Vector Field Processing on triangle meshes

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Render the Possibilities
SIGGRAPH 2016

Check out course notes!
Directional Field Synthesis, Design, and Processing

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Abstract

Direction fields and vector fields play an increasingly important role in computer graphics and geometry processing. The synthesis of directional fields on surfaces, or other spatial domains, is a fundamental step in numerous applications, such as mesh generation, deformation, texture mapping, and many more. The wide range of applications resulted in definitions for many types of directional fields: from vector and tensor fields, over line and cross fields, to frame and vector-set fields. Depending on the application at hand, researchers have used various notions of objectives and constraints to synthesize such fields. These notions are defined in terms of fairness, feature alignment, symmetry, or field topology, to mention just a few. To facilitate these objectives, various representations, discretizations, and optimization strategies have been developed. These choices come with varying strengths and weaknesses. This report provides a systematic overview of directional field synthesis for graphics applications, the challenges it poses, and the methods developed in recent years to address these challenges.

Categories and Subject Descriptors (according to ACM CCS): 1.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—

1. Introduction

There have been significant developments in directional field synthesis over the past decade. These developments have been driven...
Why Vector Fields?

© Disney/Pixar

[Fisher et al. 2007]

[Bessmeltsev and Solomon 2018]

© Disney/Pixar

Graphics
Why Vector Fields?

“Blood flow in the rabbit aortic arch and descending thoracic aorta”
Vincent et al.; J. Royal Society 2011

Biological science and imaging
Why Vector Fields?

https://disc.gsfc.nasa.gov/featured-items/airs-monitors-cold-weather

Weather modeling
Fluid modeling
Why Vector Fields?

Simulation and engineering

Plan

Crash course in theory/discretization of vector fields.
CONTINUOUS
Studying Vector fields

How to
• define a VF on a surface?
• Differentiate it?
• Integrate?
• Define its topology?

http://theanalyticpoem.net/concept-map/3-dimension/torus_vectors_oblique/
Studying Vector fields

How to

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http://theanalyticpoem.net/concept-map/3-dimension/torus_vectors_oblique/
Recall:

**Tangent Space**

\[ T_p S := \text{Image}(D\sigma_u) \]

\[ S \subset \mathbb{R}^3 \]

\[ V \cap S \]

\[ p \]

\[ \sigma \]

\[ u \]

\[ U \subset \mathbb{R}^2 \]
Recall:

Tangent Space: Coordinate-Free

\[ v \in T_p S \iff \text{there exists curve } \alpha : (-\varepsilon, \varepsilon) \to S \]
with \( \alpha(0) = p, \alpha'(0) = v \)
Some Definitions

Tangent bundle:
\[ TM := \{(p, v) : v \in T_p M\} \]

Vector field:
\[ u : M \to TM \text{ with } u(p) = (p, v), v \in T_p M \]
Studying Vector fields

How to
• define a VF on a surface?
• **Differentiate it?**
• **Integrate?**
• Define its topology?
Recall: Scalar Functions

$f \colon \rightarrow \mathbb{R}$

Map points to real numbers
Recall:
Differential of a Map

Suppose $f: S \to \mathbb{R}$ and take $p \in S$. For $v \in T_pS$, choose a curve $\alpha: (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$ and $\alpha'(0) = v$. Then the differential of $f$ is $df: T_pS \to \mathbb{R}$ with

$$(df)_p(v) := \frac{d}{dt} \bigg|_{t=0} (f \circ \alpha)(t) = (f \circ \alpha)'(0).$$

On the board (time-permitting):
- Does not depend on choice of $\alpha$
- Linear map

Following Curves and Surfaces, Montiel & Ros
Recall:

Gradient Vector Field

\[ \nabla f : S \to \mathbb{R}^3 \text{ with } \]
\[
\begin{align*}
\langle (\nabla f)(p), v \rangle &= (df)_p(v), \quad v \in T_pS \\
\langle (\nabla f)(p), N(p) \rangle &= 0
\end{align*}
\]
How do you differentiate a vector field?
Answer

THE THING IS ....

ITS COMPPLICATED

http://www.relatably.com/m/img/complicated-memes/60260587.jpg
What’s the issue?

What’s a ‘constant’ VF on a surface?

https://math.stackexchange.com/questions/2215084/parallel-transport-equations
What’s the issue?

How to identify different tangent spaces?
Many Notions of Derivative

- **Differential of covector**
  (defer for now)

- **Lie derivative**
  Weak structure, easier to compute

- **Covariant derivative**
  Strong structure, harder to compute
Vector Field Flows: Diffeomorphism

\[ \frac{d}{dt} \psi_t = V \circ \psi_t \]

Useful property: \( \psi_{t+s}(x) = \psi_t(\psi_s(x)) \)

Diffeomorphism with inverse \( \psi_{-t} \)

Group structure!
Fun example:

Killing Vector Fields (KVFs)

Wilhelm Killing
1847-1923
Germany

Preserves distances infinitesimally
Differential of Vector Field Flow

\[ d\psi_t(p) : T_p M \rightarrow T_{\psi_t(p)} M \]
Lie Derivative

\[(\mathcal{L}_V W)_p := \lim_{t \to 0} \frac{1}{t} \left[ (d\psi_{-t})_{\psi_t(p)}(W_{\psi_t(p)}) - W_p \right]\]

Fig. 9.13 The Lie derivative of a vector field
Amoeba example
Amoeba example
Amoeba example
Amoeba example
Amoeba example
Amoeba example
Amoeba example
Amoeba example
What’s Wrong with Lie Derivatives?

\[(\mathcal{L}_V W)_p := \lim_{t \to 0} \frac{1}{t} \left[ (d\psi_{-t})_{\psi_t(p)}(W_{\psi_t(p)}) - W_p \right]\]

Image courtesy A. Carapeti

Depends on structure of \( V \)
What We Want

“What is the derivative of the blue vector field in the orange direction?”

What we don’t want:
Specify *orange* direction anywhere but at $p$. 
Parallel Transport

Canonical identification of tangent spaces

More later...
Covariant Derivative (Embedded)

\[ \nabla_V W := [D_V W]^{\parallel} = \text{proj}_{T_pS}(W \circ \alpha)'(0) \]

Synonym: (Levi-Civita) Connection

Note: \([D_V W]^\perp = \mathbb{II}(V, W)N\)
Some Properties

Properties of the Covariant Derivative

As defined, $\nabla_V Y$ depends only on $V_p$ and $Y$ to first order along $c$.

Also, we have the Five Properties:

1. $C^\infty$-linearity in the $V$-slot:
   \[ \nabla_{V_1 + fV_2} Y = \nabla_{V_1} Y + f \nabla_{V_2} Y \text{ where } f : S \to \mathbb{R} \]

2. $\mathbb{R}$-linearity in the $Y$-slot:
   \[ \nabla_V (Y_1 + aY_2) = \nabla_V Y_1 + a \nabla_V Y_2 \text{ where } a \in \mathbb{R} \]

3. Product rule in the $Y$-slot:
   \[ \nabla_V (fY) = f \cdot \nabla_V Y + (\nabla_V f) \cdot Y \text{ where } f : S \to \mathbb{R} \]

4. The metric compatibility property:
   \[ \nabla_V \langle Y, Z \rangle = \langle \nabla_V Y, Z \rangle + \langle Y, \nabla_V Z \rangle \]

5. The “torsion-free” property:
   \[ \nabla_{V_1} V_2 - \nabla_{V_2} V_1 = [V_1, V_2] \]

The Lie bracket

\[ [V_1, V_2](f) := D_{V_1}D_{V_2}(f) - D_{V_2}D_{V_1}(f) \]

Defines a vector field, which is tangent to $S$ if $V_1$, $V_2$ are!
Recall:

Geodesic Equation

\[ \text{proj}_{T_{\gamma(s)} S} [\gamma''(s)] = 0 \]

- The only acceleration is out of the surface
- No steering wheel!
Intrinsic Geodesic Equation

\[ \nabla \dot{\gamma}(t) \dot{\gamma}(t) = 0 \]

- No stepping on the accelerator
- No steering wheel!
Parallel Transport

Only path-independent if domain is flat.

\[ 0 = \nabla_{\dot{\gamma}(t)} V \]

Preserves length, inner product (can be used to define covariant derivative)
Holonomy

Path dependence of parallel transport

Integrated Gaussian curvature
Studying Vector fields

How to

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http://theanalyticpoem.net/concept-map/3-dimension/torus_vectors_oblique/
Vector Field Topology

Fig. 9.8 Examples of flows near equilibrium points

Image from Smooth Manifolds, Lee
Poincaré-Hopf Theorem

\[ \sum \text{index}_{x_i}(v) = \chi(M) \]

where vector field \( v \) has isolated singularities \( \{x_i\} \).

\[ v(c(t)) = \|v(c(t))\| \begin{pmatrix} \cos \alpha(t) \\ \sin \alpha(t) \end{pmatrix} \]

Image from “Directional Field Synthesis, Design, and Processing” (Vaxman et al., EG STAR 2016)
Famous Corollary

Hairy ball theorem

© Keenan Crane
Singularities in wild
Singularities in the wild
DISCRETE VECTOR FIELDS
Vector Fields on Triangle Meshes

No consensus:

• Triangle-based
• Edge-based
• Vertex-based

$\subseteq \mathbb{R}^3$
Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based

$\subseteq \mathbb{R}^3$
Triangle-Based

- Triangle as its **own tangent plane**
- One vector per triangle
  - “Piecewise constant”
  - Discontinuous at edges/vertices
- Easy to “unfold”/“hinge”
Discrete Levi-Civita Connection

- Simple notion of **parallel transport**
- Transport around vertex: 
  Excess angle is (integrated) **Gaussian curvature** (holonomy!)

\[ K = 2\pi - \sum_a \gamma_a \]
Arbitrary Connection

Represent using angle $\theta_{edge}$ of extra rotation.
Trivial Connections

- Vector field design
- Zero holonomy on discrete cycles
  - Except for a few singularities
- Path-independent away from singularities

"Trivial Connections on Discrete Surfaces"
Crane et al., SGP 2010
Trivial Connections: Details

• Solve $\theta_{edge}$ of **extra rotation** per edge

• Linear constraint:
  **Zero holonomy** on basis cycles
  – $V+2g$ constraints: Vertex cycles plus harmonic
  – Fix curvature at chosen singularities

• Underconstrained: **Minimize $||\tilde{\theta}||$**
  – Best approximation of Levi-Civita
Result

Resulting trivial connection
(no other singularities present)
Aside:

Nice 2D Identification

\[ \vec{v} \leftrightarrow a e^{i\theta} \]
Face-Based Calculus

Vertex-based
“Conforming”
Already familiar

Edge-based
“Nonconforming”
[Wardetzky 2006]

Relationship: \( \psi_{ij} = \phi_i + \phi_j - \phi_k \)

Gradient Vector Field
Recall:

Gradient of a Hat Function

\[ \| \nabla f \| = \frac{1}{\ell_3 \sin \theta_3} = \frac{1}{h} \]

\[ \nabla f = \frac{e_{23}}{2A} \]

Length of \( e_{23} \) cancels "base" in \( A \)
Helmholtz-Hodge Decomposition

Divergence free

Harmonic

Curl free

Image courtesy K. Crane
Helmholtz-Hodge Decomposition

Divergence free
\( R \nabla f \)

Curl free
\( \nabla f \)

2\( g \)-dimensional
Harmonic

Image courtesy K. Crane
Recall:

Euler Characteristic

\[ V - E + F := \chi \]

\[ \chi = 2 - 2g \]

- \( g = 0 \) (soccer ball)
- \( g = 1 \) (donut)
- \( g = 2 \) (two-lobed pretzel)
Discrete Helmholtz-Hodge

\[ 2 - 2g = V - E + F \]

\[ \implies 2F = (V - 1) + (E - 1) + 2g \]

Either
- Vertex-based gradients
- Edge-based rotated gradients

or
- Edge-based gradients
- Vertex-based rotated gradients

“Mixed” finite elements
Vector Fields on Triangle Meshes

No consensus:

• Triangle-based
• Edge-based
• Vertex-based

Not covered here! Google ‘Discrete Exterior Calculus’
Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based
Vertex-Based Fields

- **Pros**
  - Possibility of higher-order differentiation

- **Cons**
  - Vertices don’t have natural tangent spaces
  - Gaussian curvature concentrated
2D (Planar) Case: Easy

Piecewise-linear \((x,y)\) components
3D Case: Ambiguous

Interpolate vector and tangent space!
Recent Method for Continuous Fields

Discrete Connection and Covariant Derivative for Vector Field Analysis and Design

Beibei Liu and Yiyung Tong
Michigan State University
and
Fernando de Goes and Mathieu Desbrun
California Institute of Technology

In this paper, we introduce a discrete definition of connection on simplicial manifolds, involving closed-form continuous expressions within simplices and finite rotations across simplices. The finite-dimensional parameters of this connection are optimally computed by minimizing a quadratic measure of the deviation to the (discontinuous) Levi-Civita connection induced by the embedding of the input triangle mesh, or to any metric connection with arbitrary cone singularities at vertices. From this discrete connection, a covariant derivative is constructed through exact differentiation, leading to explicit expressions for local integrals of first-order derivatives (such as divergence, curl and the Cauchy-Riemann operator), and for $L_2$-based energies (such as the Dirichlet energy). We finally demonstrate the utility, flexibility, and accuracy of our discrete formulations for the design and analysis of vector, $n$-vector, and $n$-direction fields.

digital geometry processing, with applications ranging from texture synthesis to shape analysis, meshing, and simulation. However, existing discrete counterparts of such a differential operator acting on simplicial manifolds can either approximate local derivatives (such as divergence and curl) or estimate global integrals (such as the Dirichlet energy), but not both simultaneously.

In this paper, we present a unified discretization of the covariant derivative that offers closed-form expressions for both local and global first-order derivatives of vertex-based tangent vector fields on triangulations. Our approach is based on a new construction of discrete connections that provides consistent interpolation of tangent vectors within and across mesh simplices, while minimizing the deviation to the Levi-Civita connection induced by the 3D embedding of the input mesh—or more generally, to any metric connection with arbitrary cone singularities at vertices. We demonstrate the relevance of our contributions by providing new computational tools to design and edit vector and $n$-direction fields.


CCS Concepts: •Computing methodologies $\rightarrow$ Mesh models;
Vector Fields on Triangle Meshes

No consensus:

• Triangle-based
• Edge-based
• Vertex-based
Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based
- ... others?
An Operator Approach to Tangent Vector Field Processing

Omri Azencot\textsuperscript{1} and Mirela Ben-Chen\textsuperscript{1} and Frédéric Chazal\textsuperscript{2} and Maks Ovsjanikov\textsuperscript{3}

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\textsuperscript{2}Geometria, INRIA
\textsuperscript{3}LIX, École Polytechnique

Figure 1: Using our framework various vector field design goals can be easily posed as linear constraints. Here, given three symmetry maps: rotational (S1), bilateral (S2) and front/back (S3), we can generate a symmetric vector field using only S1 (left), S1 + S2 (center) and S1 + S2 + S3 (right). The top row shows the front of the 3D model, and the bottom row shows the back.

Abstract

In this paper, we introduce a novel coordinate-free method for manipulating and analyzing vector fields on surfaces. Unlike the commonly used representations of a vector field as an assignment of vectors to the mesh, or as real values on edges, we argue that vector fields can also be naturally viewed as operators whose domain and range are functions defined on the mesh. Although this point of view is common in differential geometry, it has so far not been adopted in geometry processing applications. We recall the theoretical properties of vector field operators, consider the relationship with linear PDEs, and demonstrate how they can be efficiently computed.
### Extension: Direction Fields

<table>
<thead>
<tr>
<th>Field Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-vector field</td>
<td>One vector, classical “vector field”</td>
</tr>
<tr>
<td>2-direction field</td>
<td>Two directions with $\pi$ symmetry, “line field”, “2-RoSy field”</td>
</tr>
<tr>
<td>$1^3$-vector field</td>
<td>Three independent vectors, “3-polyvector field”</td>
</tr>
<tr>
<td>4-vector field</td>
<td>Four vectors with $\pi/2$ symmetry, “non-unit cross field”</td>
</tr>
<tr>
<td>4-direction field</td>
<td>Four directions with $\pi/2$ symmetry, “unit cross field”, “4-RoSy field”</td>
</tr>
<tr>
<td>$2^2$-vector field</td>
<td>Two pairs of vectors with $\pi$ symmetry each, “frame field”</td>
</tr>
<tr>
<td>$2^2$-direction field</td>
<td>Two pairs of directions with $\pi$ symmetry each, “non-ortho. cross field”</td>
</tr>
<tr>
<td>6-direction field</td>
<td>Six directions with $\pi/3$ symmetry, “6-RoSy”</td>
</tr>
<tr>
<td>$2^3$-vector field</td>
<td>Three pairs of vectors with $\pi$ symmetry each</td>
</tr>
</tbody>
</table>

“Directional Field Synthesis, Design, and Processing” (Vaxman et al., EG STAR 2016)
One encoding of direction fields

\[ \{u_0, u_1, \ldots, u_k\} \]
\[ \mapsto f(z) := (z - u_0) \cdots (z - u_k) \]
\[ \mapsto f(z) = z^{k+1} + a_k z^k + \cdots + a_1 z + a_0 \]
\[ \mapsto \{a_0, \ldots, a_k\} \]