

# IFT 6113 CURVES

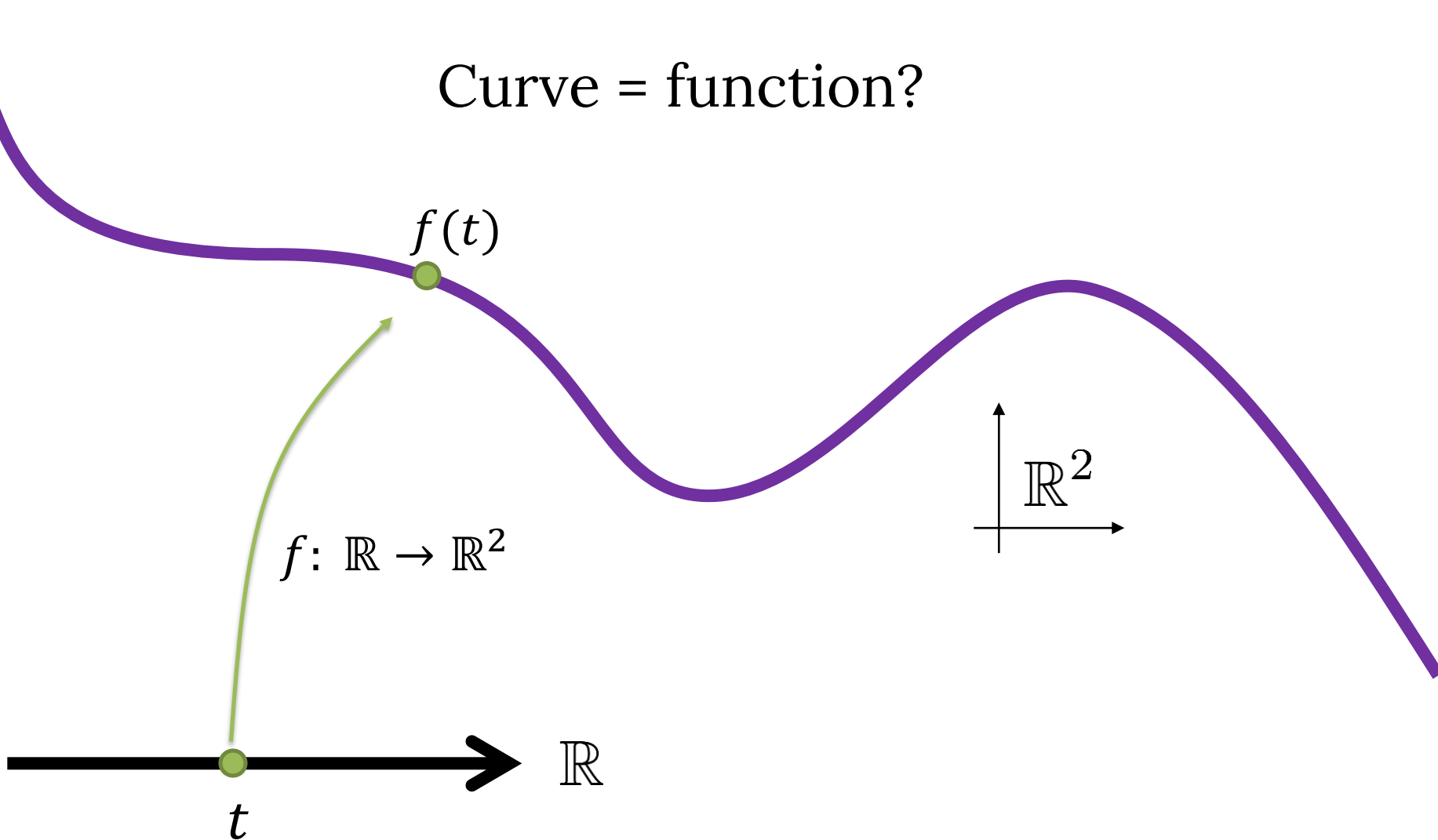


Mikhail Bessmeltsev

<http://tiny.cc/6113>

# DEFINING “CURVE”

Curve = function?

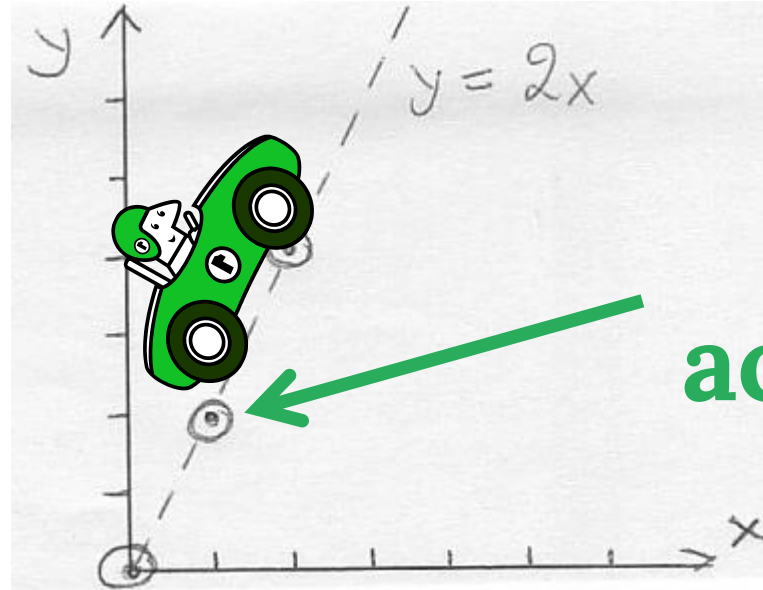


# SUBTLETY

$$\gamma(t) = (0,0)$$

Not all functions are curves!

# DIFFERENT FROM CALCULUS



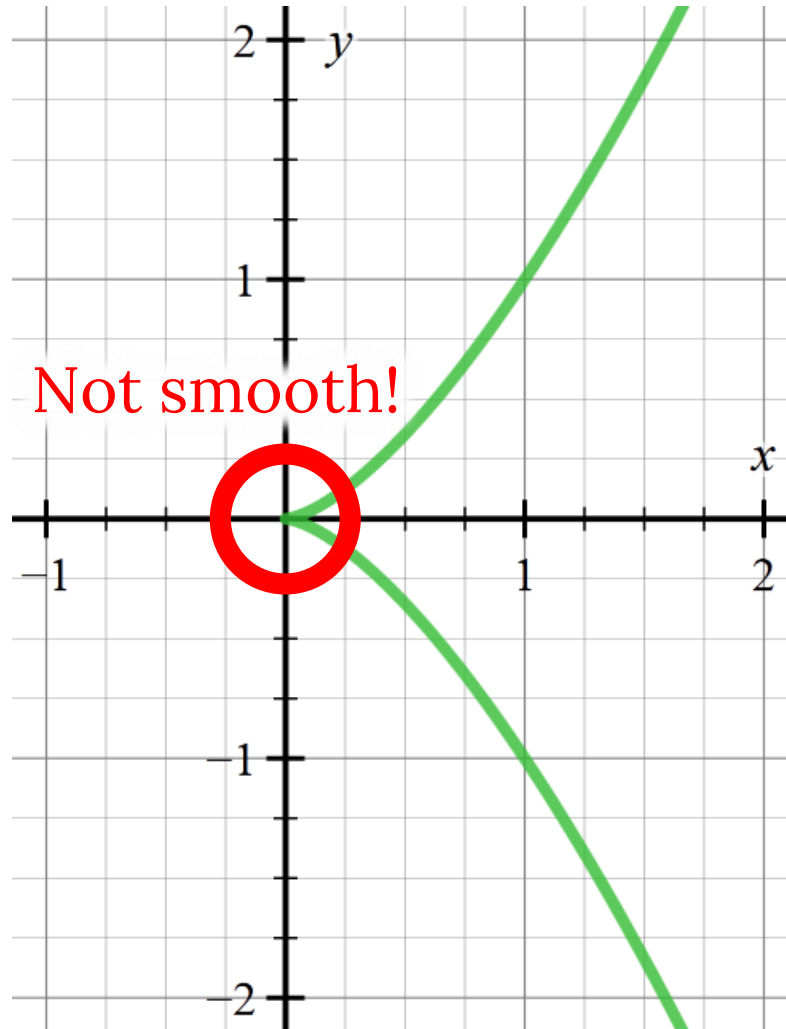
**Sudden  
acceleration**

Same curves?

$$f_1(t) = (t, 2t)$$

$$f_2(t) = \begin{cases} (t, 2t) & t \leq 1 \\ (2(t - \frac{1}{2}), 4(t - \frac{1}{2})) & t > 1 \end{cases}$$

# GRAPHS OF SMOOTH FUNCTIONS



$$f(t) = (t^2, t^3)$$

Smooth functions

# GEOMETRY OF A CURVE

A curve is a  
**set of points**  
with certain properties.

It is not a function.

# GEOMETRIC DEFINITION



Set of points that locally **looks like a line.**

# GEOMETRIC DEFINITION

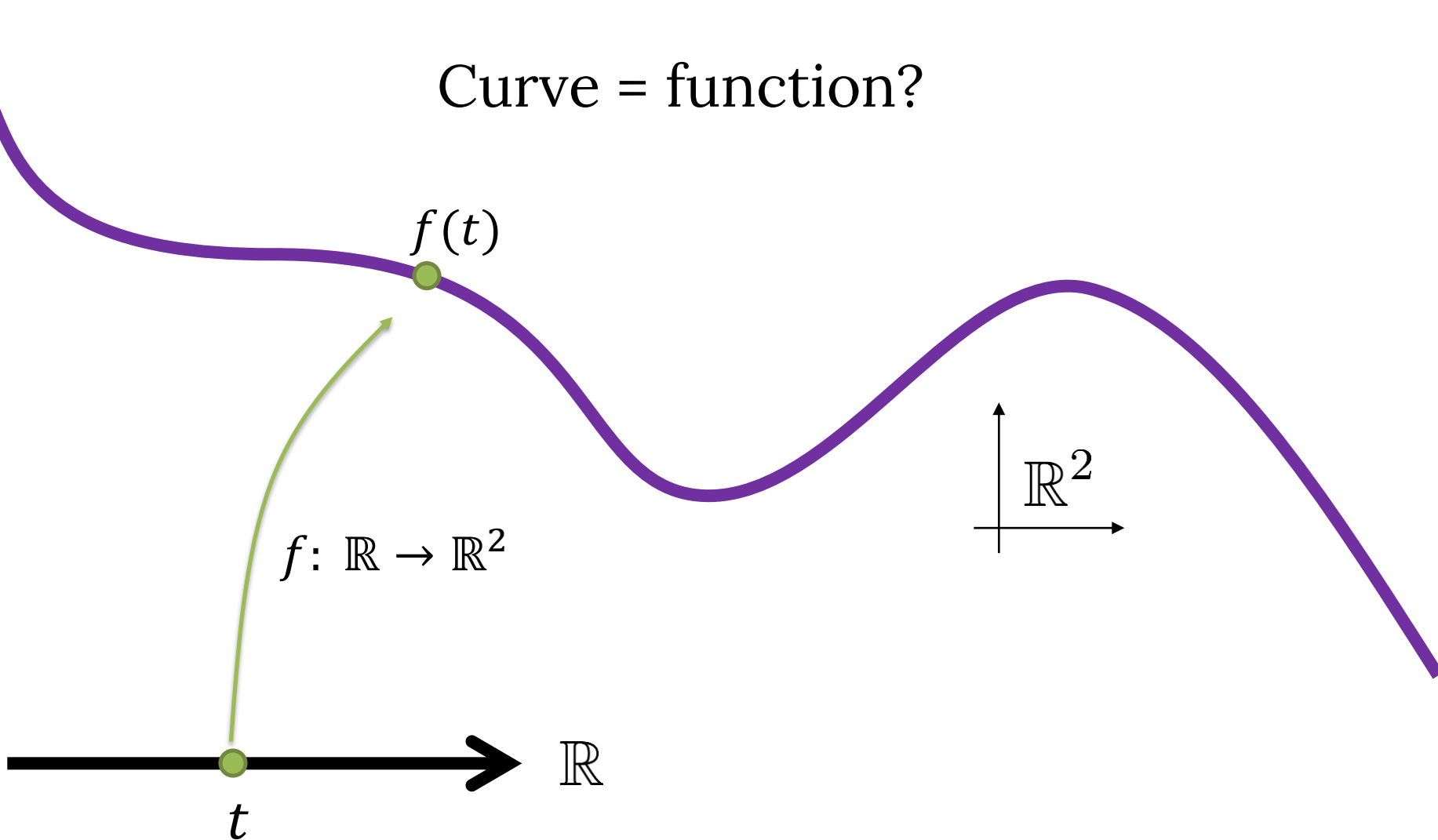


**Too restrictive?**



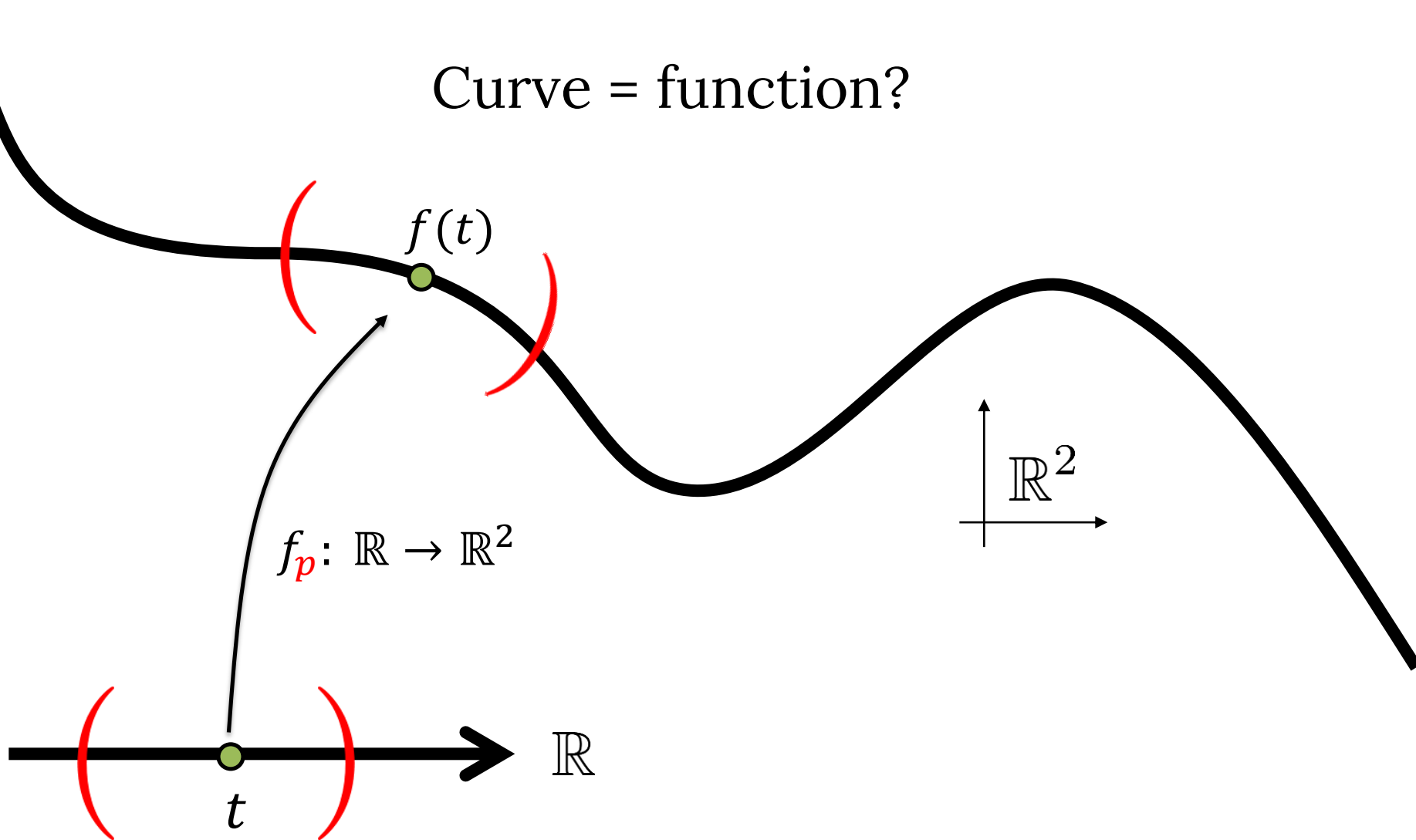
# DEFINING “CURVE”

Curve = function?

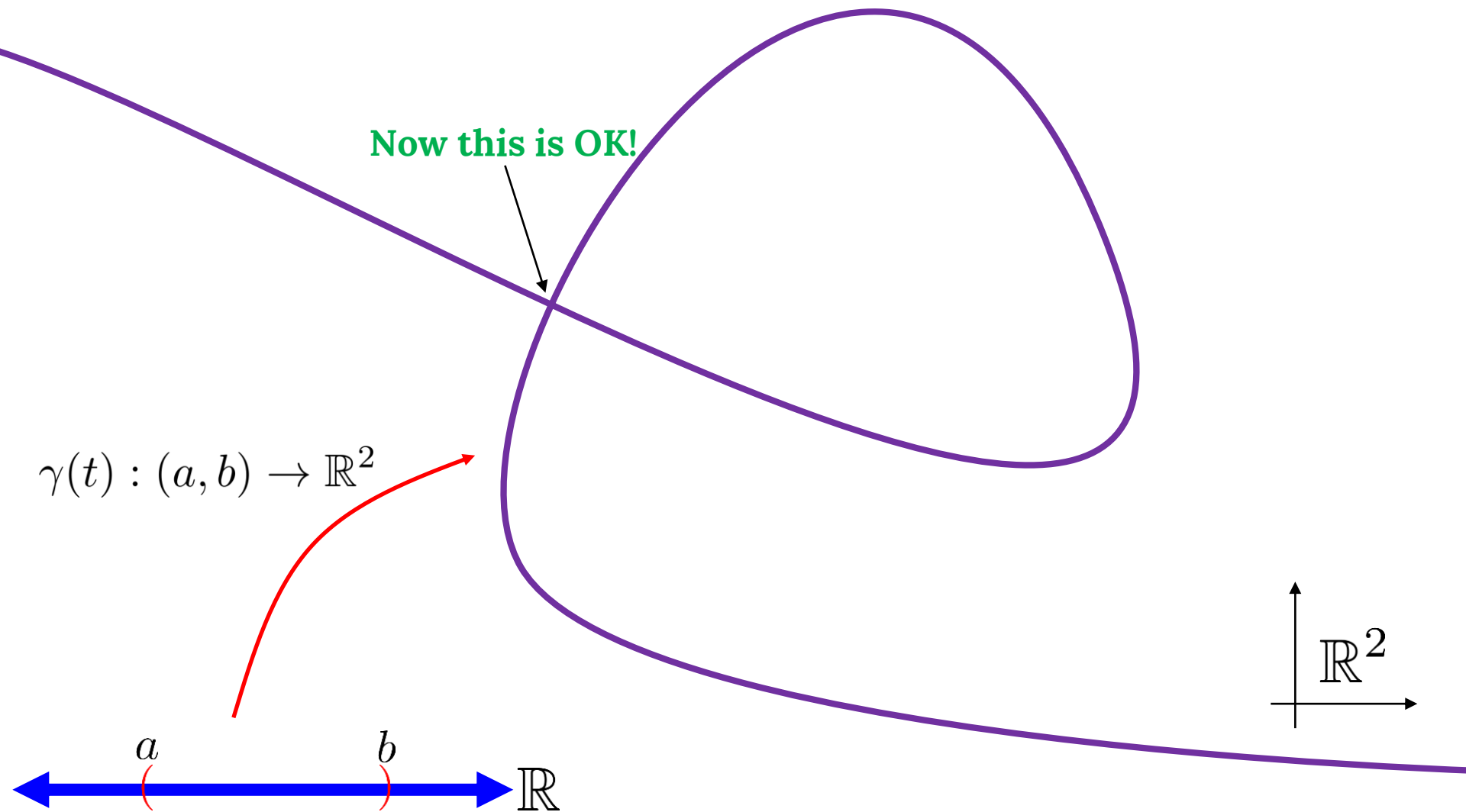


# DEFINING “CURVE”

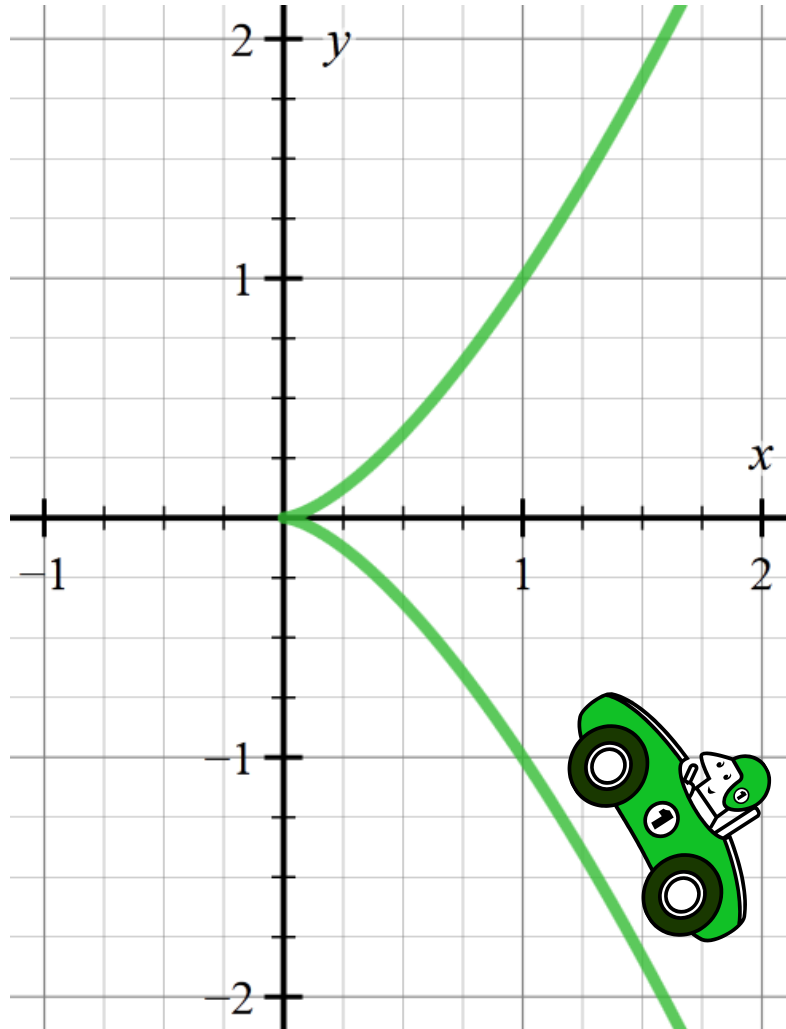
Curve = function?



# PARAMETERIZED CURVE



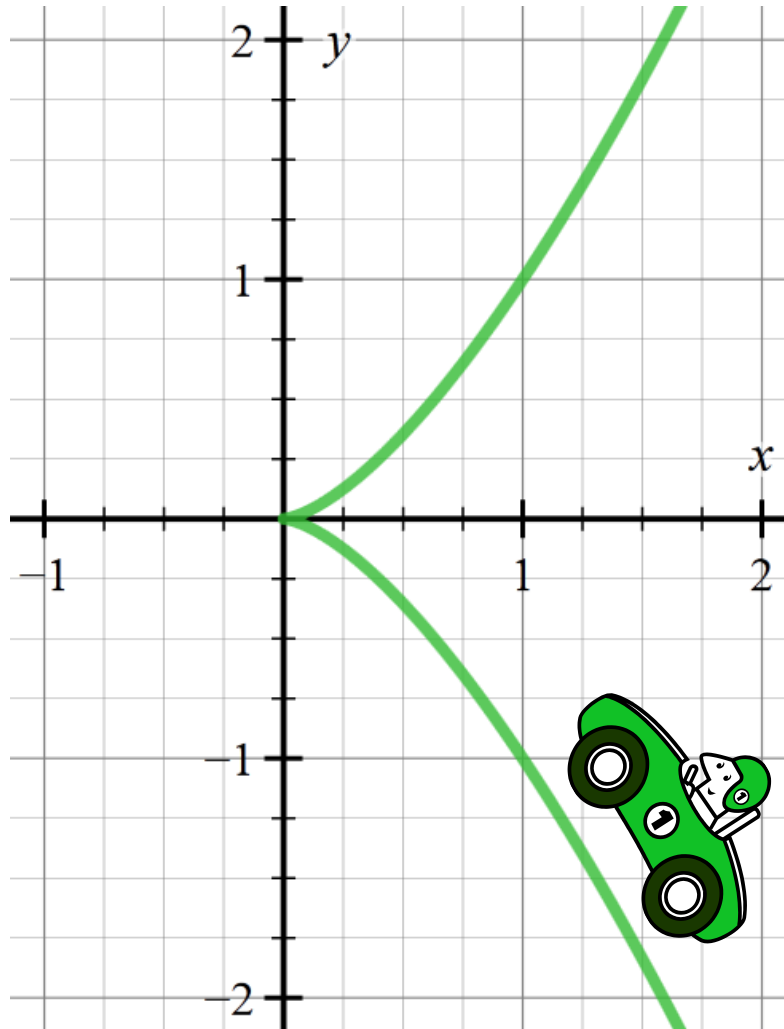
# WHAT WAS THE PROBLEM HERE?



$$f(t) = (t^2, t^3)$$

Smooth functions

# FIXING



$f'(t) \neq 0$   
Non-zero velocity!

# DEPENDENCE OF VELOCITY

$$\tilde{\gamma}(s) := \gamma(\phi(s))$$

*On the board:*

Effect on velocity and acceleration.

# CHANGE OF PARAMETER

Geometric measurements should be  
**invariant**  
to changes of parameter.



# ARC LENGTH

$$\int_a^b \|\gamma'(t)\| dt$$

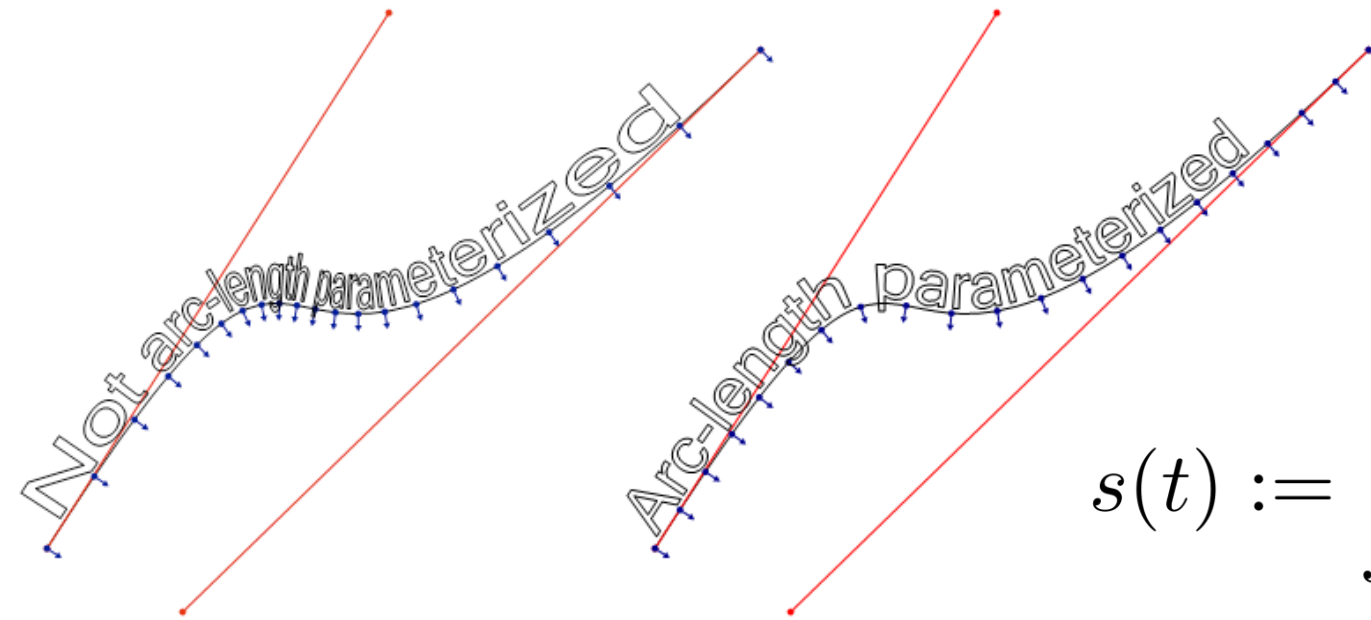
**On the board:**

Independence of parameter



# PARAMETERIZATION BY ARC LENGTH

<http://www.planetclegg.com/projects/WarpingTextToSplines.html>



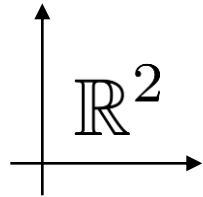
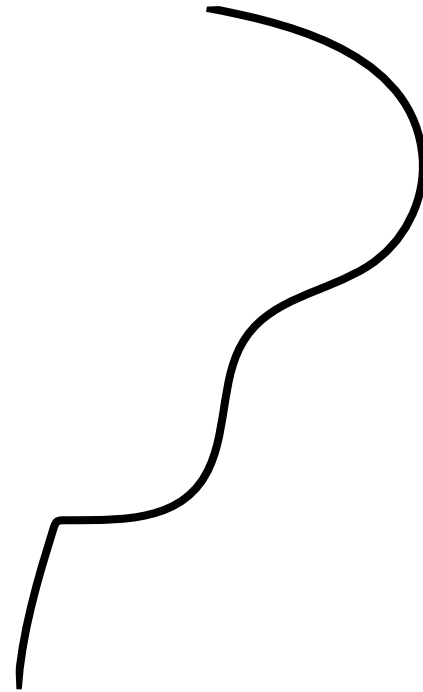
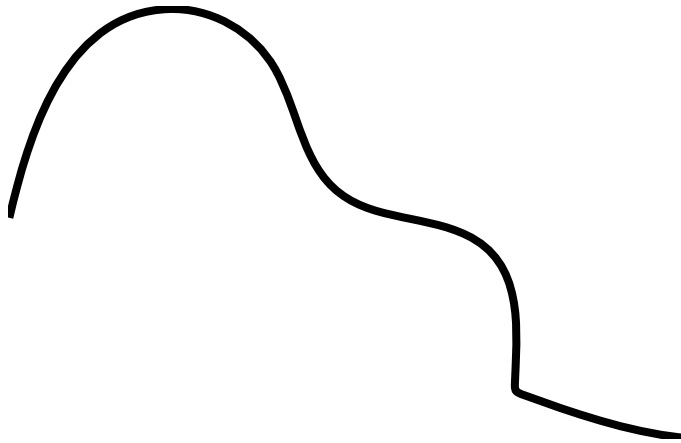
$$s(t) := \int_{t_0}^t \|\gamma'(t)\| dt$$

$$t(s) := \text{inverse of } s(t)$$

$$\gamma(s) := \gamma(s(t))$$

Constant-speed parameterization

# SAME CURVE?

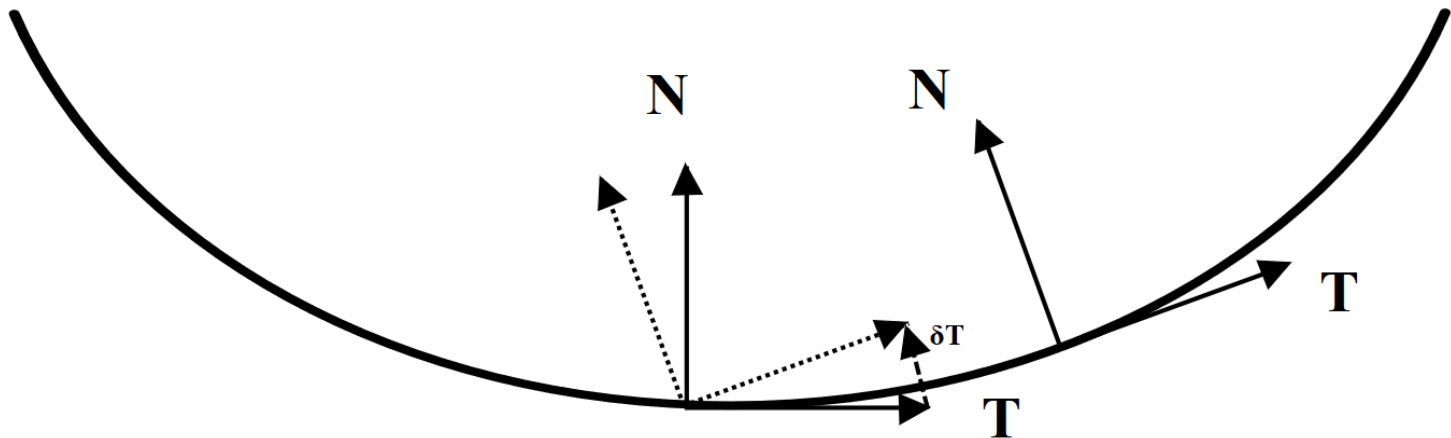


# MOVING FRAME IN 2D

Tangent:  $\tau(s) := \gamma'(s)$

$$\|\tau(s)\| = 1$$

Normal:  $n(s) := \text{Rot}\left(\frac{\pi}{2}\right)\tau(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\tau(s)$



# PHILOSOPHICAL POINT

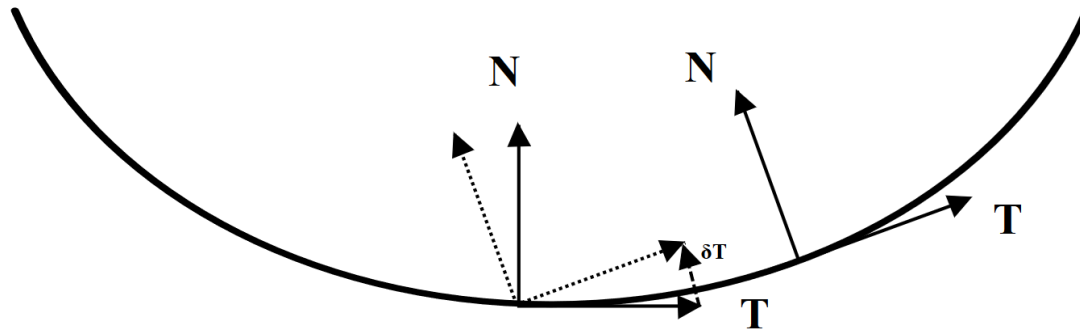
Differential geometry “should” be  
**coordinate-invariant.**

Referring to  $x$  and  $y$  is a hack!  
*(but sometimes convenient...)*

# FRENET-SERRET FORMULAS

On the board:

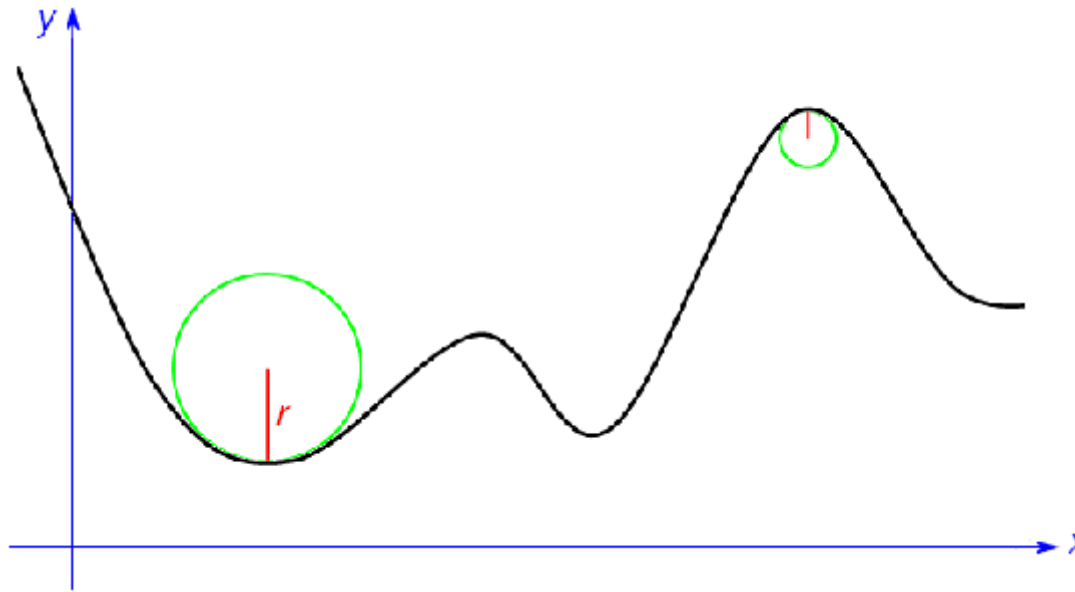
$$\frac{d}{ds} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix} := \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix}$$



[https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret\\_formulas](https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas)

Use coordinates *from* the curve  
to express its shape!

# RADIUS OF CURVATURE



$$r(s) := \frac{1}{\kappa(s)}$$

# RADIUS OF CURVATURE



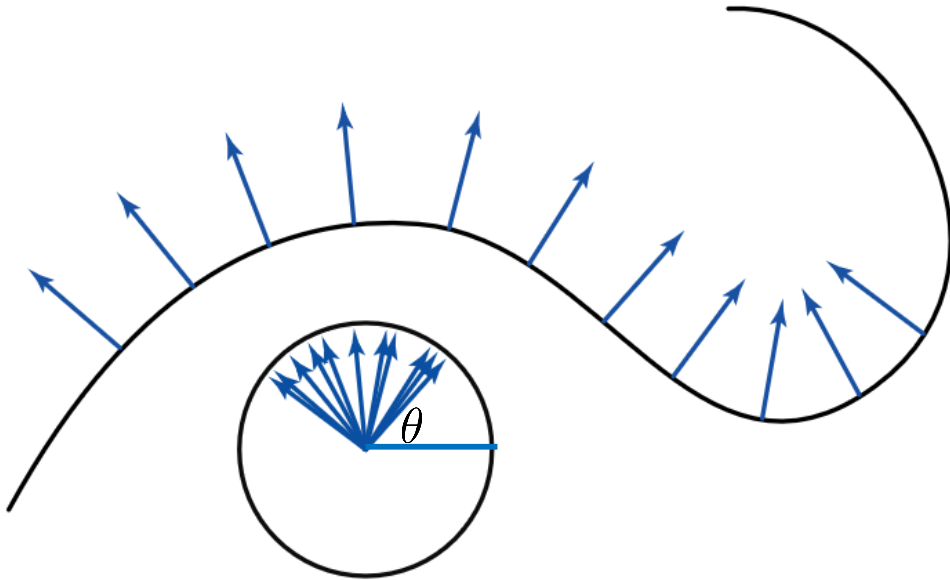
$$r(s) := \frac{1}{\kappa(s)}$$

Fundamental theorem of the local theory of plane curves:

**$k(s)$  characterizes a planar curve up to rigid motion.**



# IDEA OF PROOF



$$T(s) := (\cos \theta(s), \sin \theta(s))$$
$$\implies k(s) := \theta'(s)$$

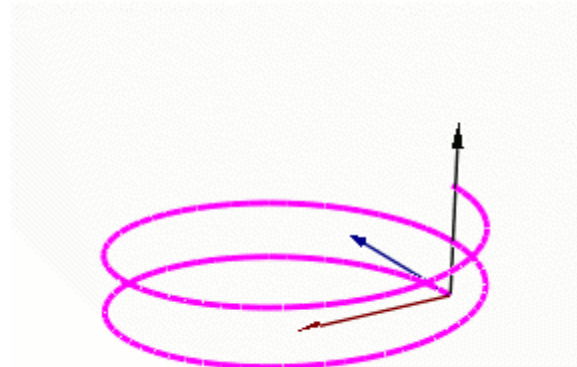
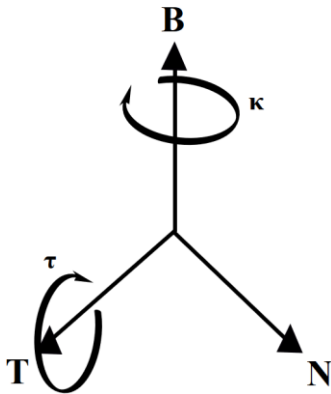
Image from DDG course notes by E. Grinspun

# Provides intuition for curvature

# FRENET FRAME: CURVES IN $\mathbb{R}^3$

$$\frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

- **Binormal:**  $T \times N$
- **Curvature:** In-plane motion
- **Torsion:** Out-of-plane motion

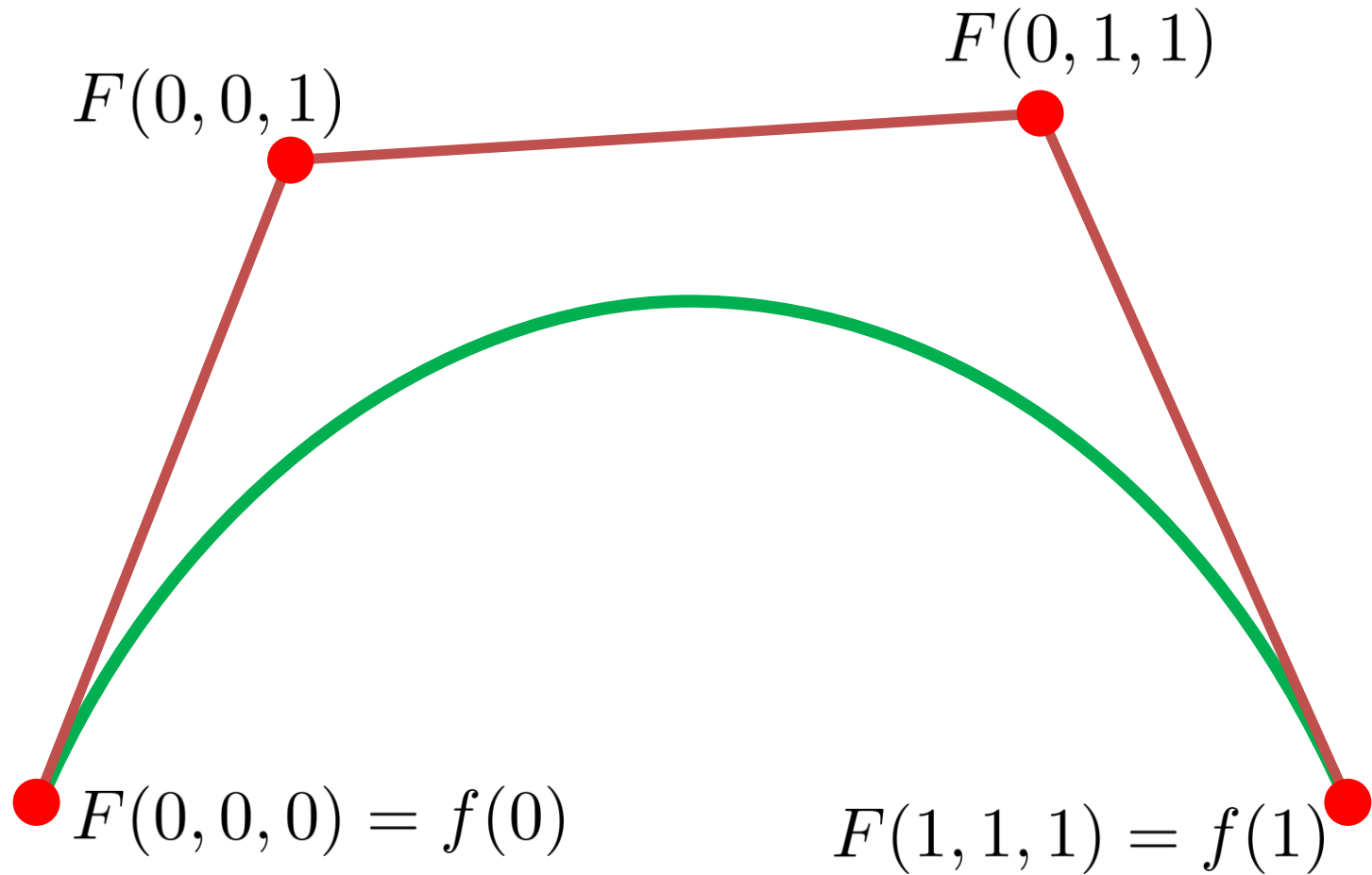


Fundamental theorem of the  
local theory of space curves:

**Curvature and torsion  
characterize a 3D curve up  
to rigid motion.**

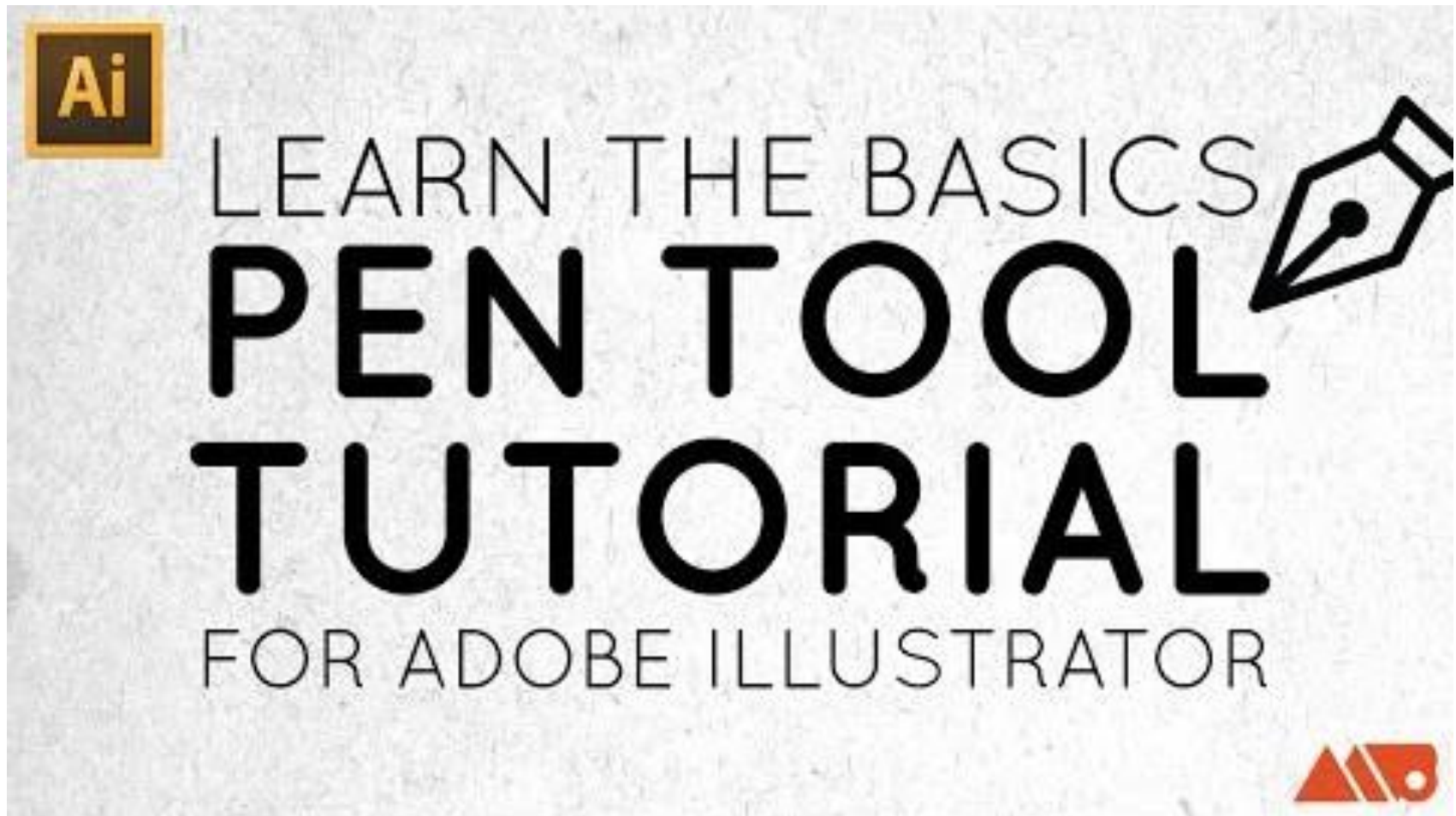
# Representing curves digitally

# TRADITIONAL APPROACH



Piecewise smooth approximations

# MODELING CURVES



# NOT THE ONLY MODELING TOOL

## $\kappa$ -Curves: Interpolation at Local Maximum Curvature

ZHIPEI YAN, Texas A&M University

STEPHEN SCHILLER, Adobe Research

GREGG WILENSKY, Adobe

NATHAN CARR, Adobe Research

SCOTT SCHAEFER, Texas A&M University

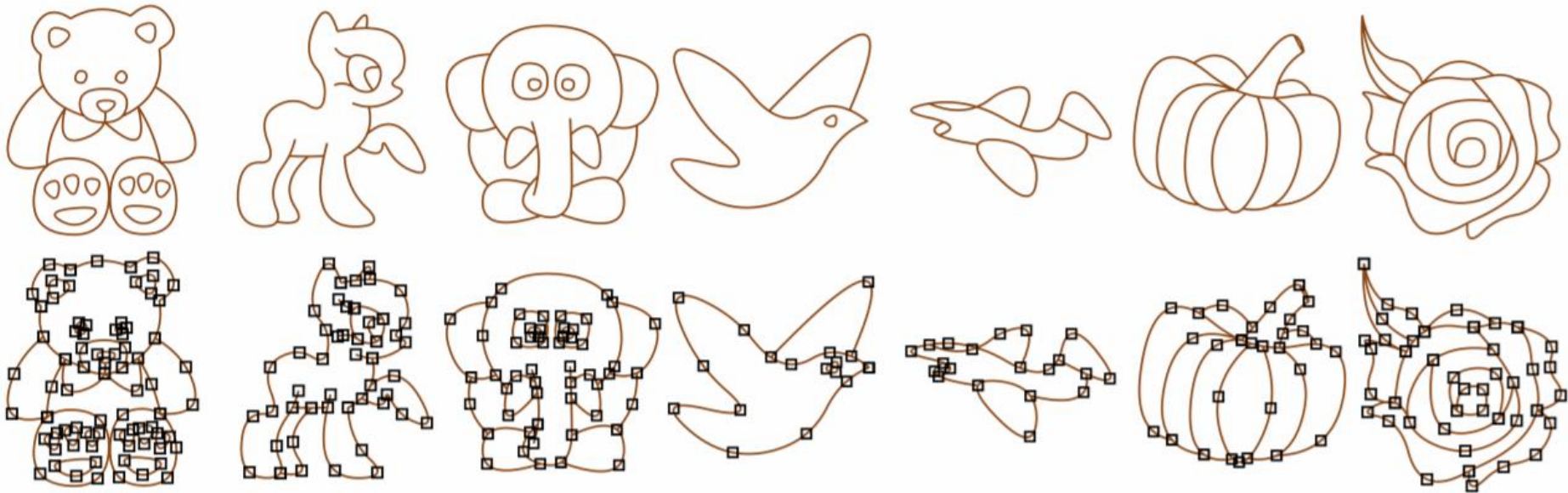


Fig. 1. Top row shows example shapes made from the control points below. In all cases, local maxima of curvature only appear at the control points, and the curves are  $G^2$  almost everywhere.

# QUESTION

What is the arc length of a cubic Bézier curve?

$$\int_a^b \|\gamma'(t)\| dt$$



# QUESTION

What is the arc length of a cubic Bézier curve?

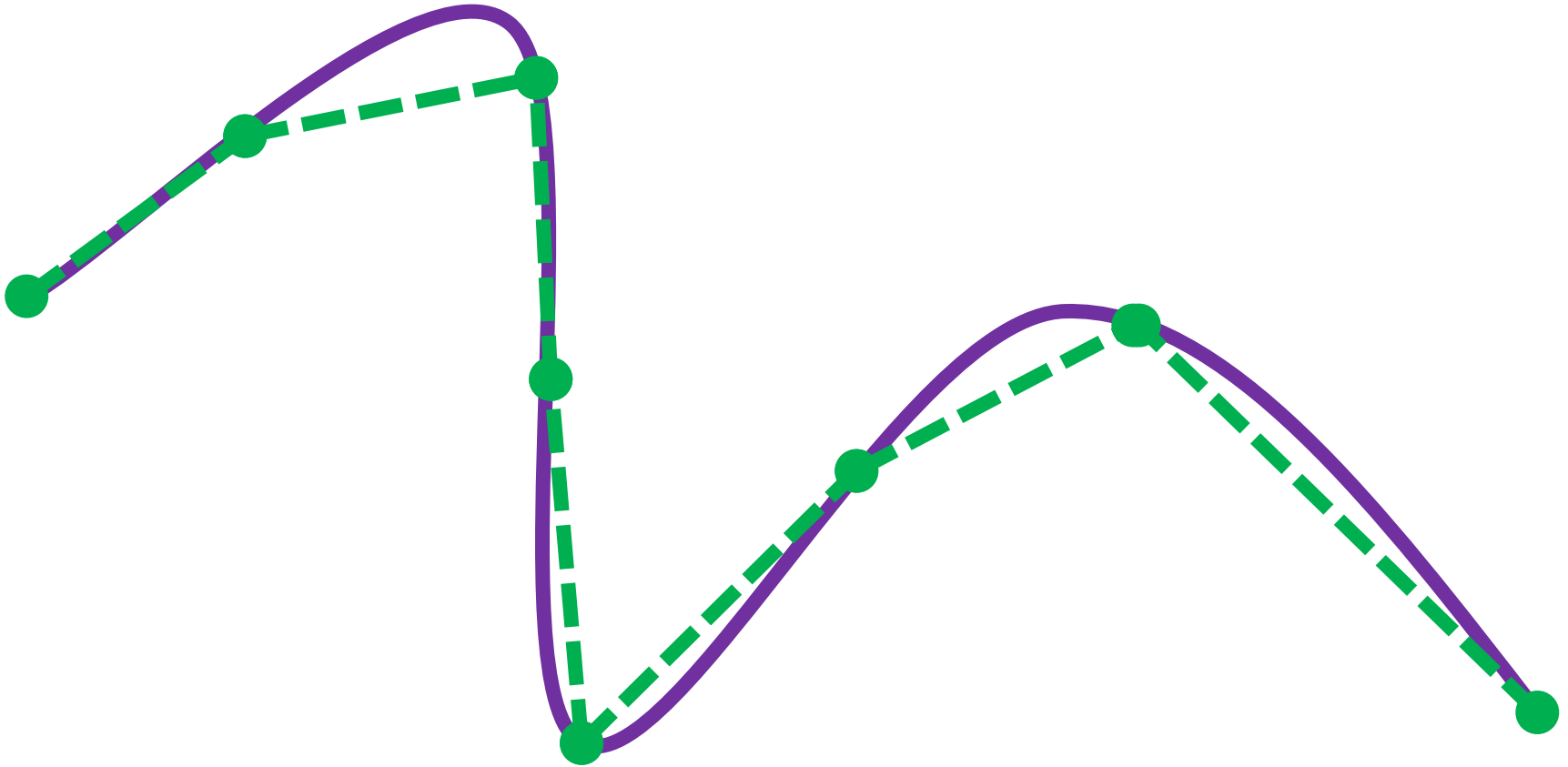
$$\int_a^b \|\gamma'(t)\| dt$$

**Not known in closed form.**

# ONLY APPROXIMATIONS ANYWAY

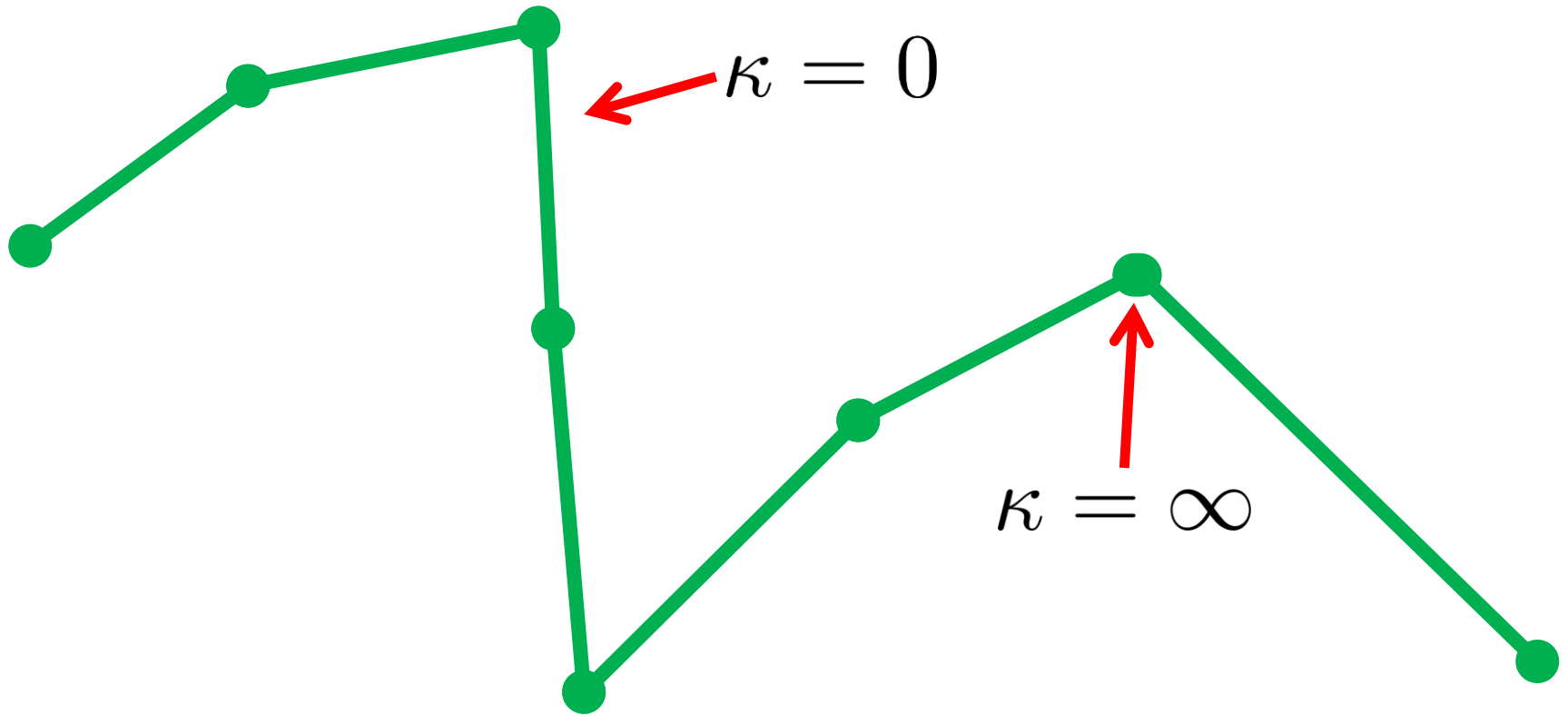
$$\{\text{Bézier curves}\} \subsetneq \{\gamma : \mathbb{R} \rightarrow \mathbb{R}^3\}$$

# ANOTHER REASONABLE APPROXIMATION



Piecewise linear

# BIG PROBLEM



**Boring differential structure**

# FINITE DIFFERENCE APPROACH

$$f'(x) \approx \frac{1}{h} [f(x+h) - f(x)]$$

THEOREM: As  $\Delta h \rightarrow 0$ , [insert statement].

# REALITY CHECK

$$f'(x) \approx \frac{1}{h} [f(x+h) - f(x)]$$

THEOREM  statement].

# TWO KEY CONSIDERATIONS

- Convergence to continuous theory
- Discrete behavior

# GOAL

Examine discrete theories  
of differentiable curves.

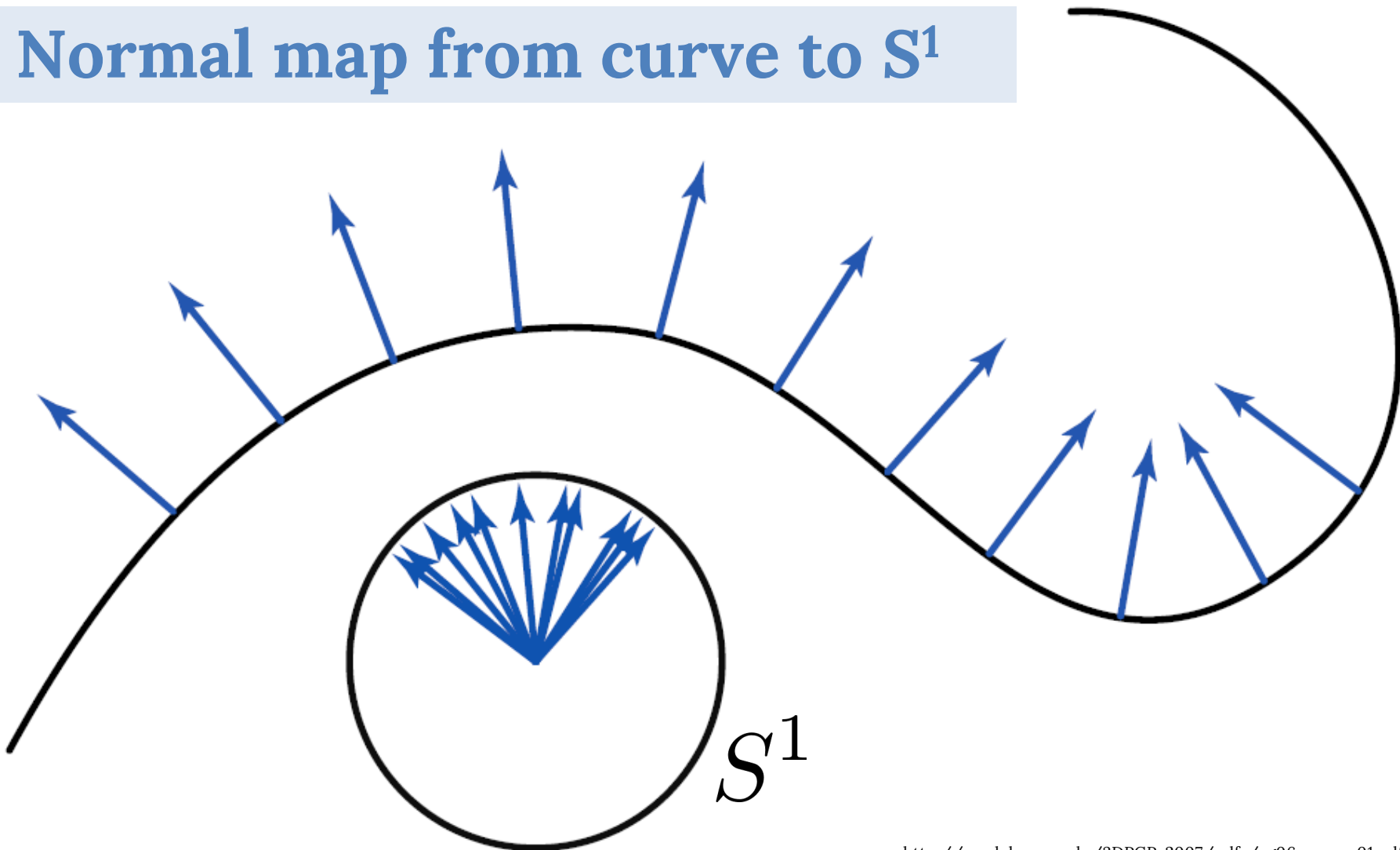


# GOAL

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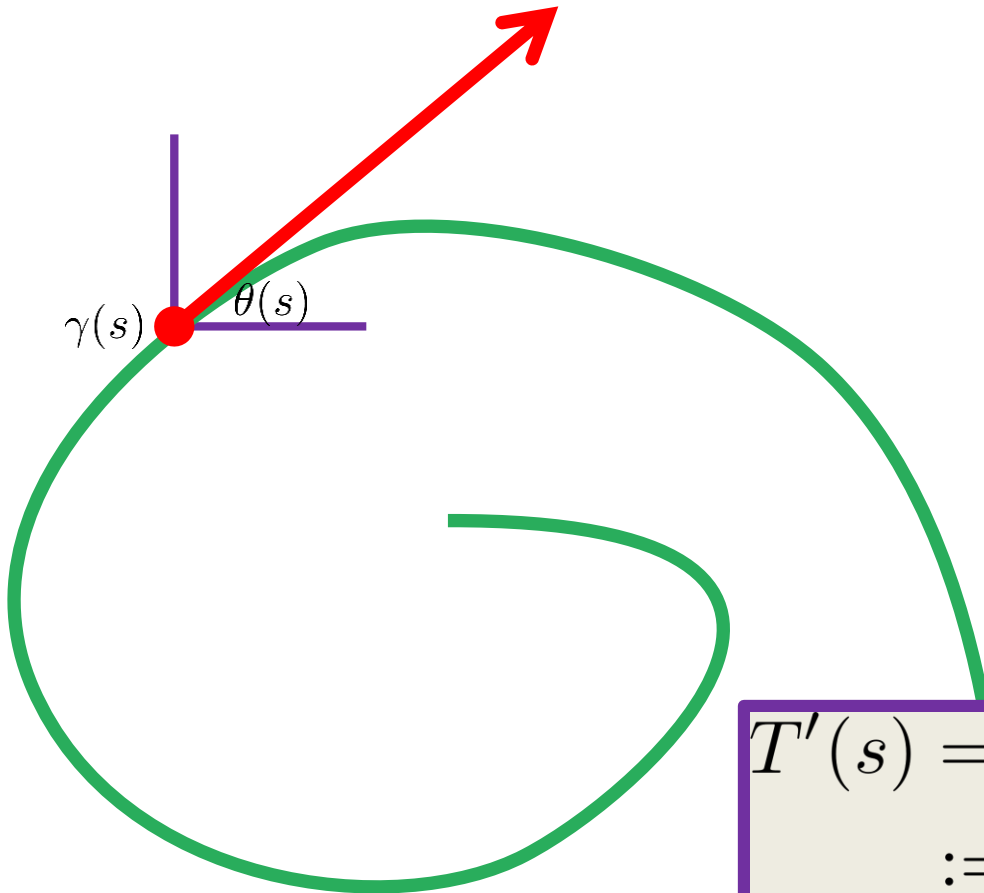
# GAUSS MAP

Normal map from curve to  $S^1$



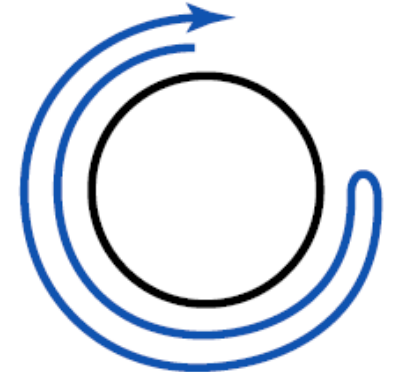
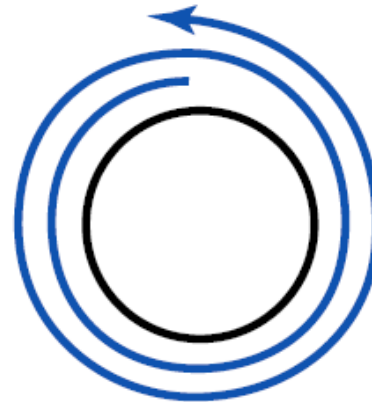
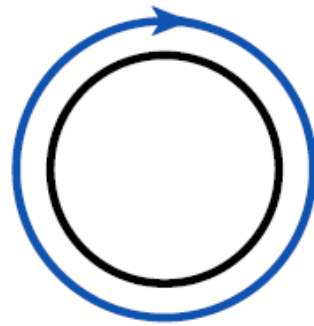
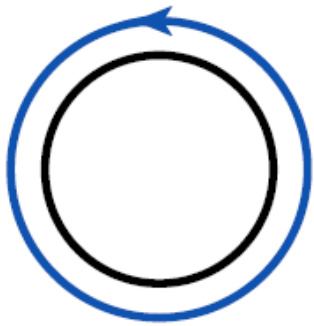
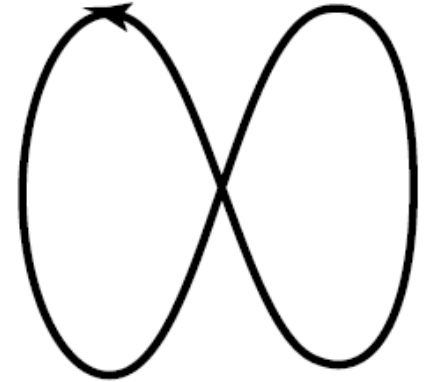
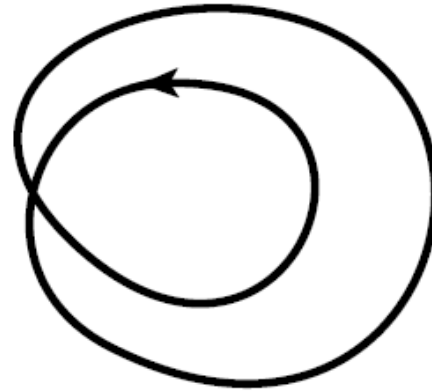
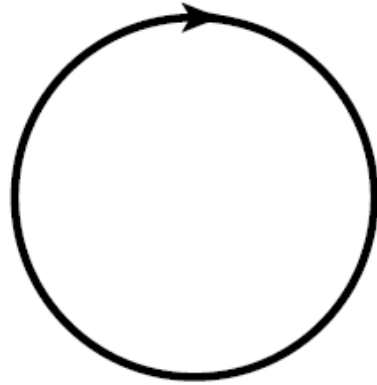
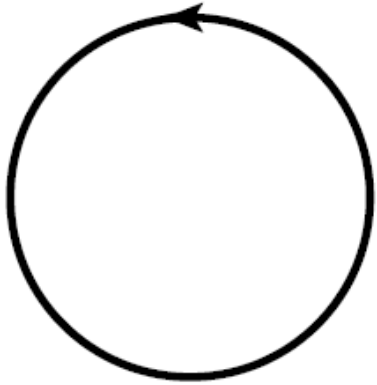
# SIGNED CURVATURE ON PLANE CURVES

$$T(s) = (\cos \theta(s), \sin \theta(s))$$



$$\begin{aligned} T'(s) &= \theta'(s)(-\sin \theta(s), \cos \theta(s)) \\ &:= \kappa(s)N(s) \end{aligned}$$

# TURNING NUMBERS



+1

-1

+2

0

# RECOVERING THETA

$$\theta'(s) = \kappa(s)$$



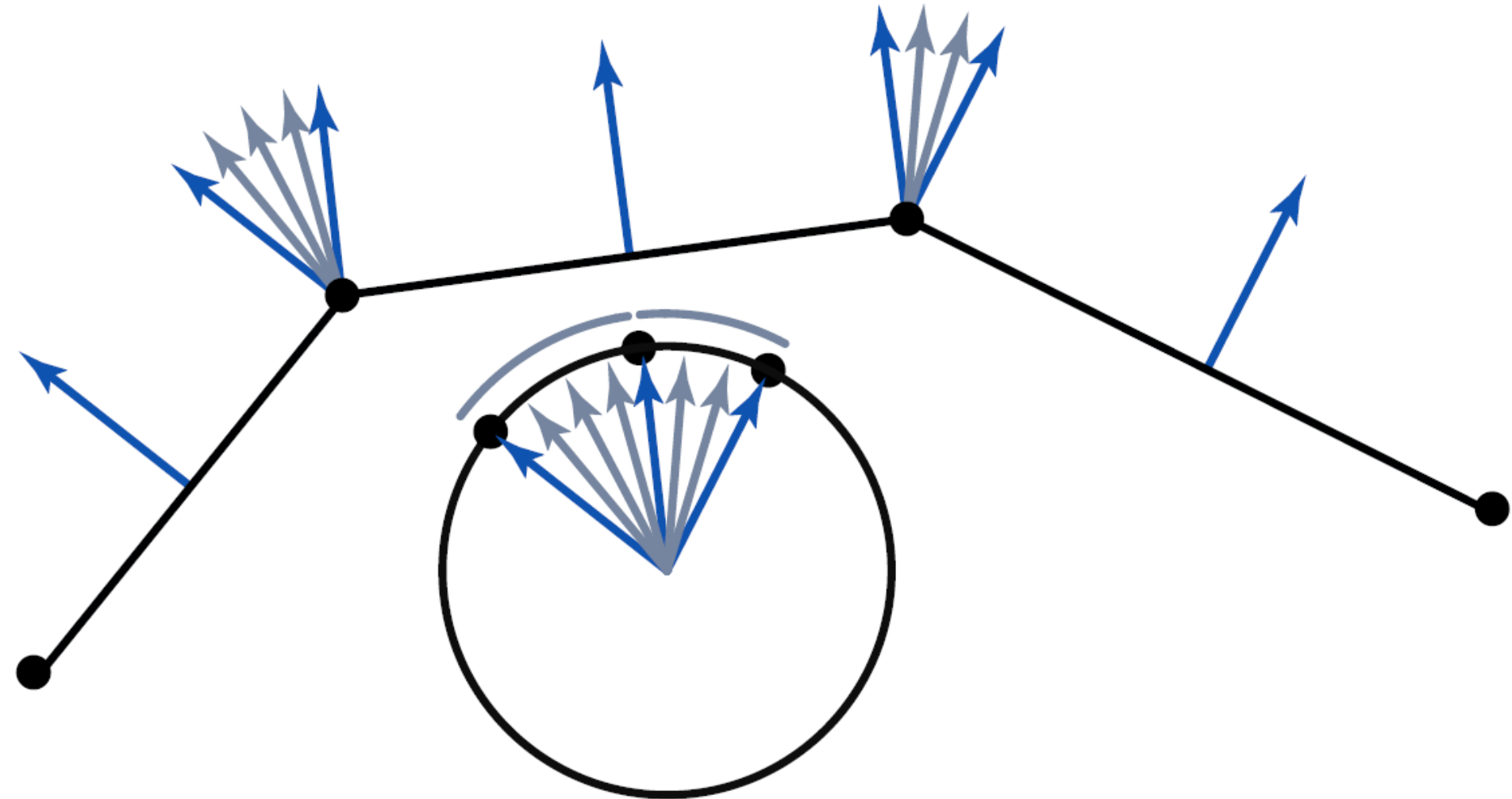
$$\Delta\theta = \int_{s_0}^{s_1} \kappa(s) ds$$

# TURNING NUMBER THEOREM

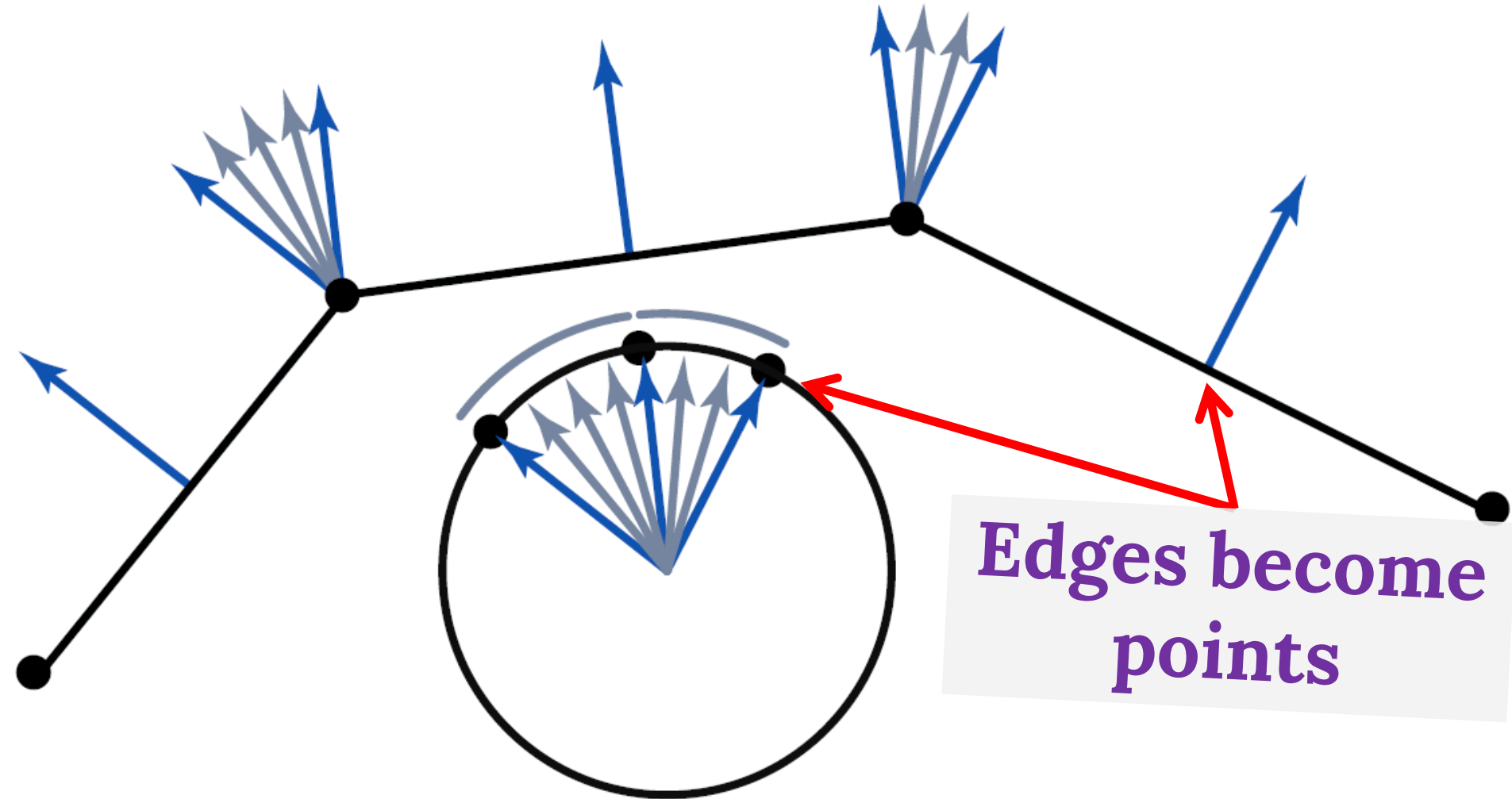
$$\int_{\Omega} \kappa(s) ds = 2\pi k$$

A “global” theorem!

# DISCRETE GAUSS MAP

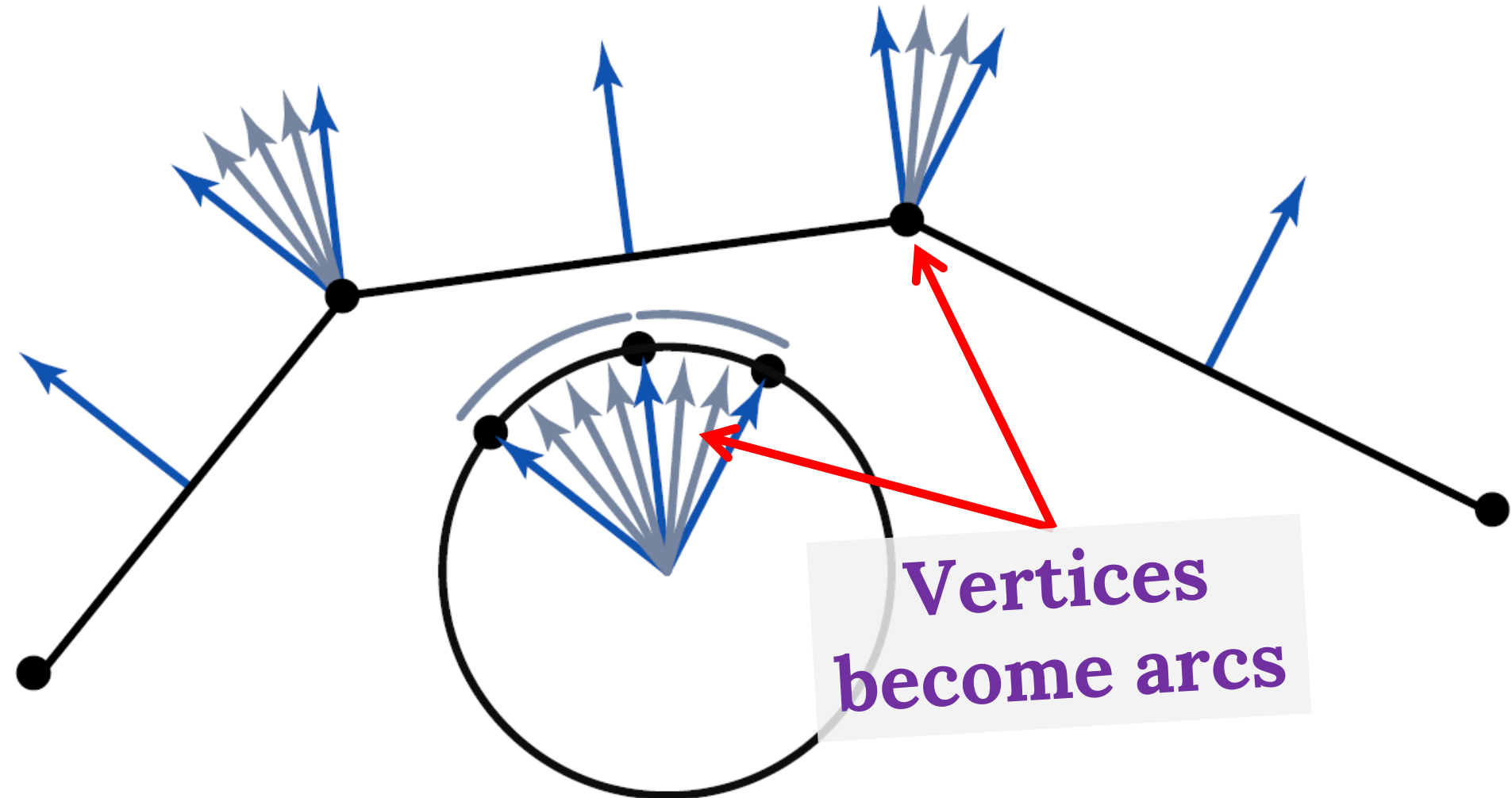


# DISCRETE GAUSS MAP

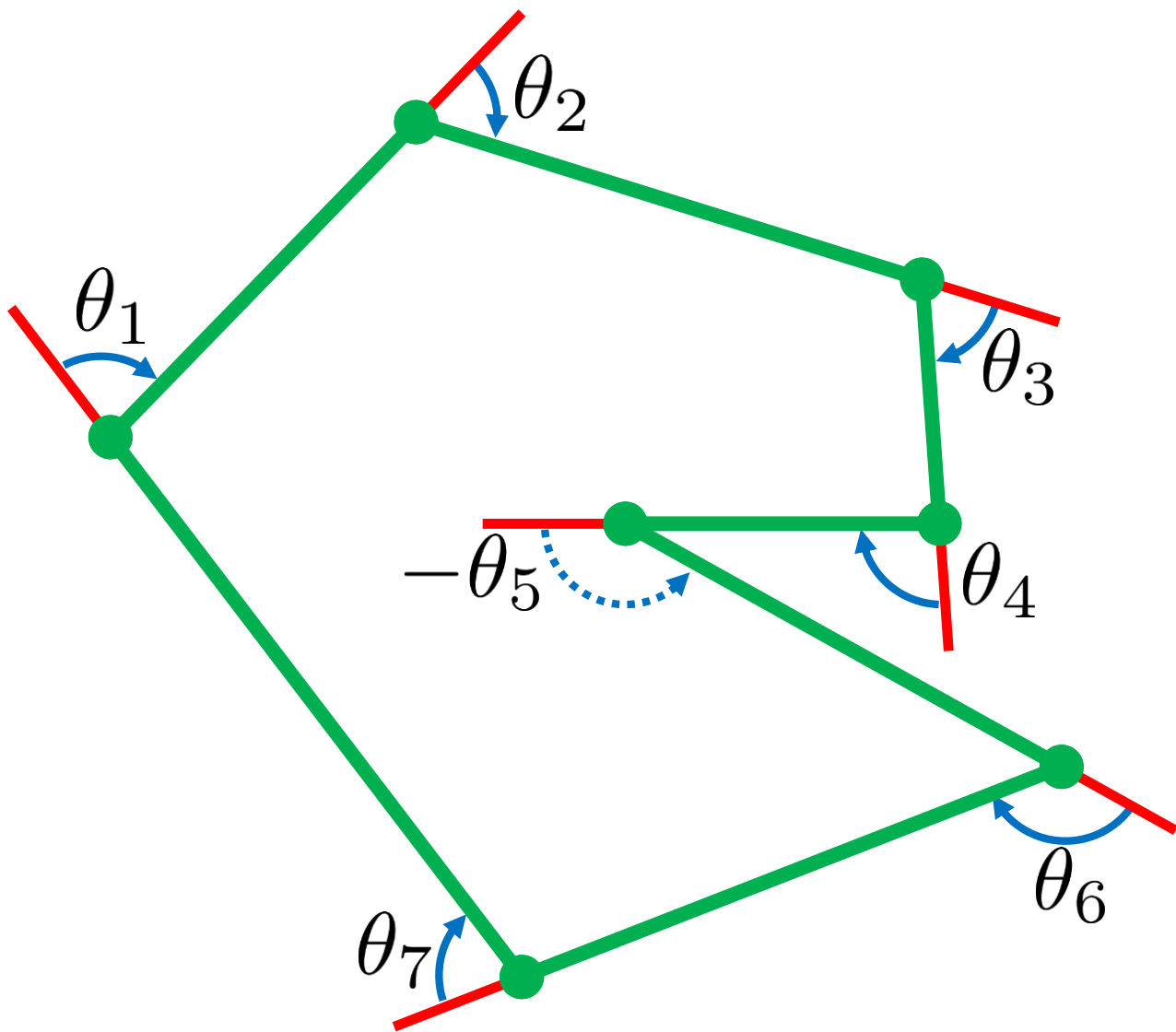




# DISCRETE GAUSS MAP

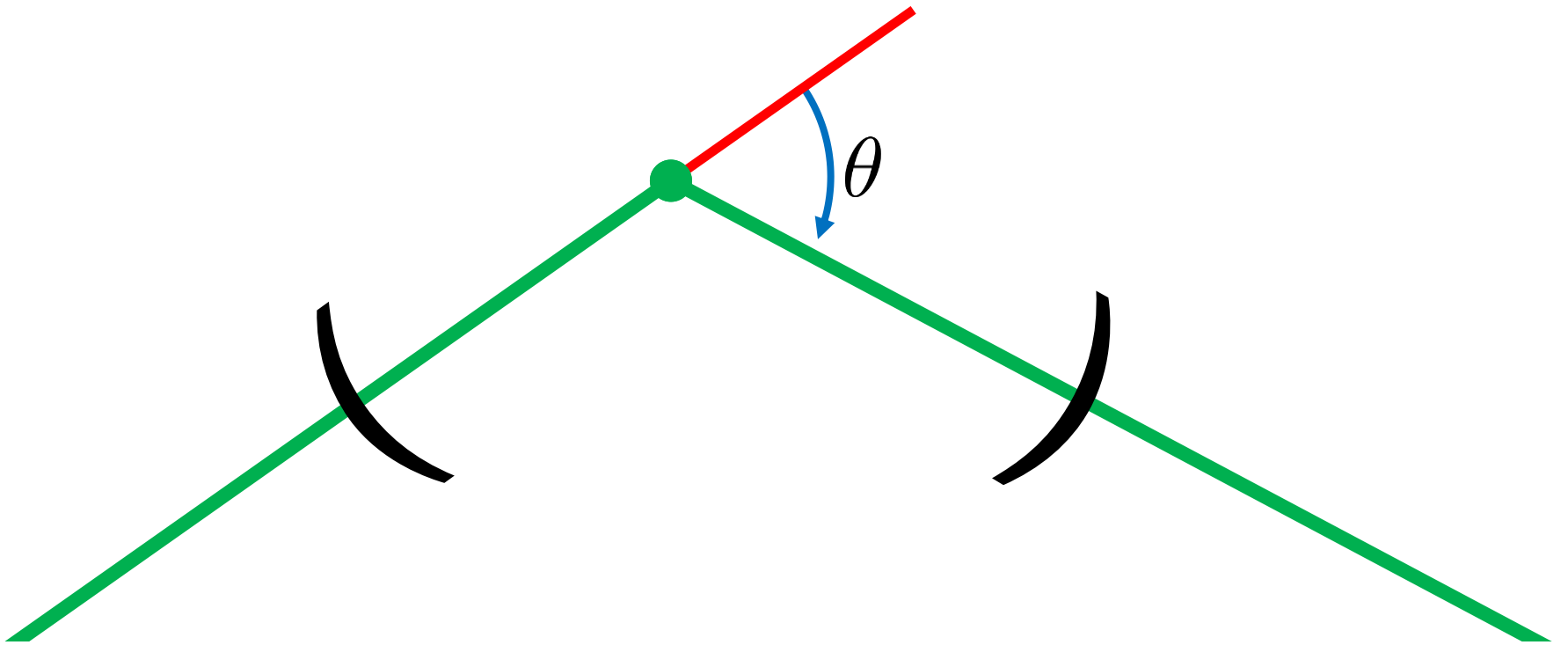


# KEY OBSERVATION



$$\sum_i \theta_i = 2\pi k$$

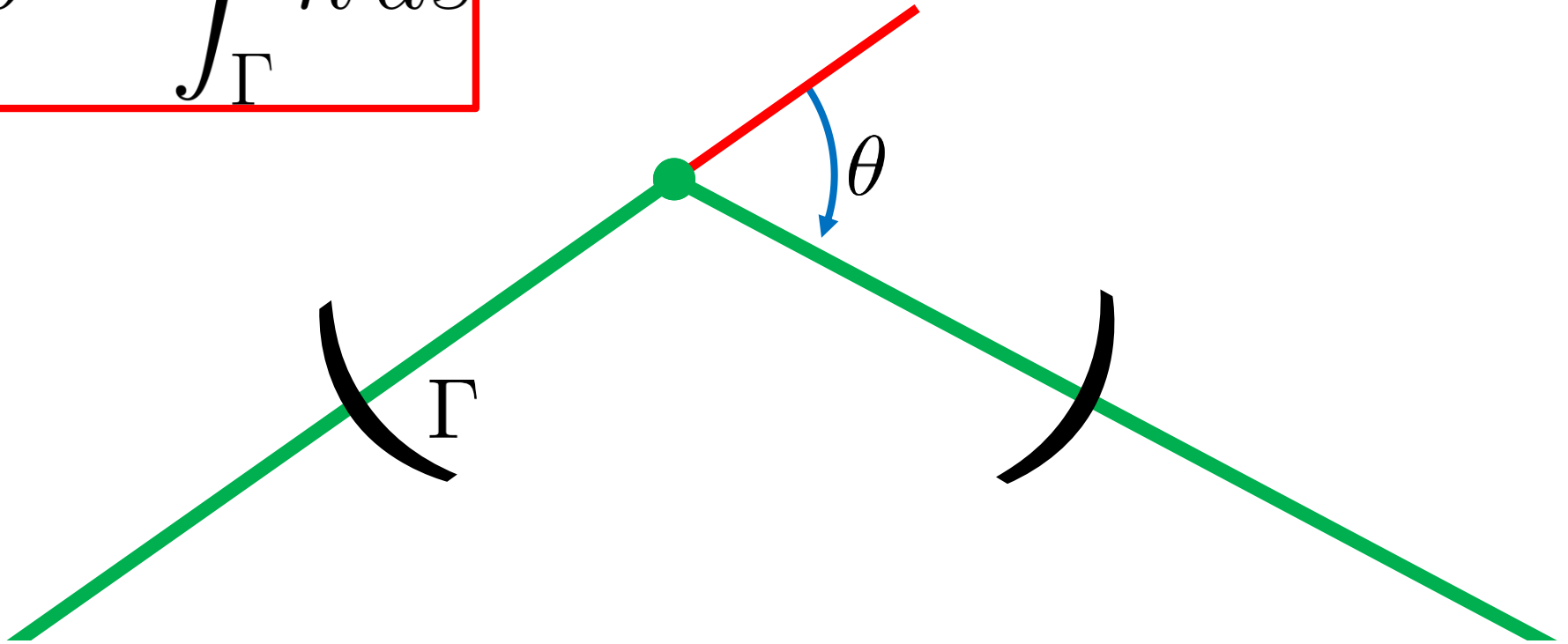
# WHAT'S GOING ON?



Total change in curvature

# WHAT'S GOING ON?

$$\theta = \int_{\Gamma} \kappa ds$$

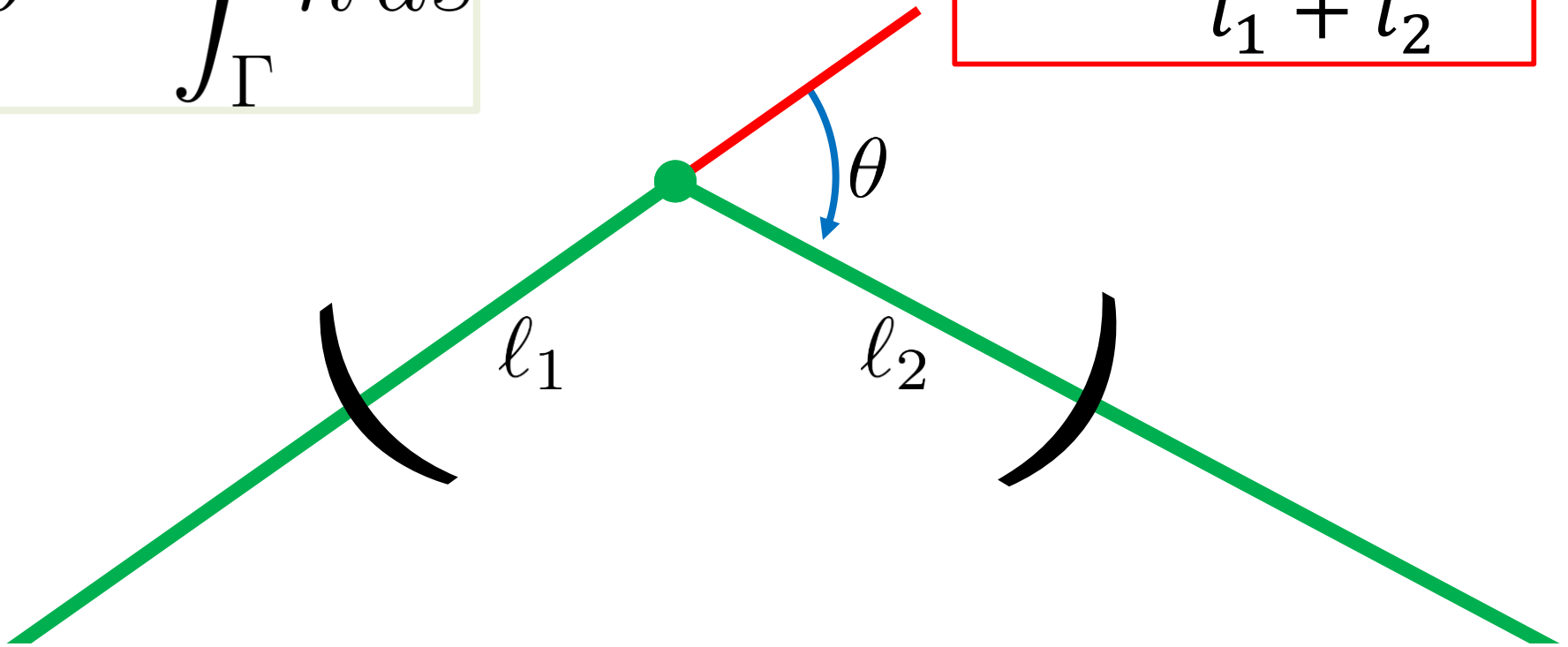


Total change in curvature

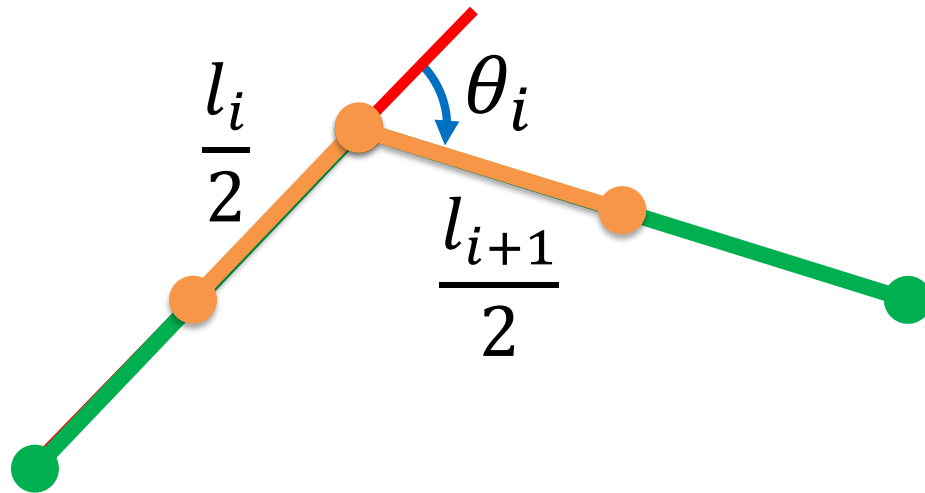
# WHAT'S GOING ON?

$$\theta = \int_{\Gamma} \kappa ds$$

$$\kappa \approx \frac{2\theta}{l_1 + l_2}$$



Total change in curvature



Continuous:

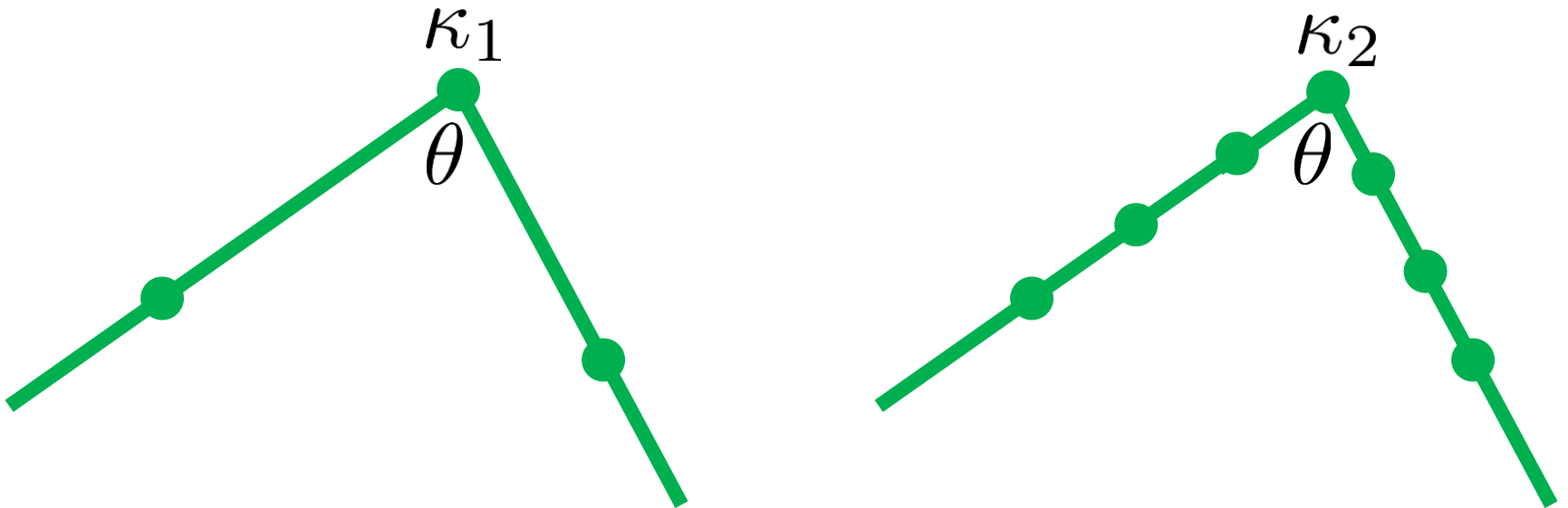
$$k(s) = \theta'(s)$$

Discrete (*finite difference*):

$$k_i = \frac{\theta_i}{(l_i + l_{i+1})/2}$$

# INTERESTING DISTINCTION

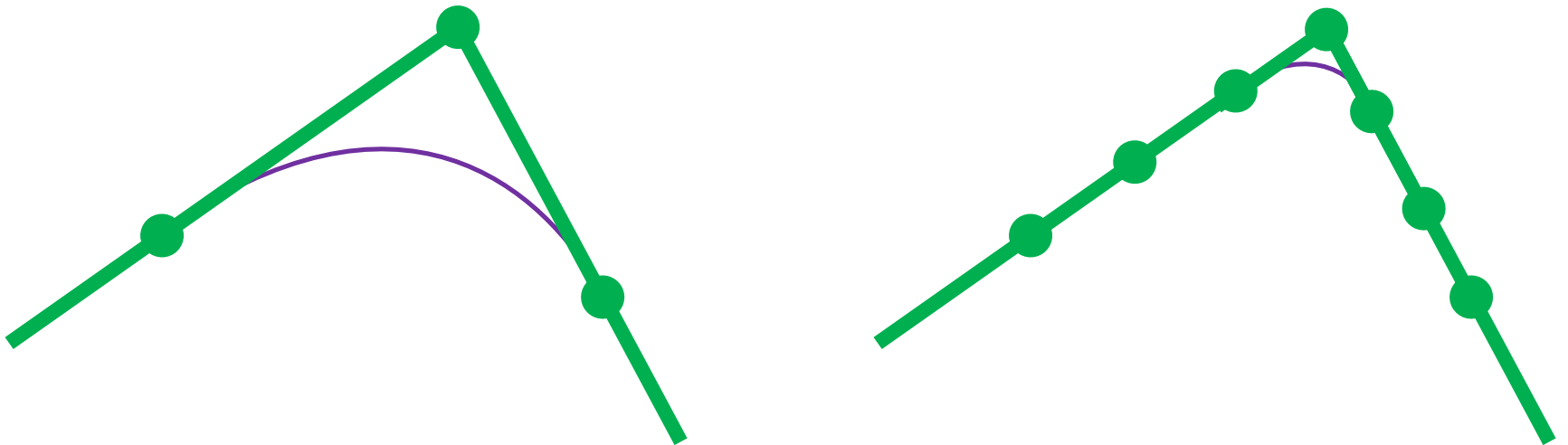
$$\kappa_1 \neq \kappa_2$$



Same integrated curvature

# INTERESTING DISTINCTION

$$\kappa_1 \neq \kappa_2$$



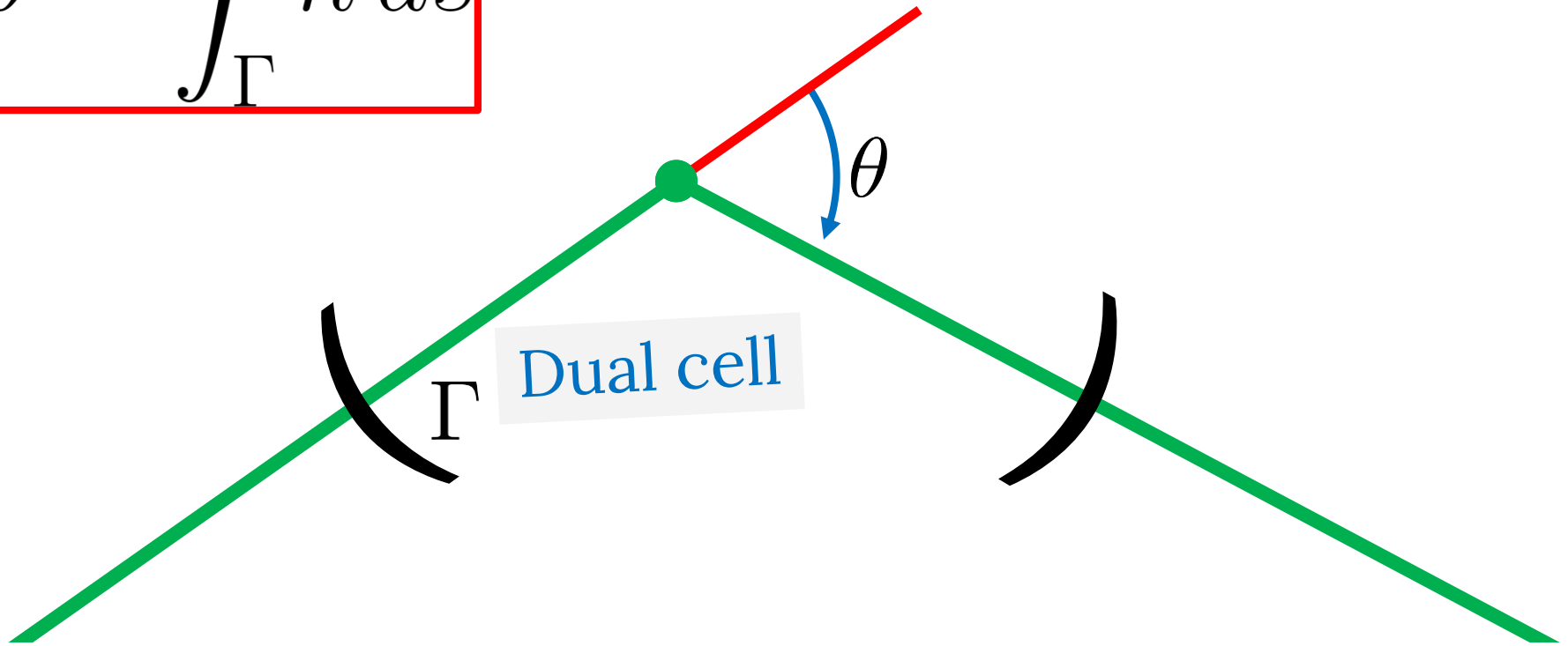
Same integrated curvature



# WHAT'S GOING ON?

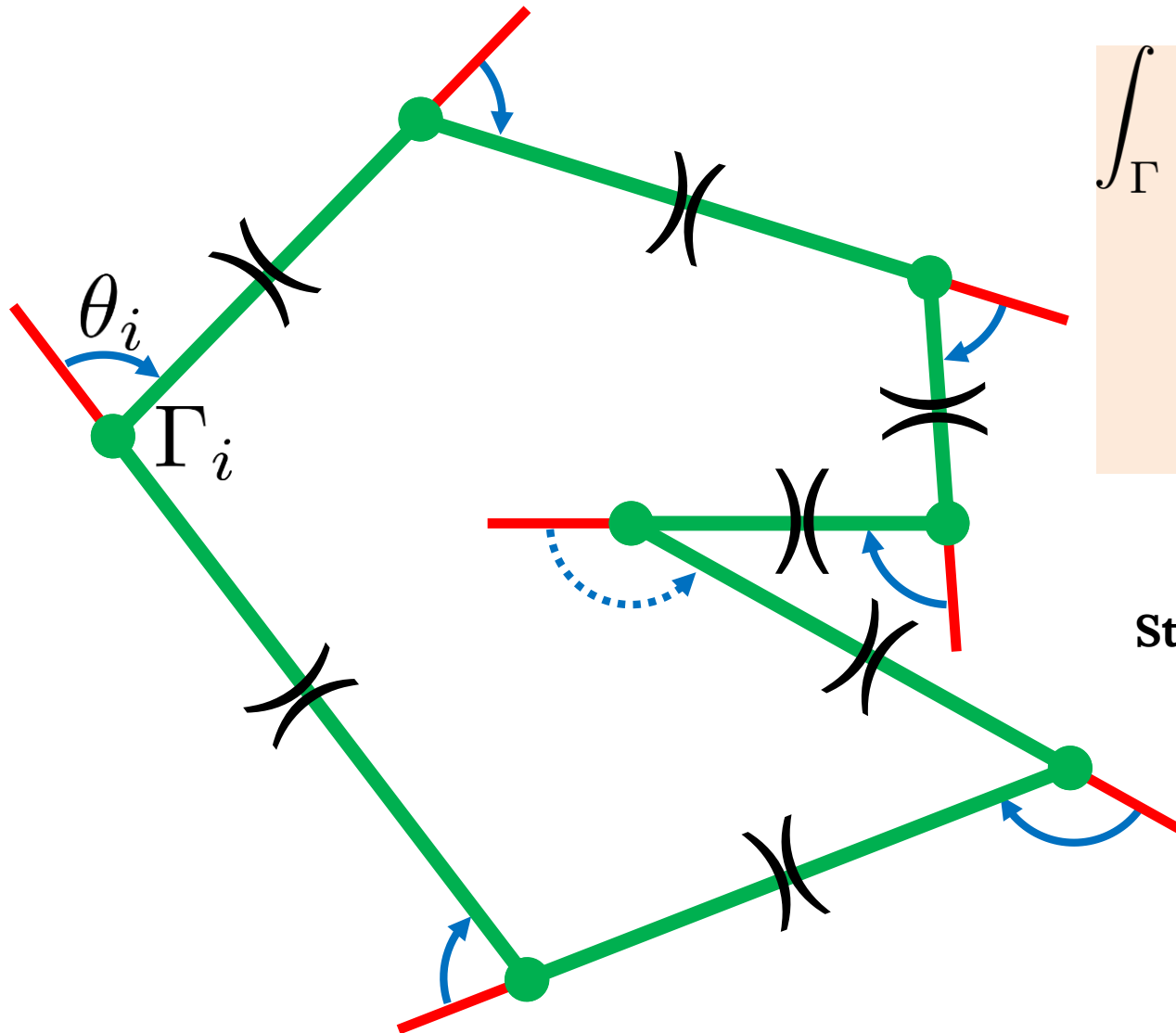
$$\theta = \int_{\Gamma} \kappa ds$$

Integrated quantity



Total change in curvature

# DISCRETE TURNING ANGLE THEOREM

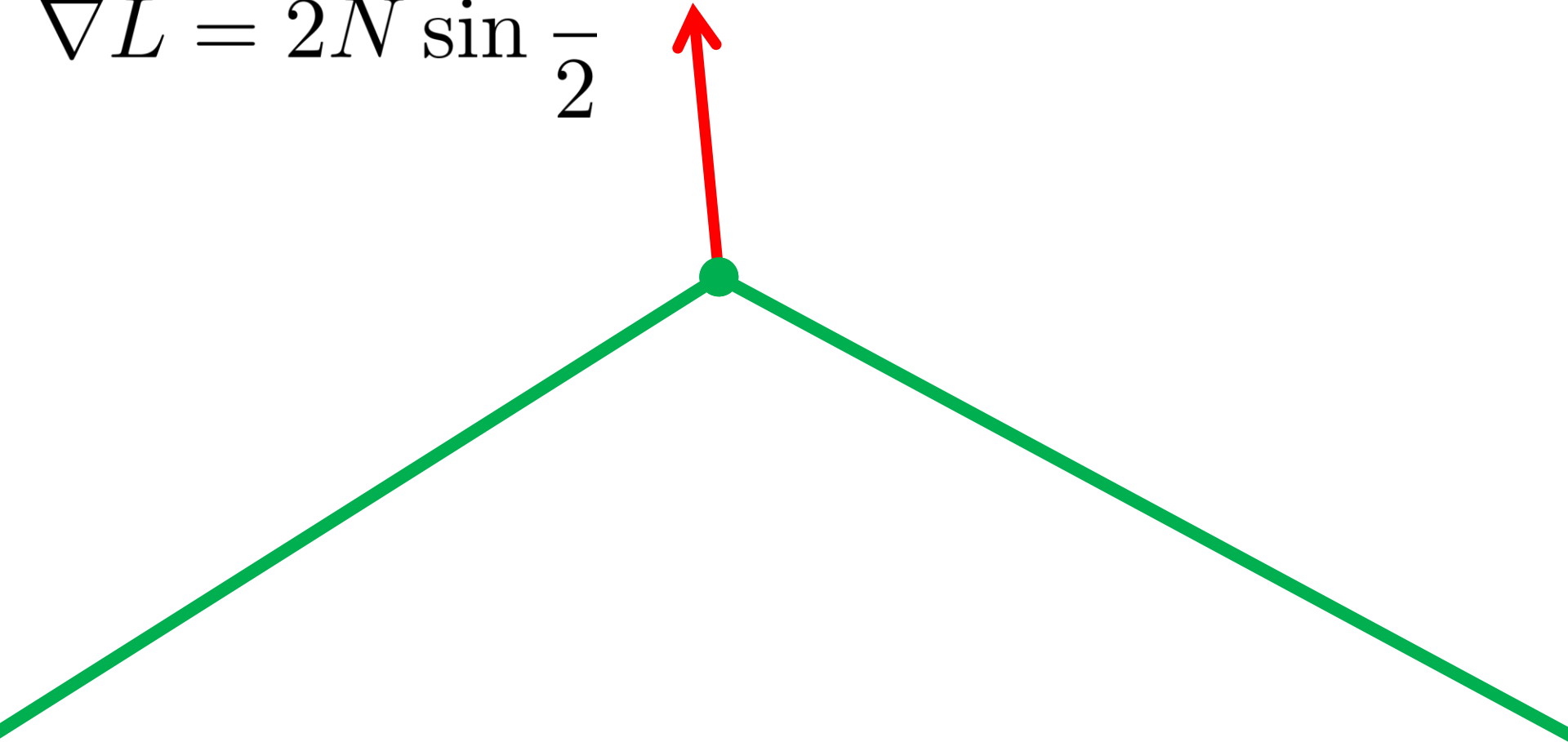


$$\begin{aligned}\int_{\Gamma} \kappa ds &= \sum_i \int_{\Gamma_i} \kappa ds \\ &= \sum_i \theta_i \\ &= 2\pi k\end{aligned}$$

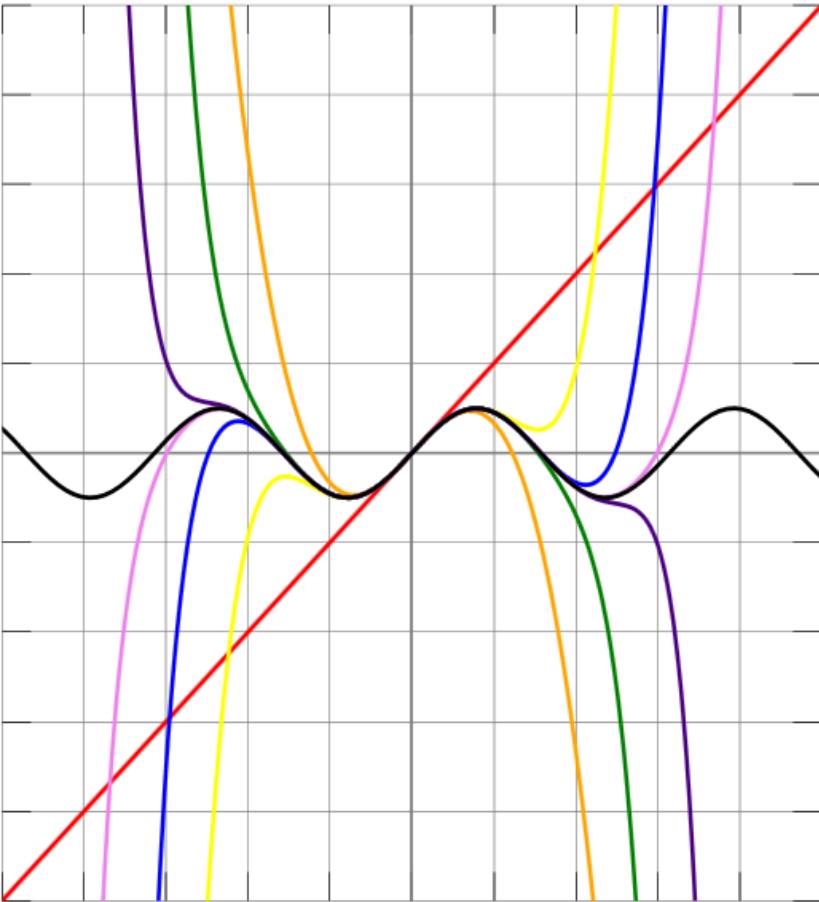
**Structure Preservation!**

# ANOTHER DISCRETIZATION

$$\nabla L = 2N \sin \frac{\theta}{2}$$



# FOR SMALL $\theta$



$$\begin{aligned} 2 \sin \frac{\theta}{2} &\approx 2 \cdot \frac{\theta}{2} \\ &= \theta \end{aligned}$$

Same behavior in the limit

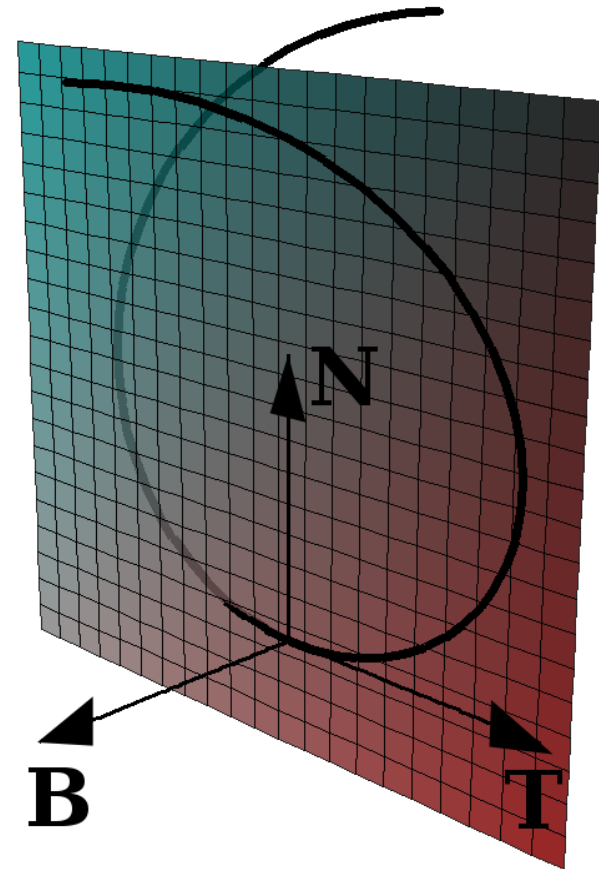
Does discrete curvature  
converge in limit?

*Yes!*

...workflow

# Discretizing 3D curves

# FRENET FRAME

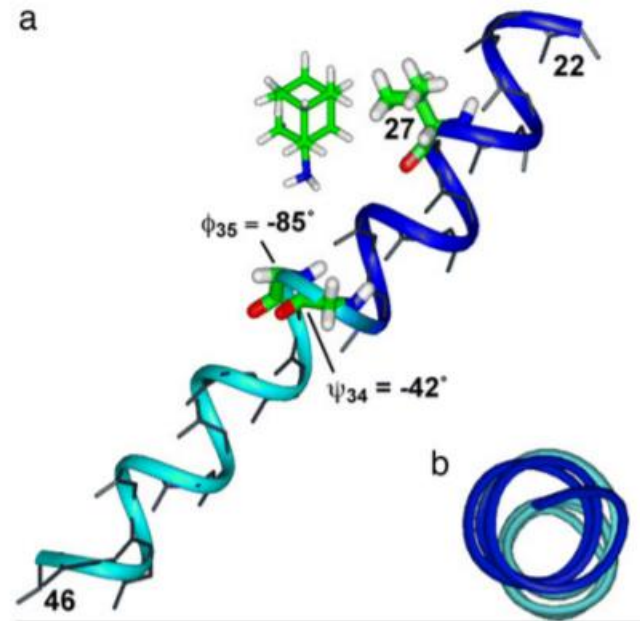


$$\frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

# APPLICATION



**NMR  
scanner**



**Kinked alpha helix**

**Structure Determination of Membrane Proteins Using Discrete Frenet Frame  
and Solid State NMR Restraints**

Achuthan and Quine

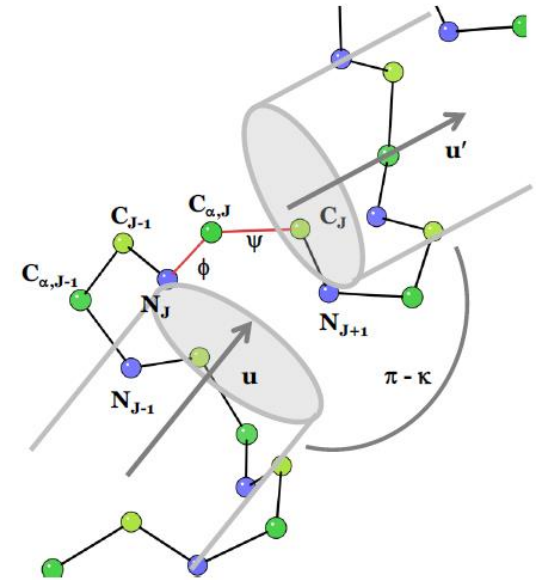
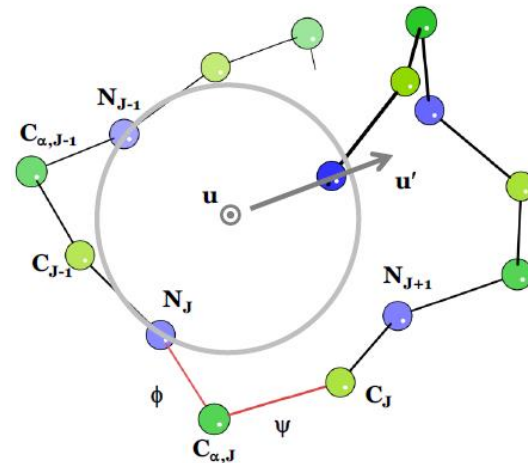
*Discrete Mathematics and its Applications*, ed. M. Sethumadhavan (2006)



# APPLICATION



**NMR  
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Achuthan and Quine

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# POTENTIAL DISCRETIZATION

$$T_j = \frac{p_{j+1} - p_j}{\|p_{j+1} - p_j\|}$$

$$B_j = t_{j-1} \times t_j$$

$$N_j = b_j \times t_j$$

**Discrete Frenet  
frame**

$$T_k = R(B_k, \theta_k) T_{k-1}$$

$$B_{k+1} = R(T_k, \phi_k) B_k$$

**“Bond and torsion  
angles”**

**(derivatives converge to  
 $\kappa$  and  $\tau$ , resp.)**

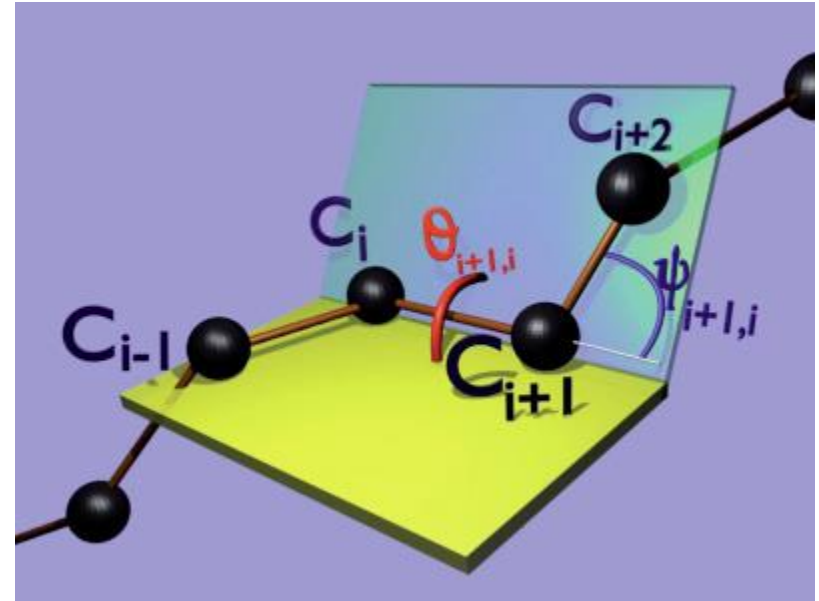
Discrete frame introduced in:

**The resultant electric moment of complex molecules**

Eyring, Physical Review, 39(4):746–748, 1932.

# TRANSFER MATRIX

$$\begin{pmatrix} T_{i+1} \\ N_{i+1} \\ B_{i+1} \end{pmatrix} = R_{i+1,i} \begin{pmatrix} T_i \\ N_i \\ B_i \end{pmatrix}$$



**Discrete construction that works for fractal curves and converges in continuum limit.**

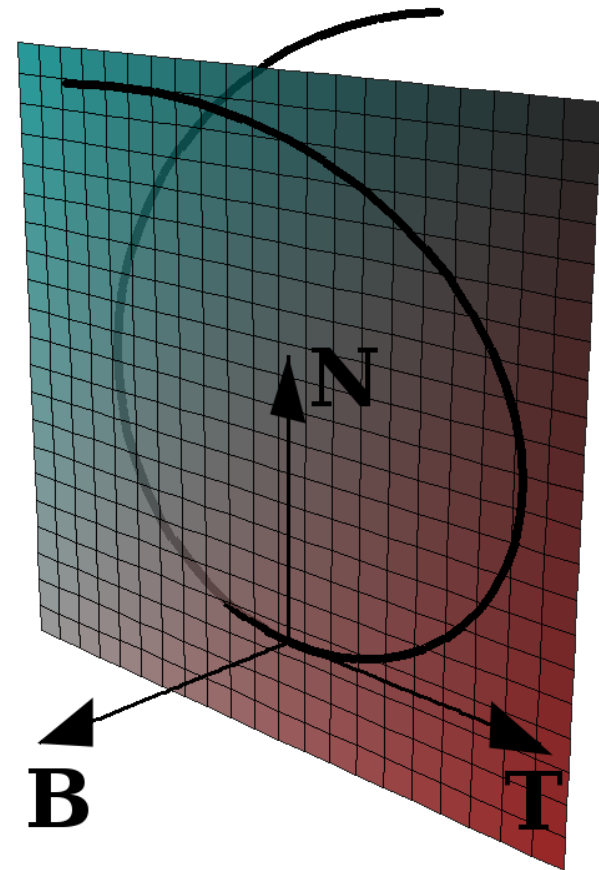
**Discrete Frenet Frame, Inflection Point Solitons, and Curve Visualization with Applications to Folded Proteins**

Hu, Lundgren, and Niemi  
*Physical Review E* 83 (2011)

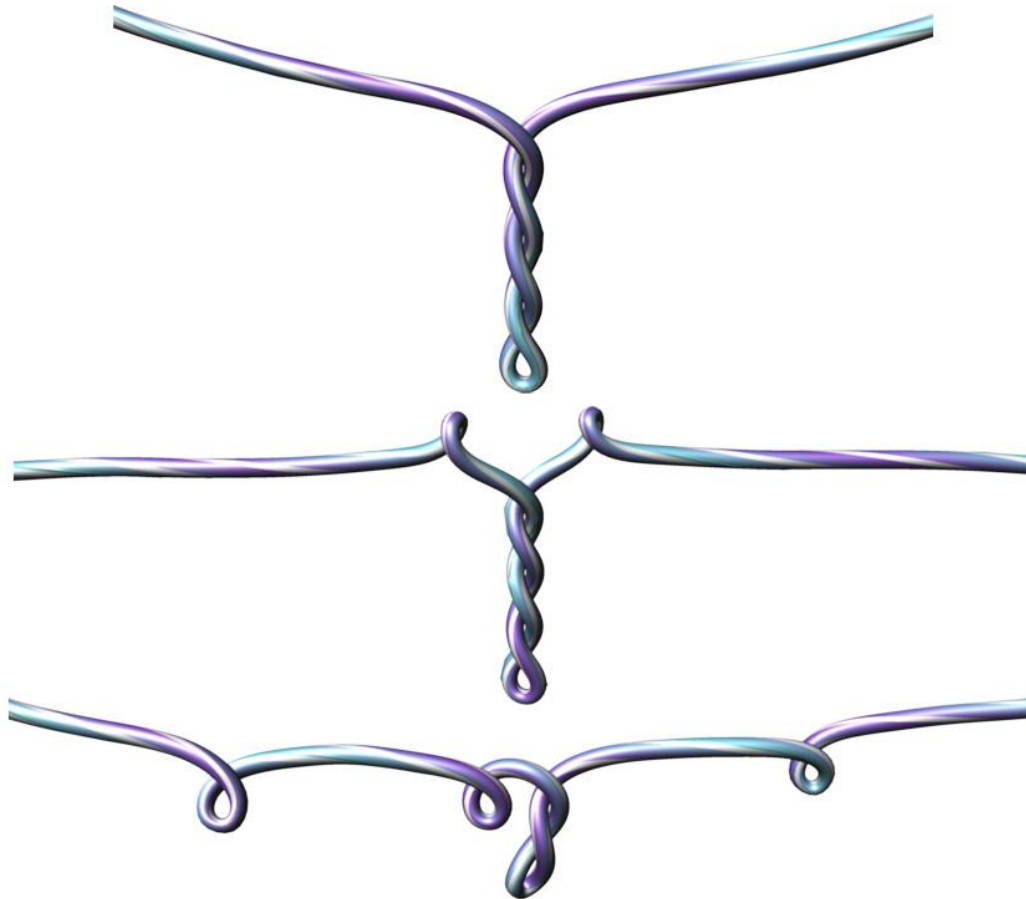
# FRENET FRAME: ISSUE

$$\kappa = 0?$$

$$\frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \boxed{\kappa} & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$



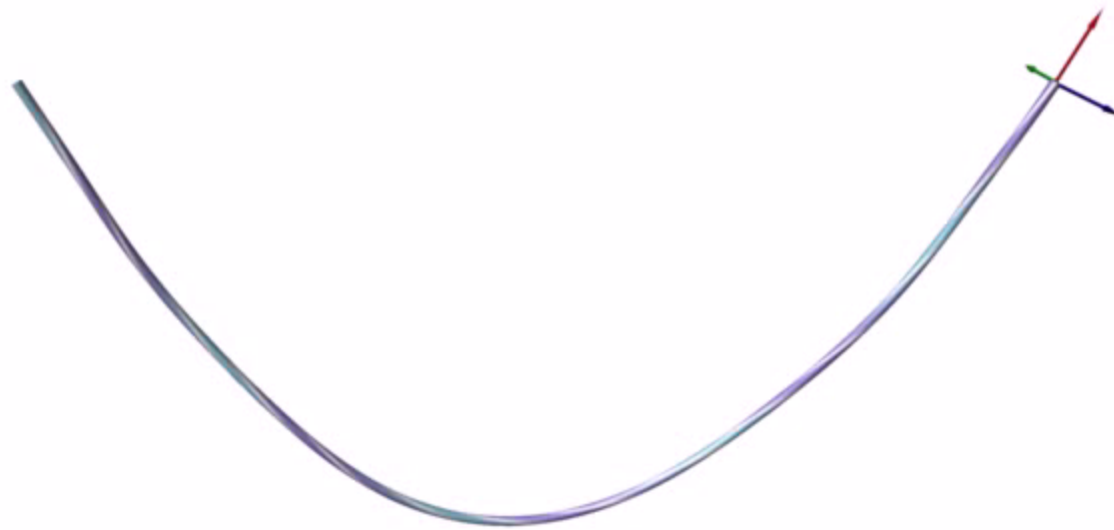
# SEGMENTS NOT ALWAYS ENOUGH



## Discrete Elastic Rods

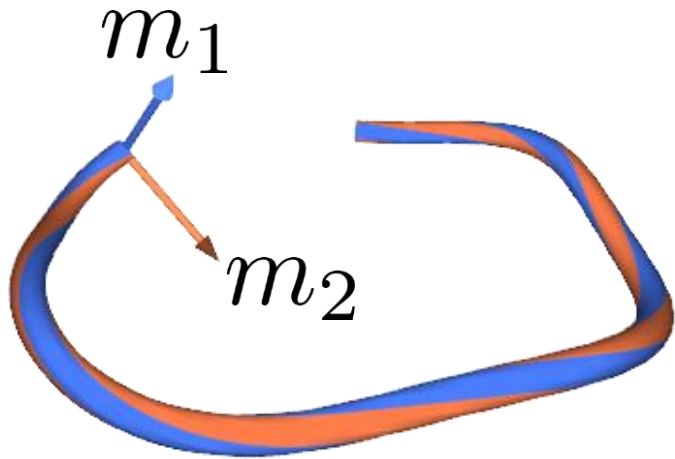
Bergou, Wardetzky, Robinson, Audoly, and Grinspun  
SIGGRAPH 2008

# SIMULATION GOAL





# ADAPTED FRAMED CURVE



$$\Gamma = \{\gamma(s); T, m_1, m_2\}$$

Material  
frame

Normal part encodes twist



# BENDING ENERGY

$$E_{\text{bend}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \alpha \kappa^2 ds$$

Punish turning the steering wheel

---

$$\begin{aligned} \kappa N &= T' \\ &= (T' \cdot T)T + (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \\ &= (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \\ &:= \omega_1 m_1 + \omega_2 m_2 \end{aligned}$$

# BENDING ENERGY

$$E_{\text{bend}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \alpha(\omega_1^2 + \omega_2^2) ds$$

Punish turning the steering wheel

---

$$\kappa N = T'$$

$$= (T' \cdot T)T + (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2$$

$$= (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2$$

$$:= \omega_1 m_1 + \omega_2 m_2$$

# TWISTING ENERGY

$$E_{\text{twist}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \beta m^2 ds$$

Punish non-tangent change in material frame

---

$$\begin{aligned} m &:= m'_1 \cdot m_2 \\ &= \frac{d}{dt} (m_1 \cdot m_2) - m_1 \cdot m'_2 \\ &= -m_1 \cdot m'_2 \end{aligned}$$

# TWISTING ENERGY

$$E_{\text{twist}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \beta m^2 ds$$

Punish non-tangent change in material frame

---

Swapping  $m_1$  and  $m_2$  does  
not affect  $E_{\text{twist}}$ !

# WHICH BASIS TO USE

## THERE IS MORE THAN ONE WAY TO FRAME A CURVE

RICHARD L. BISHOP

The Frenet frame of a 3-times continuously differentiable (that is,  $C^3$ ) non-degenerate curve in euclidean space has long been the standard vehicle for analysing properties of the curve invariant under euclidean motions. For arbitrary moving frames, that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the curve parameter in terms of the frame itself, and due to orthonormality the coefficient matrix is always skew-symmetric. Thus it generally has three nonzero entries. The Frenet frame gains part of its special significance from the fact that one of the three derivatives is always zero. Another feature of the Frenet frame is that it is *adapted* to the curve: the members are either tangent to or perpendicular to the curve. It is the purpose of this paper to show that there are other frames which have these same advantages and to compare them with the Frenet frame.

**1. Relatively parallel fields.** We say that a normal vector field  $M$  along a curve is *relatively parallel* if its derivative is tangential. Such a field turns only whatever amount is necessary for it to remain normal, so it is as close to being parallel as possible without losing normality. Since its derivative is perpendicular to it, a relatively parallel normal field has constant length. Such fields occur classically in the discussion of curves which are said to be parallel to the given curve. Indeed, if

# BISHOP FRAME

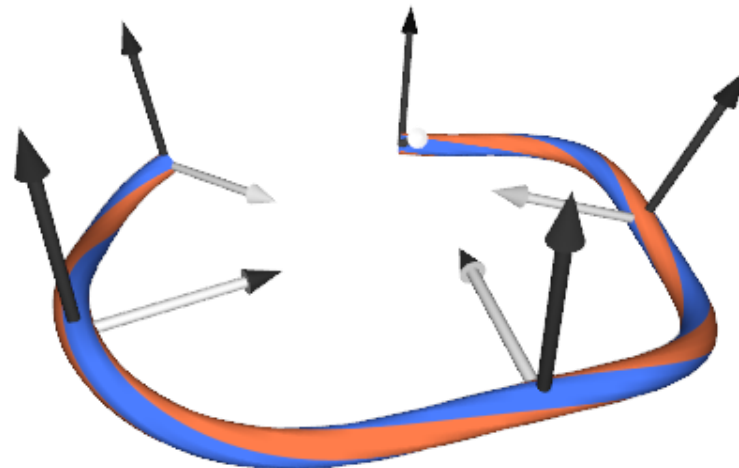
$$T' = \Omega \times T$$

$$u' = \Omega \times u$$

$$v' = \Omega \times v$$

$$u' \cdot v \equiv 0$$

No twist  
("parallel transport")



# BISHOP FRAME

$$T' = \Omega \times T$$

$$u' = \Omega \times u$$

$$v' = \Omega \times v$$

$$\Omega := \kappa B \text{ (“curvature binormal”)}$$

Darboux vector

$$u' \cdot v \equiv 0$$

No twist  
 (“parallel  
 transport”)

# CURVE-ANGLE REPRESENTATION

$$m_1 = u \cos \theta + v \sin \theta$$

$$m_2 = -u \sin \theta + v \cos \theta$$

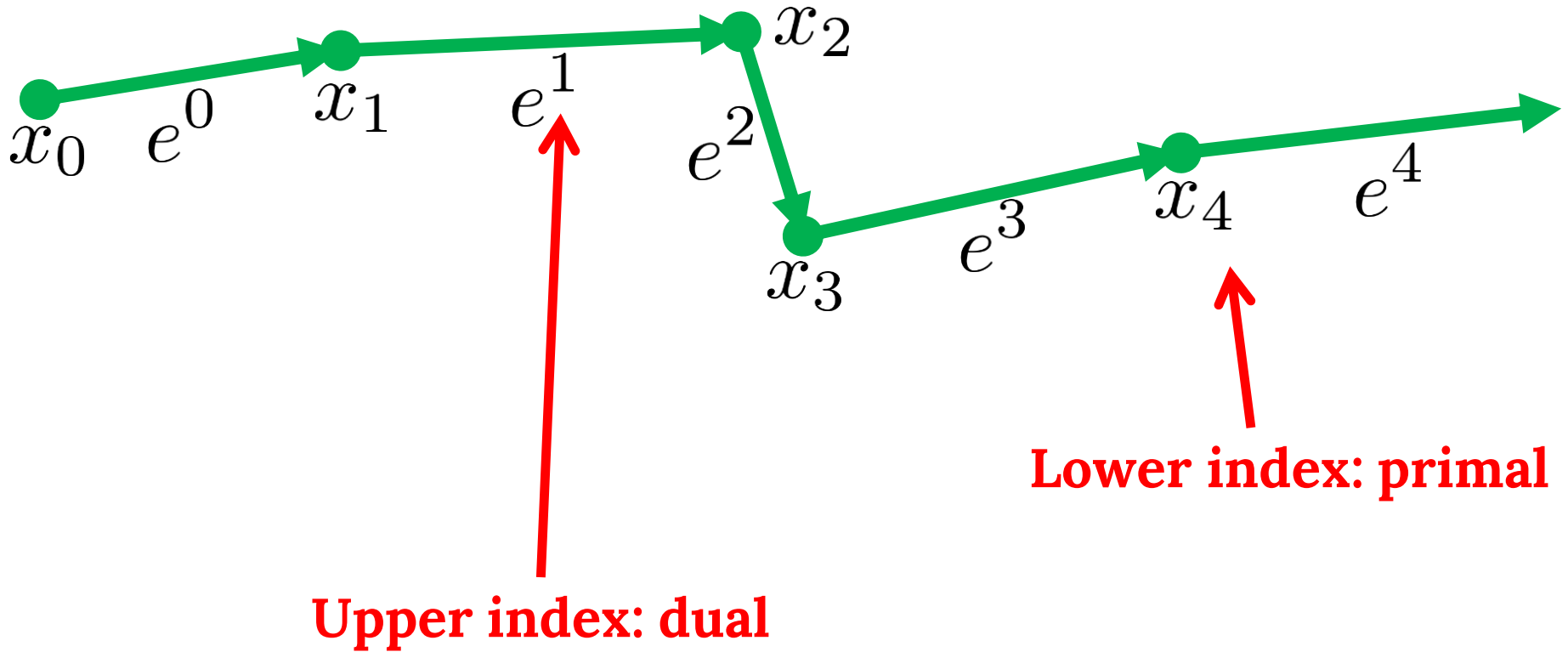
$$E_{\text{twist}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \beta(\theta')^2 ds$$

**Degrees of freedom for elastic energy:**

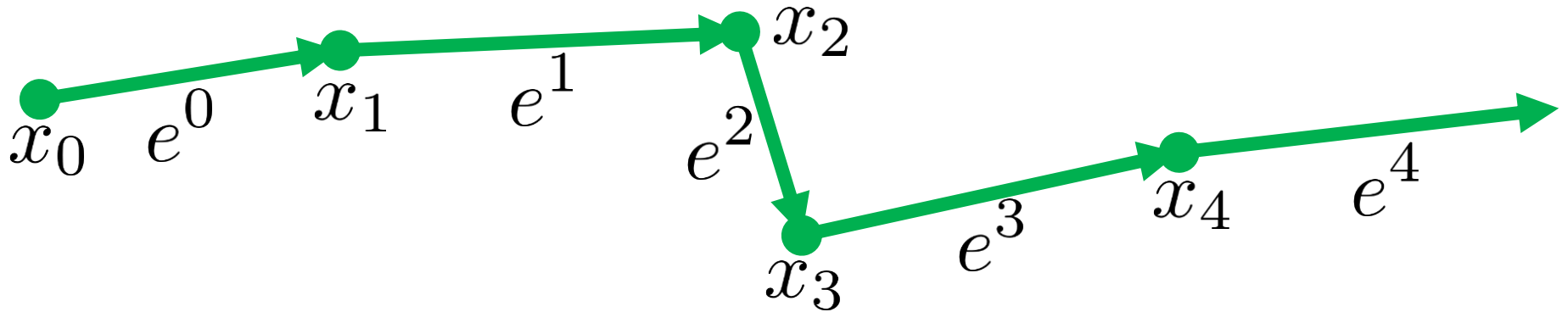
- Shape of curve
- Twist angle  $\theta$



# DISCRETE KIRCHHOFF RODS



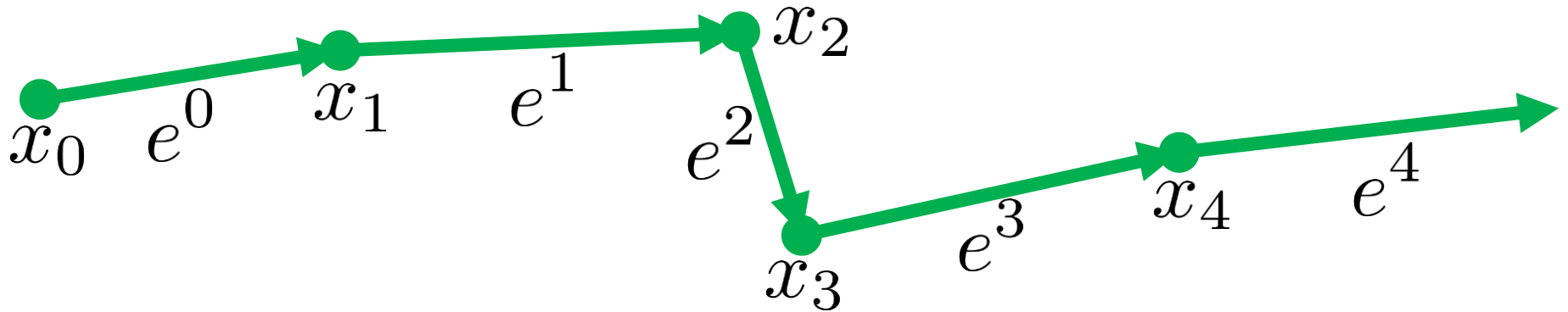
# DISCRETE KIRCHHOFF RODS



$$T^i := \frac{e^i}{\|e^i\|}$$

Tangent unambiguous on edge

# DISCRETE KIRCHHOFF RODS



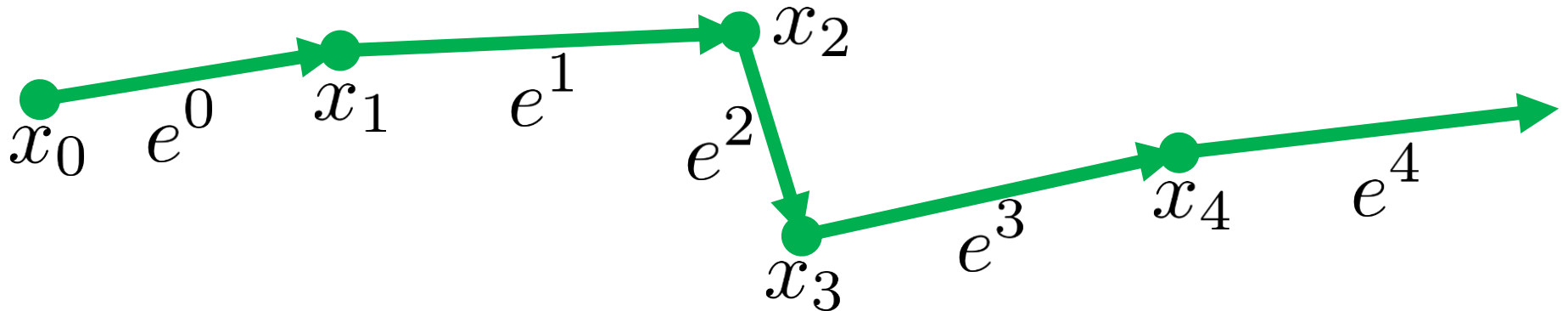
$$\kappa_i := 2 \tan \frac{\phi_i}{2}$$

Turning angle

**Yet another  
curvature!**

Integrated curvature

# DISCRETE KIRCHHOFF RODS



$$\kappa_i := 2 \tan \frac{\phi_i}{2}$$

**Yet another  
curvature!**

$$(\kappa B)_i := \frac{2e^{i-1} \times e^i}{\|e^{i-1}\| \|e^i\| + e^{i-1} \cdot e^i}$$

**Orthogonal to osculating plane,  
norm  $\kappa_i$**

Darboux vector

# BENDING ENERGY

$$E_{\text{bend}}(\Gamma) := \frac{\alpha}{2} \sum_i \left( \frac{(\kappa B)_i}{l_i/2} \right)^2 \frac{l_i}{2}$$
$$= \alpha \sum_i \frac{\|(\kappa B)_i\|^2}{l_i}$$

Can extend for  
natural bend

Convert to pointwise and integrate

# DISCRETE PARALLEL TRANSPORT

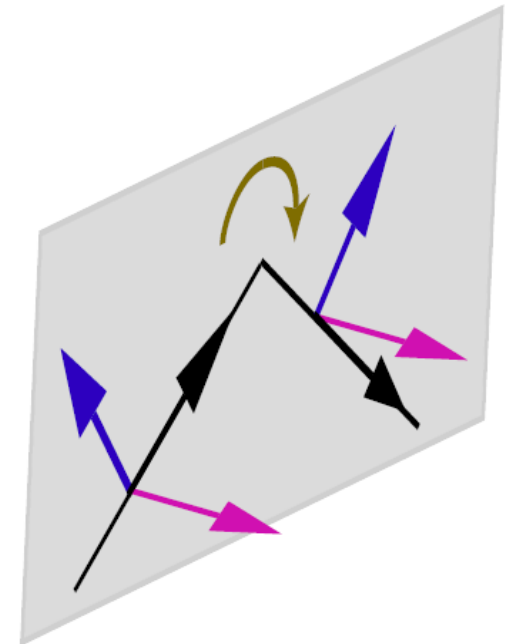
$$P_i(T^{i-1}) = T^i$$

$$P_i(T^{i-1} \times T^i) = T^{i-1} \times T^i$$

- **Map tangent to tangent**
- **Preserve binormal**
- **Orthogonal**

$$u^i = P_i(u^{i-1})$$

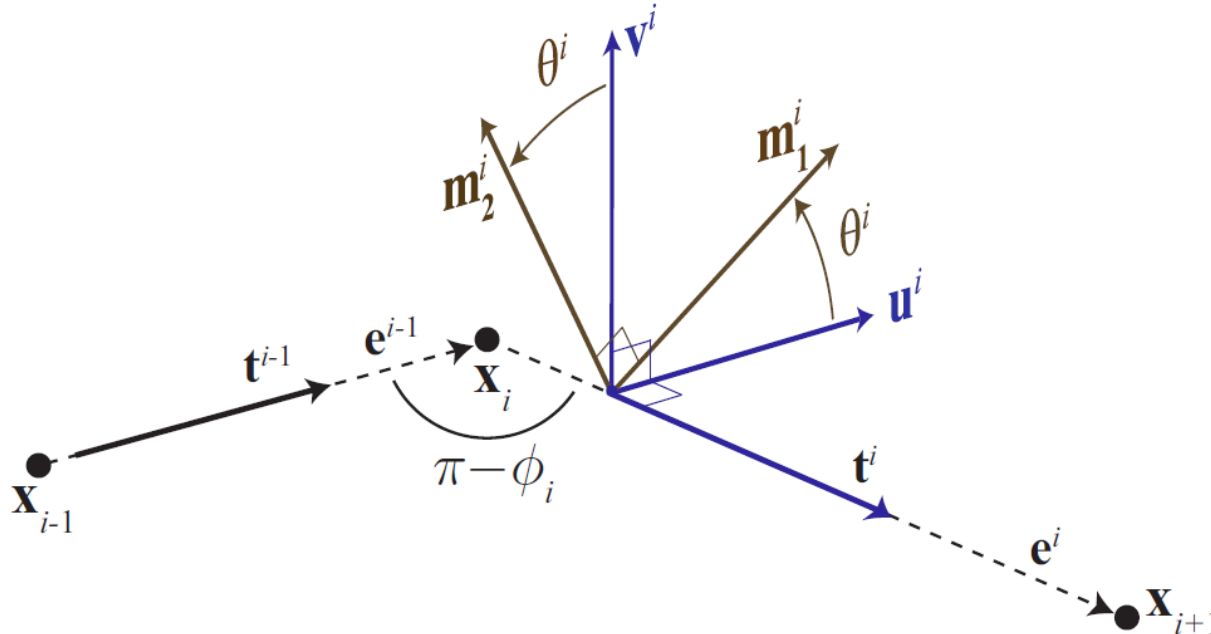
$$v^i = T^i \times u^i$$



# DISCRETE MATERIAL FRAME

$$m_1^i = u^i \cos \theta^i + v^i \sin \theta^i$$

$$m_2^i = -u^i \sin \theta^i + v^i \cos \theta^i$$



# DISCRETE TWISTING ENERGY

$$E_{\text{twist}}(\Gamma) := \beta \sum_i \frac{(\theta^i - \theta^{i-1})^2}{l_i}$$

Note  $\theta_0$  can be arbitrary



# SIMULATION

`\omit{physics}`

*Worth reading!*

# EXTENSION AND SPEEDUP

## Discrete Viscous Threads

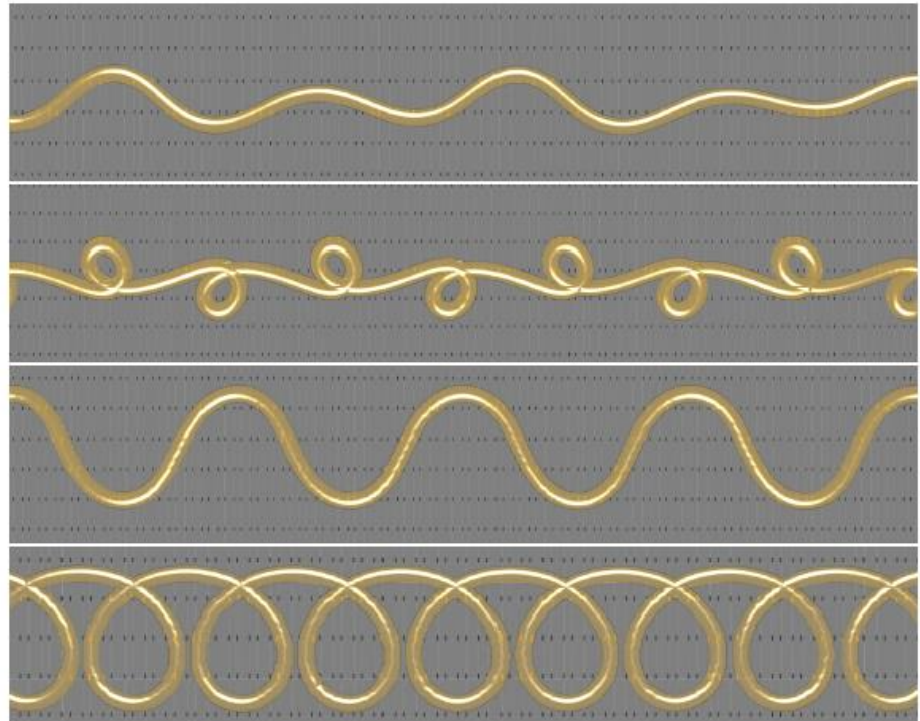
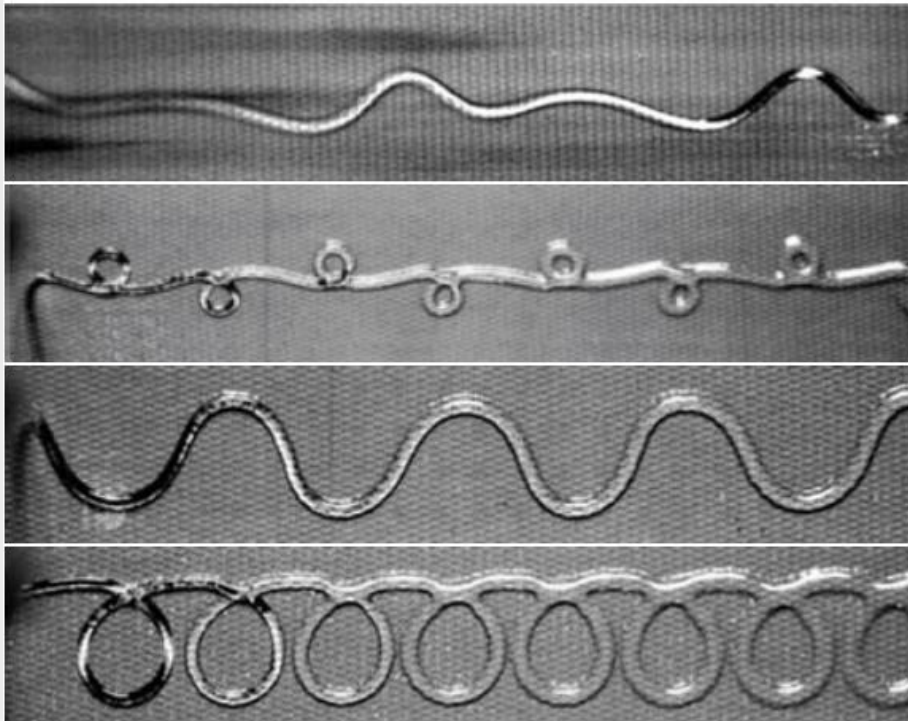
Miklós Bergou  
Columbia University

Basile Audoly  
UPMC Univ. Paris 06 & CNRS

Etienne Vouga  
Columbia University

Max Wardetzky  
Universität Göttingen

Eitan Grinspun  
Columbia University



# EXTENSION AND SPEEDUP

## Discrete Viscous Threads

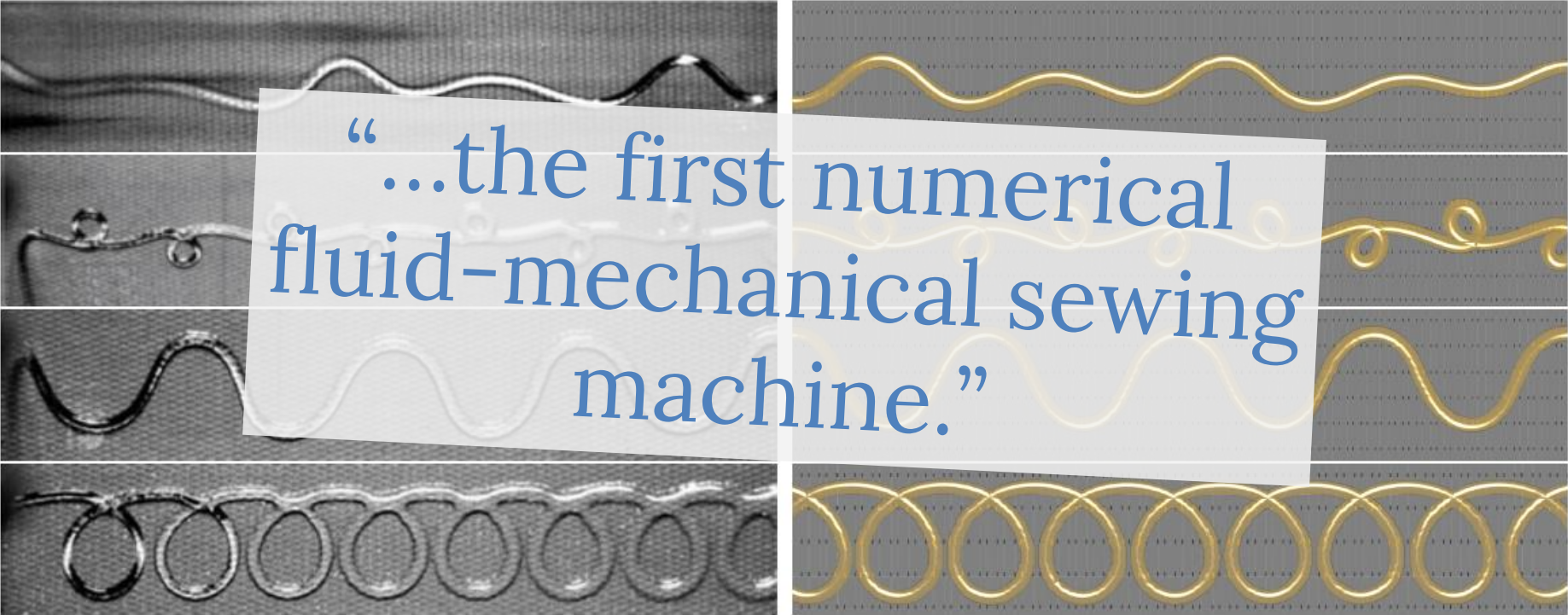
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“...the first numerical  
fluid-mechanical sewing  
machine.”

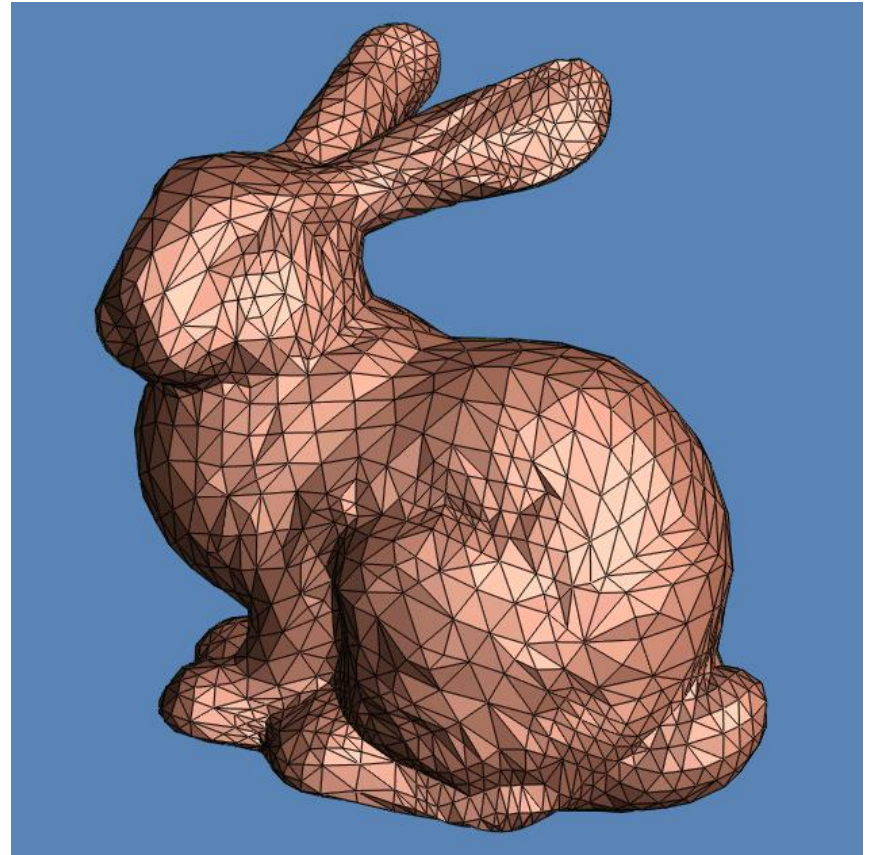
# Three different curvature discretizations

$$\theta \qquad 2 \sin \frac{\theta}{2} \qquad 2 \tan \frac{\theta}{2}$$

Easy theoretical object,  
hard to use.

$$\frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

# NEXT



<http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg>  
<http://www.stat.washington.edu/wxs/images/BUNMID.gif>

## Surfaces