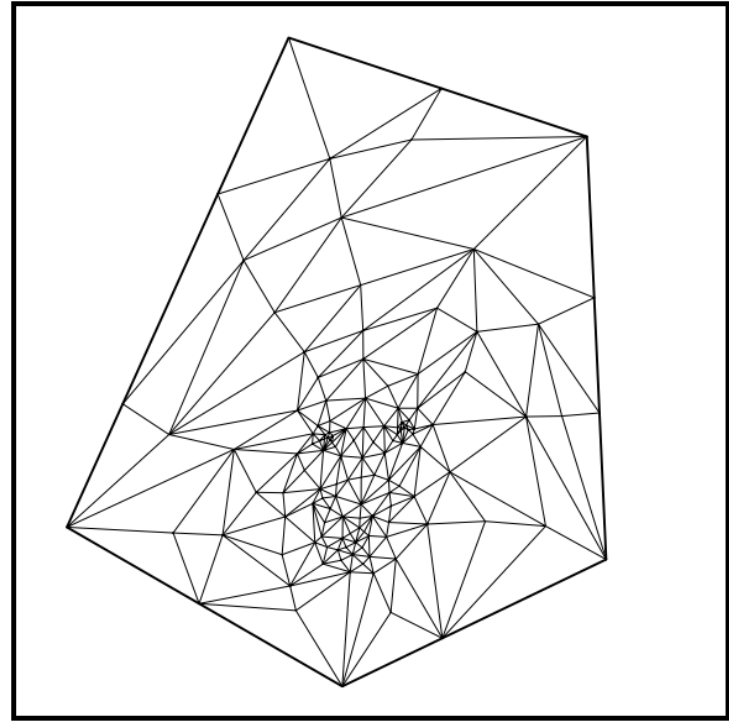
Unconditionally stable, but not necessarily accurate for large T

Implicit time stepping

Parameterization: Harmonic Map



(a) Original mesh tile



(b) Harmonic embedding

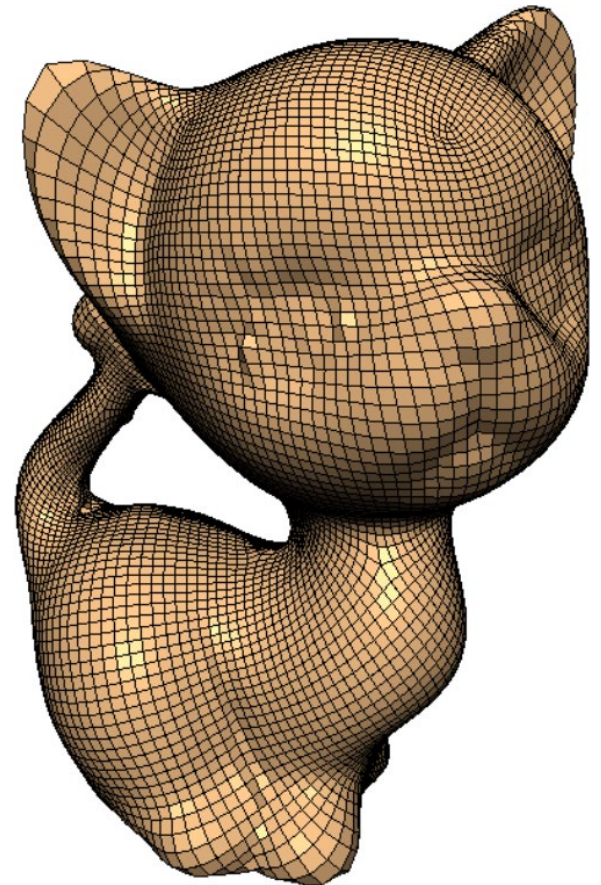
*Recall:
Mean value principle*

“Multiresolution analysis of arbitrary meshes”

Eck et al., 1995 (and many others!)

Others

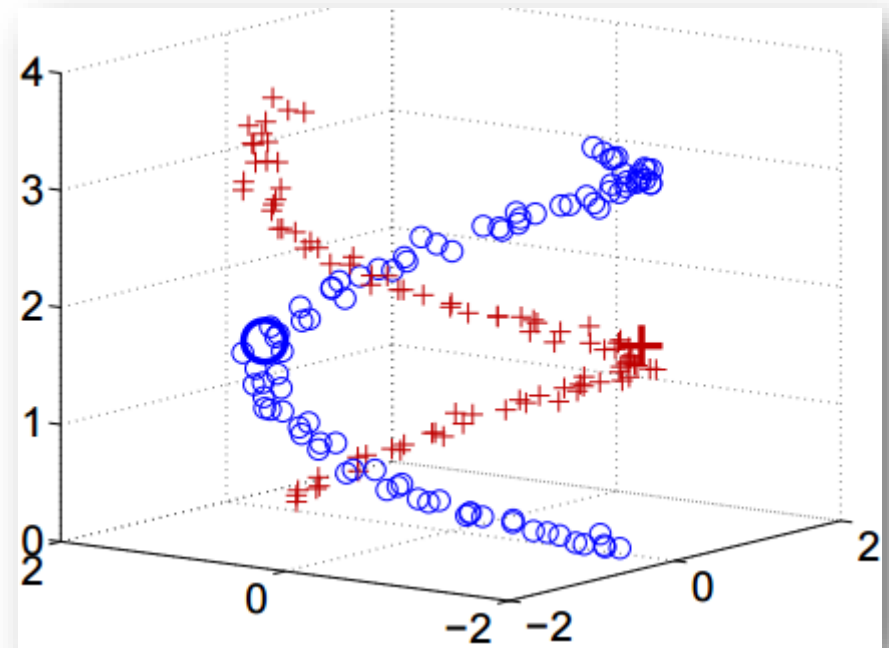
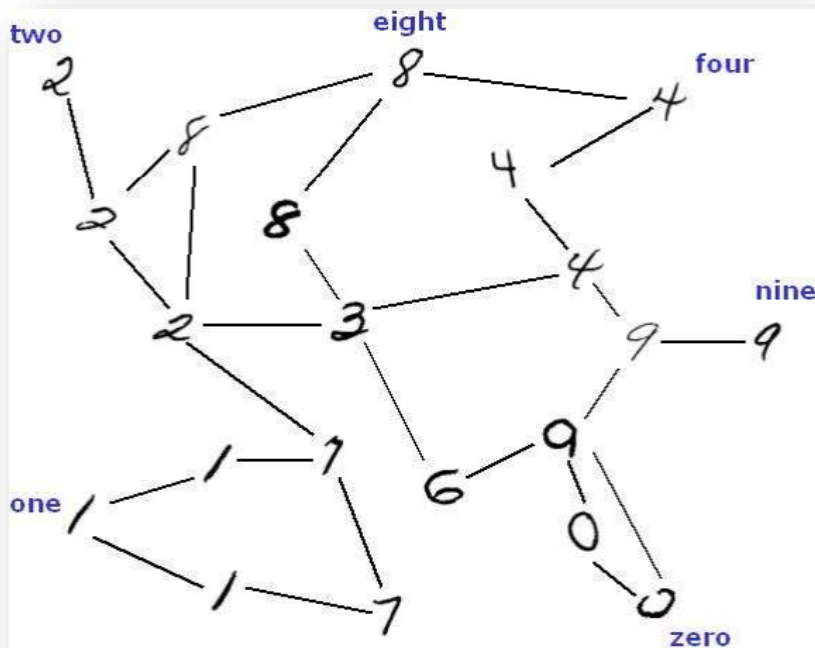
- **Shape retrieval from Laplacian eigenvalues**
“Shape DNA” [Reuter et al., 2006]
- **Quadrangulation**
Nodal domains [Dong et al., 2006]
- **Surface deformation**
“As-rigid-as-possible” [Sorkine & Alexa, 2007]



How can we use L ?

- (useful properties of the Laplacian)
- In Computer Graphics and
Geometry Modeling/Processing
- In Machine Learning

Semi-Supervised Learning



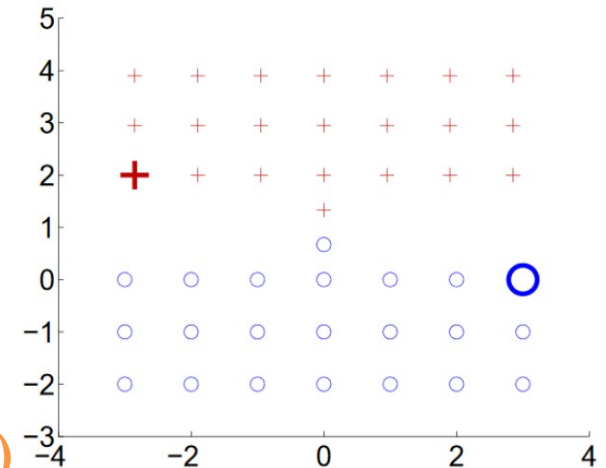
“Semi-supervised learning using Gaussian fields and harmonic functions”
Zhu, Ghahramani, & Lafferty 2003

Semi-Supervised Technique

Given: ℓ labeled points $(x_1, y_1), \dots, (x_\ell, y_\ell); y_i \in \{0, 1\}$
 u unlabeled points $x_{\ell+1}, \dots, x_{\ell+u}; \ell \ll u$

$$\min \frac{1}{2} \sum_{ij} w_{ij} (f(i) - f(j))^2$$

$$\text{s.t. } f(k) \text{ fixed } \forall k \leq \ell$$



Dirichlet energy \rightarrow Linear system of equations (Poisson)

Related Method

- **Step 1:**
Build k -NN graph
- **Step 2:**
Compute p smallest Laplacian eigenvectors
- **Step 3:**
Solve semi-supervised problem in subspace

Manifold Regularization

Regularized learning: $\arg \min_{f \in \mathcal{H}} \frac{1}{\ell} \sum_{i=1}^{\ell} V(f(x_i), y_i) + \gamma \|f\|^2$

Loss function Regularizer

Dirichlet energy

$$\|f\|_I^2 := \int \|\nabla f(x)\|^2 dx \approx f^\top L f$$

“Manifold Regularization:

A Geometric Framework for Learning from Labeled and Unlabeled Examples”

Belkin, Niyogi, and Sindhwani; JMLR 2006

Examples of Manifold Regularization

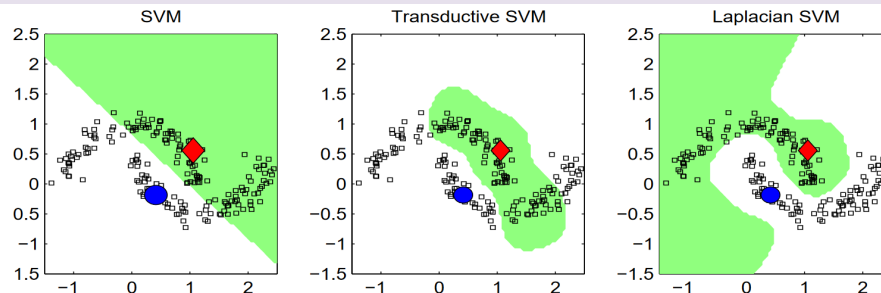
- Laplacian-regularized least squares (**LapRLS**)

$$\arg \min_{f \in \mathcal{H}} \frac{1}{\ell} \sum_{i=1}^{\ell} (f(x_i) - y_i)^2 + \gamma \|f\|_I^2 + \text{Other}[f]$$

- Laplacian support vector machine (**LapSVM**)

$$\arg \min_{f \in \mathcal{H}} \frac{1}{\ell} \sum_{i=1}^{\ell} \max(0, 1 - y_i f(x_i)) + \gamma \|f\|_I^2 + \text{Other}[f]$$

“On Manifold Regularization”
Belkin, Niyogi, Sindhwani; AISTATS 2005



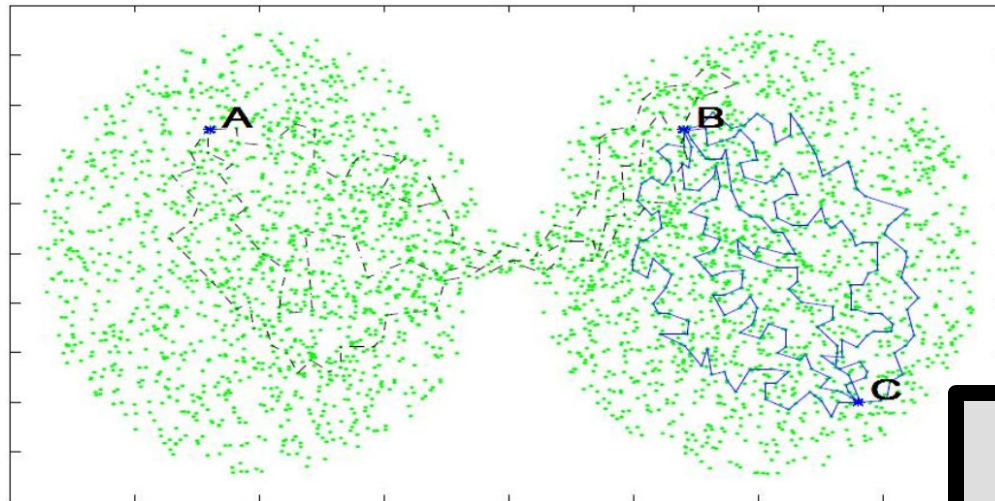
Diffusion Maps

Embedding from first k eigenvalues/vectors:

$$\Psi_t(x) := \left(\lambda_1^t \psi_1(x), \lambda_2^t \psi_2(x), \dots, \lambda_k^t \psi_k(x) \right)$$

Roughly:

$|\Psi_t(x) - \Psi_t(y)|$ is probability that x, y diffuse to the same point in time t .



**Robust to
sampling and
noise**

“Diffusion Maps”

Coifman and Lafon; Applied and Computational Harmonic Analysis, 2006