DEFINING “CURVE”

Curve = function?

$f(t)$

$f : \mathbb{R} \to \mathbb{R}^2$
SUBTLETY

\[ \gamma(t) = (0,0) \]

Not all functions are curves!
DIFFERENT FROM CALCULUS

Sudden acceleration

Same curves?

\[ f_1(t) = (t, 2t) \]

\[ f_2(t) = \begin{cases} (t, 2t) & t \leq 1 \\ (2(t - \frac{1}{2}), 4(t - \frac{1}{2}) & t > 1 \end{cases} \]
GRAPHS OF SMOOTH FUNCTIONS

Not smooth!

\[ f(t) = (t^2, t^3) \]

Smooth functions

GEOMETRY OF A CURVE

A curve is a set of points with certain properties.

It is not a function.
GEOMETRIC DEFINITION

Set of points that locally looks like a line.
GEOMETRIC DEFINITION

Too restrictive?
DEFINING “CURVE”

Curve = function?

$f(t)$

$f: \mathbb{R} \to \mathbb{R}^2$
DEFINING “CURVE”

Curve = function?

\( f_p : \mathbb{R} \rightarrow \mathbb{R}^2 \)

\( f(t) \)

\( t \)
PARAMETERIZED CURVE

$\gamma(t) : (a, b) \to \mathbb{R}^2$

Now this is OK!
WHAT WAS THE PROBLEM HERE?

\[ f(t) = (t^2, t^3) \]

Smooth functions

$f'(t) \neq 0$

Non-zero velocity!
DEPENDENCE OF VELOCITY

\[ \tilde{\gamma}(s) := \gamma(\phi(s)) \]

On the board:
Effect on velocity and acceleration.
CHANGE OF PARAMETER

Geometric measurements should be invariant to changes of parameter.
ARC LENGTH

\[ \int_a^b \left\| \gamma'(t) \right\| \, dt \]

On the board:
Independence of parameter
PARAMETERIZATION BY ARC LENGTH

Not arc-length parameterized

Arc-length parameterized

\[ s(t) := \int_{t_0}^{t} \| \gamma'(t) \| \, dt \]

\[ t(s) := \text{inverse of } s(t) \]

\[ \gamma(s) := \gamma(s(t)) \]

Constant-speed parameterization

SAME CURVE?
MOVING FRAME IN 2D

Tangent: $\tau(s) := \gamma'(s)$
$$\|\tau(s)\| = 1$$

Normal: $n(s) := Rot\left(\frac{\pi}{2}\right)\tau(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\tau(s)$
PHILOSOPHICAL POINT

Differential geometry “should” be coordinate-invariant.

Referring to $x$ and $y$ is a hack!

(but sometimes convenient...)
FRENET-SERRET FORMULAS

On the board:

\[
\frac{d}{ds} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix} := \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix}
\]

Use coordinates from the curve to express its shape!

https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas
RADIUS OF CURVATURE

\[ r(s) := \frac{1}{k(s)} \]

RADIUS OF CURVATURE

\[ r(s) := \frac{1}{k(s)} \]

Fundamental theorem of the local theory of plane curves:

\( k(s) \) characterizes a planar curve up to rigid motion.
IDEA OF PROOF

$T(s) := (\cos \theta(s), \sin \theta(s))$

$\implies k(s) := \theta'(s)$

Provides intuition for curvature

Image from DDG course notes by E. Grinspun
FRENET FRAME: CURVES IN $\mathbb{R}^3$

\[ \frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \]

- **Binormal**: $T \times N$
- **Curvature**: In-plane motion
- **Torsion**: Out-of-plane motion
Fundamental theorem of the local theory of space curves:

Curvature and torsion characterize a 3D curve up to rigid motion.
Representing curves digitally
TRADITIONAL APPROACH

Piecewise smooth approximations

$F(0, 0, 1)$

$F(0, 0, 0) = f(0)$

$F(0, 1, 1)$

$F(1, 1, 1) = f(1)$
NOT THE ONLY MODELING TOOL

κ-Curves: Interpolation at Local Maximum Curvature

ZHIPEI YAN, Texas A&M University
STEPHEN SCHILLER, Adobe Research
GREGG WILENSKY, Adobe
NATHAN CARR, Adobe Research
SCOTT SCHAEFER, Texas A&M University

Fig. 1. Top row shows example shapes made from the control points below. In all cases, local maxima of curvature only appear at the control points, and the curves are $G^2$ almost everywhere.
What is the arc length of a cubic Bézier curve?

$$\int_a^b \| \gamma'(t) \| \, dt$$
What is the arc length of a cubic Bézier curve?

\[ \int_a^b \| \gamma'(t) \| \, dt \]

Not known in closed form.
ONLY APPROXIMATIONS ANYWAY

\{ \text{Bézier curves} \} \subset \{ \gamma : \mathbb{R} \to \mathbb{R}^3 \}
ANOTHER REASONABLE APPROXIMATION

Piecewise linear
BIG PROBLEM

$\kappa = 0$

$\kappa = \infty$

Boring differential structure
FINITE DIFFERENCE APPROACH

\[ f'(x) \approx \frac{1}{h} [f(x + h) - f(x)] \]

THEOREM: As \( \Delta h \to 0 \), [insert statement].
REALITY CHECK

THEOREM: \( f'(x) \approx \frac{1}{h} [f(x + h) - f(x)] \)

\( h > 0 \)
TWO KEY CONSIDERATIONS

• Convergence to continuous theory

• Discrete behavior
Examine discrete theories of differentiable curves.
Examine discrete theories of differentiable curves.
GAUSS MAP

Normal map from curve to $S^1$
Signed Curvature on Plane Curves

\[ T(s) = (\cos \theta(s), \sin \theta(s)) \]

\[ T'(s) = \theta'(s)(-\sin \theta(s), \cos \theta(s)) \]
\[ := \kappa(s)N(s) \]
RECOVERING THETA

\[ \theta'(s) = \kappa(s) \]

\[ \Delta \theta = \int_{s_0}^{s_1} \kappa(s) \, ds \]
TURNING NUMBER THEOREM

A “global” theorem!

\[ \int_{\Omega} \kappa(s) \, ds = 2\pi k \]
DISCRETE GAUSS MAP

http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf
DISCRETE GAUSS MAP

Edges become points
KEY OBSERVATION

\[ \sum_{i} \theta_i = 2\pi k \]
WHAT’S GOING ON?

Total change in curvature

\[ \theta \]
WHAT’S GOING ON?

\[ \theta = \int_{\Gamma} \kappa \, ds \]

Total change in curvature
WHAT'S GOING ON?

\[ \theta = \int \kappa \, ds \]

\[ \kappa \approx \frac{2\theta}{l_1 + l_2} \]

Total change in curvature
Continuous:

\[ k(s) = \theta'(s) \]

Discrete (finite difference):

\[ k_i = \frac{\theta_i}{(l_i + l_{i+1})/2} \]
INTERESTING DISTINCTION

\[ \kappa_1 \neq \kappa_2 \]

\[ \theta \]

Same integrated curvature
INTERESTING DISTINCTION

\[ \kappa_1 \neq \kappa_2 \]

Same integrated curvature
WHAT'S GOING ON?

\[ \theta = \int_{\Gamma} \kappa \, ds \]

Integrated quantity

Dual cell

Total change in curvature
DISCRETE TURNING ANGLE THEOREM

\[ \int_{\Gamma} \kappa \, ds = \sum_i \int_{\Gamma_i} \kappa \, ds \]
\[ = \sum_i \theta_i \]
\[ = 2\pi k \]

Structure Preservation!
\n\n\[ \n\n\n\n\]
FOR SMALL $\theta$

$$2 \sin \frac{\theta}{2} \approx 2 \cdot \frac{\theta}{2}$$

$$= \theta$$

Same behavior in the limit

http://en.wikipedia.org/wiki/Taylor_series
Does discrete curvature converge in limit?

Yes!
Discretizing 3D curves
FRENET FRAME

\[ \frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \]
APPLICATION

NMR scanner

Kinked alpha helix

Structure Determination of Membrane Proteins Using Discrete Frenet Frame and Solid State NMR Restraints
Achuthan and Quine
APPLICATION

Structure Determination of Membrane Proteins Using Discrete Frenet Frame and Solid State NMR Restraints
Achuthan and Quine
POTENTIAL DISCRETIZATION

\[
T_j = \frac{p_{j+1} - p_j}{\|p_{j+1} - p_j\|}
\]

\[
B_j = t_{j-1} \times t_j
\]

\[
N_j = b_j \times t_j
\]

Discrete Frenet frame

\[
T_k = R(B_k, \theta_k)T_{k-1}
\]

\[
B_{k+1} = R(T_k, \phi_k)B_k
\]

“Bond and torsion angles”

(derivatives converge to \( \kappa \) and \( \tau \), resp.)

Discrete frame introduced in:
The resultant electric moment of complex molecules
Discrete construction that works for fractal curves and converges in continuum limit.

Discrete Frenet Frame, Inflection Point Solitons, and Curve Visualization with Applications to Folded Proteins
Hu, Lundgren, and Niemi
Physical Review E 83 (2011)
Frenet Frame: Issue

\[
\frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}
\]

\[\kappa = 0?\]
SEGMENTS NOT ALWAYS ENOUGH

Discrete Elastic Rods
Bergou, Wardetzky, Robinson, Audoly, and Grinspun
SIGGRAPH 2008

http://www.cs.columbia.edu/cg/rods/
SIMULATION GOAL

http://www.cs.columbia.edu/cg/rods/
ADAPTED FRAMED CURVE

\[ \Gamma = \{ \gamma(s); T, m_1, m_2 \} \]

Material frame

Normal part encodes twist

http://www.cs.columbia.edu/cg/rods/
**BENDING ENERGY**

\[ E_{\text{bend}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \alpha \kappa^2 \, ds \]

Punish turning the steering wheel

\[ \kappa N = T' \]

\[ = (T' \cdot T)T + (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \]

\[ = (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \]

\[ := \omega_1 m_1 + \omega_2 m_2 \]
BENDING ENERGY

\[ E_{\text{bend}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \alpha (\omega_1^2 + \omega_2^2) \, ds \]

Punish turning the steering wheel

\[ \kappa N = T' \]
\[ = (T' \cdot T)T + (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \]
\[ = (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \]
\[ := \omega_1 m_1 + \omega_2 m_2 \]
TWISTING ENERGY

\[ E_{\text{twist}}(\Gamma) := \frac{1}{2} \int_\Gamma \beta m^2 \, ds \]

Punish non-tangent change in material frame

\[ m := m'_1 \cdot m_2 \]

\[ = \frac{d}{dt} (m_1 \cdot m_2) - m_1 \cdot m'_2 \]

\[ = -m_1 \cdot m'_2 \]
TWISTING ENERGY

\[ E_{\text{twist}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \beta m^2 \, ds \]

Punish non-tangent change in material frame

Swapping \( m_1 \) and \( m_2 \) does not affect \( E_{\text{twist}} \)!
WHICH BASIS TO USE

THERE IS MORE THAN ONE WAY TO FRAME A CURVE

RICHARD L. BISHOP

The Frenet frame of a 3-times continuously differentiable (that is, $C^3$) non-degenerate curve in euclidean space has long been the standard vehicle for analysing properties of the curve invariant under euclidean motions. For arbitrary moving frames, that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the curve parameter in terms of the frame itself, and due to ortho-normality the coefficient matrix is always skew-symmetric. Thus it generally has three nonzero entries. The Frenet frame gains part of its special significance from the fact that one of the three derivatives is always zero. Another feature of the Frenet frame is that it is adapted to the curve: the members are either tangent to or perpendicular to the curve. It is the purpose of this paper to show that there are other frames which have these same advantages and to compare them with the Frenet frame.

1. Relatively parallel fields. We say that a normal vector field $M$ along a curve is relatively parallel if its derivative is tangential. Such a field turns only whatever amount is necessary for it to remain normal, so it is as close to being parallel as possible without losing normality. Since its derivative is perpendicular to it, a relatively parallel normal field has constant length. Such fields occur classically in the discussion of curves which are said to be parallel to the given curve. Indeed, if
BISHOP FRAME

\[ T' = \Omega \times T \]
\[ u' = \Omega \times u \]
\[ v' = \Omega \times v \]

\[ u' \cdot v \equiv 0 \]

No twist
(“parallel transport”)
BISHOP FRAME

\[ T' = \Omega \times T \]
\[ u' = \Omega \times u \]
\[ v' = \Omega \times v \]

\[ \Omega := \kappa B \] ("curvature binormal")

No twist ("parallel transport")

\[ u' \cdot v \equiv 0 \]

Darboux vector

http://www.cs.columbia.edu/cg/rods/
CURVE-ANGLE REPRESENTATION

Degrees of freedom for elastic energy:

- Shape of curve
- Twist angle $\theta$

$$m_1 = u \cos \theta + v \sin \theta$$

$$m_2 = -u \sin \theta + v \cos \theta$$

$$E_{\text{twist}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \beta (\theta')^2 \, ds$$
DISCRETE KIRCHHOFF RODS

$x_0 e_0 x_1 e^1 x_2 e^2 x_3 e^3 x_4 e^4$

Lower index: primal

Upper index: dual
Tangent unambiguous on edge
Yet another curvature!

Integrated curvature
DISCRETE KIRCHHOFF RODS

\[ \kappa_i := 2 \tan \frac{\phi_i}{2} \]

Yet another curvature!

Orthogonal to osculating plane, norm \( \kappa_i \)

\[ (\kappa B)_i := \frac{2e^{i-1} \times e^i}{\| e^{i-1} \| \| e^i \| + e^{i-1} \cdot e^i} \]

Darboux vector
BENDING ENERGY

\[ E_{\text{bend}}(\Gamma) := \frac{\alpha}{2} \sum_i \left( \frac{(\kappa B)_i}{\ell_i/2} \right)^2 \frac{\ell_i}{2} \]

\[ = \alpha \sum_i \frac{\| (\kappa B)_i \|^2}{\ell_i} \]

Can extend for natural bend

Convert to pointwise and integrate
DISCRETE PARALLEL TRANSPORT

\[ P_i(T^{i-1}) = T^i \]

\[ P_i(T^{i-1} \times T^i) = T^{i-1} \times T^i \]

- Map tangent to tangent
- Preserve binormal
- Orthogonal

\[ u^i = P_i(u^{i-1}) \]

\[ v^i = T^i \times u^i \]

http://www.cs.columbia.edu/cg/rods/
DISCRETE MATERIAL FRAME

\[
m_1^i = u^i \cos \theta^i + v^i \sin \theta^i
\]

\[
m_2^i = -u^i \sin \theta^i + v^i \cos \theta^i
\]
DISCRETE TWISTING ENERGY

\[ E_{\text{twist}}(\Gamma) := \beta \sum_i \frac{(\theta^i - \theta^{i-1})^2}{l_i} \]

Note \( \theta_0 \) can be arbitrary
SIMULATION

\textit{Worth reading!}
EXTENSION AND SPEEDUP

Discrete Viscous Threads

Miklós Bergou
Columbia University

Basile Audoly
UPMC Univ. Paris 06 & CNRS

Etienne Vouga
Columbia University

Max Wardetzky
Universität Göttingen

Eitan Grinspun
Columbia University

http://www.cs.columbia.edu/cg/threads/
EXTENSION AND SPEEDUP

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Eitan Grinspun
Columbia University

“...the first numerical fluid-mechanical sewing machine.”
Three different curvature discretizations

\[ \theta \quad 2 \sin \frac{\theta}{2} \quad 2 \tan \frac{\theta}{2} \]
Easy theoretical object, hard to use.

\[ \frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \]
Surfaces