IFT 6113

A (VERY) SHORT INTRO TO COMPUTATIONAL GEOMETRY

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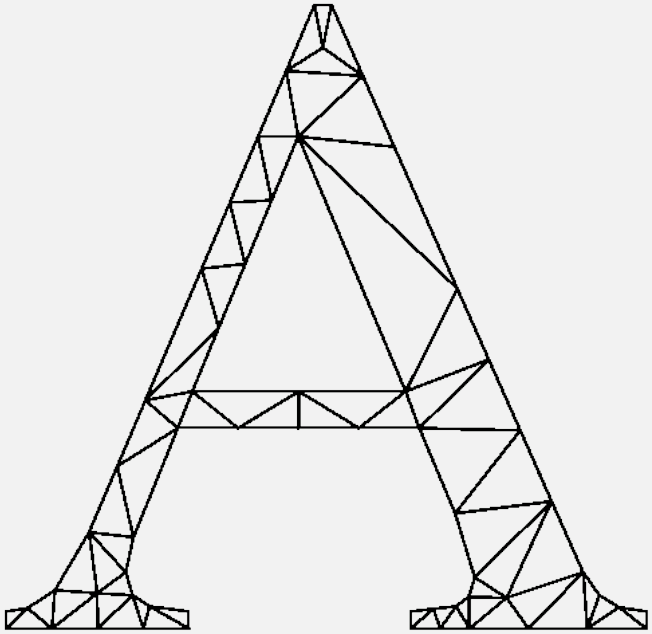


Pic from https://www.cs.cmu.edu/~quake/triangle.defs.html

Mikhail Bessmeltsev

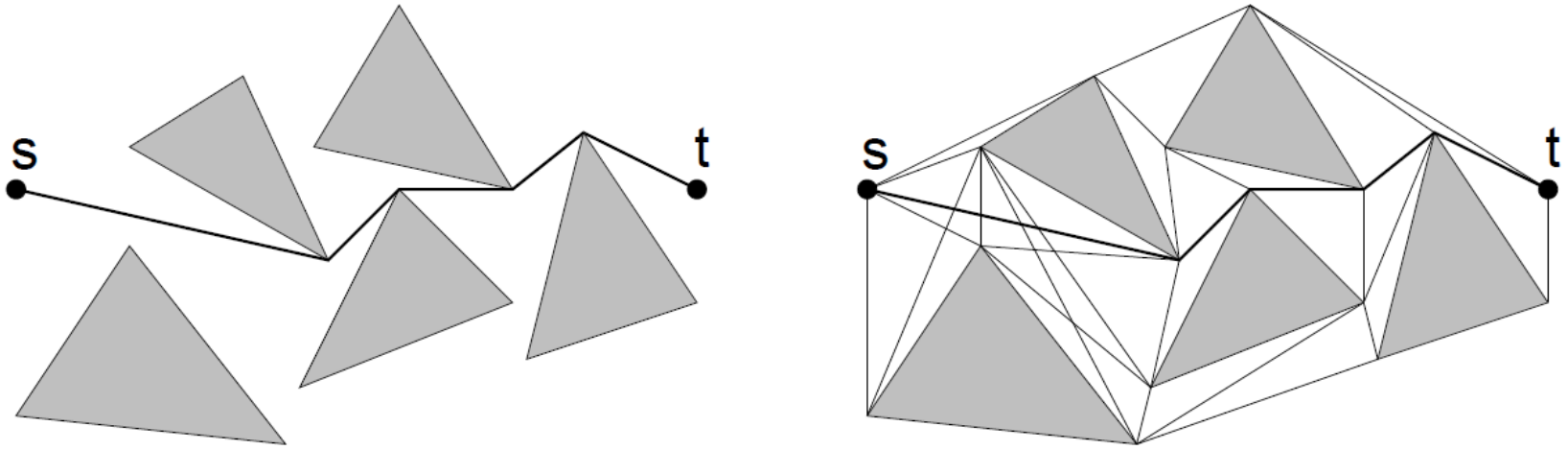
Today

- Intro
- Orientation and convex hulls
- Line segment intersection
- Polygons and triangulations
- Voronoi diagrams
- Delaunay triangulations

Credit: most of images and ideas from Dave Mount's lecture notes
http://www.cs.umd.edu/~mount/754/Lects/754lects.pdf
And David Kirkpatrick's lectures

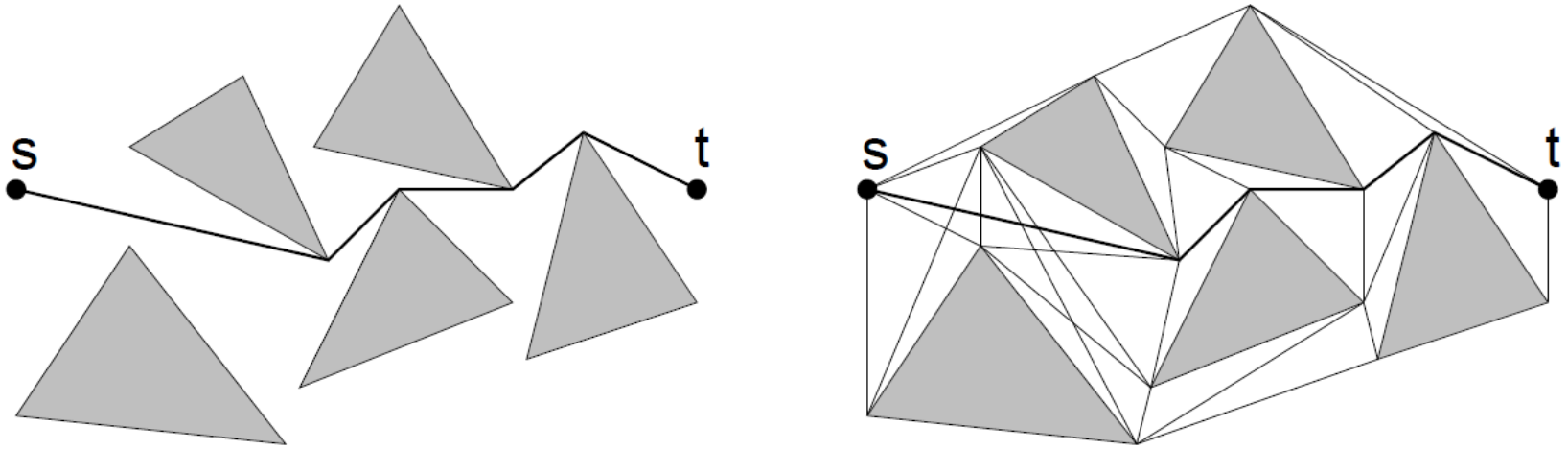
Focus: 2D algorithms

- Typical problem: shortest paths



Focus: 2D algorithms

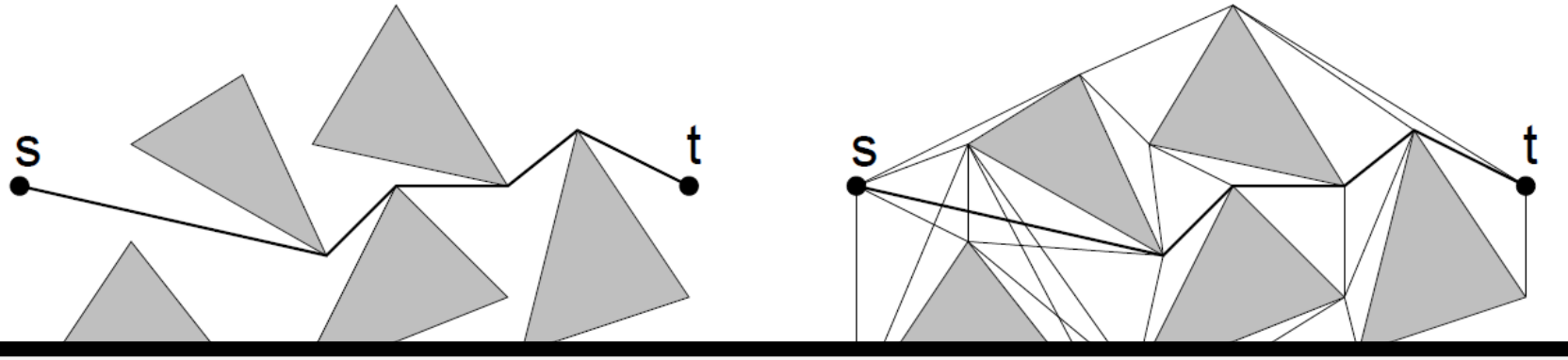
- Typical problem: shortest paths



→ graph → Dijkstra algorithm?

Focus: 2D algorithms

- Typical problem: shortest paths



Use geometric ideas!→
more efficient algorithms

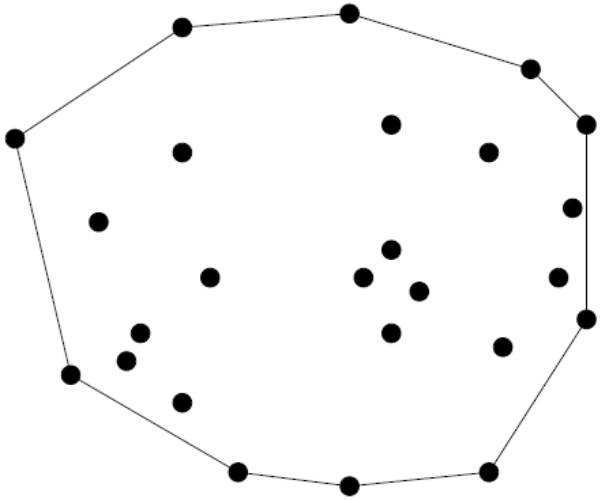
graph → Dijkstra algorithm?

Complexity Analysis

- Big O notation
- Mostly worst case (*sometimes* average)
- Less attention to constants…

Main objects

Convex hulls



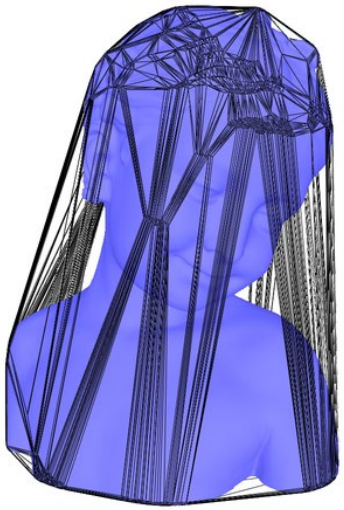
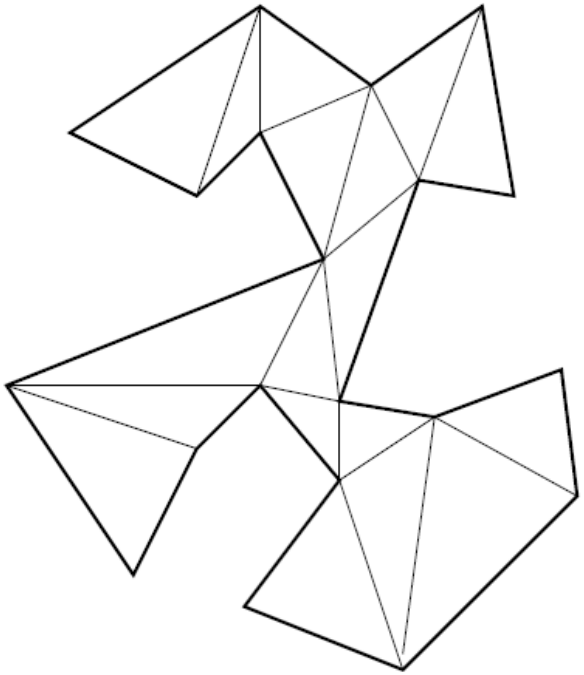


Image from CGAL manual

Main objects

Triangulations



Main objects

Line arrangements

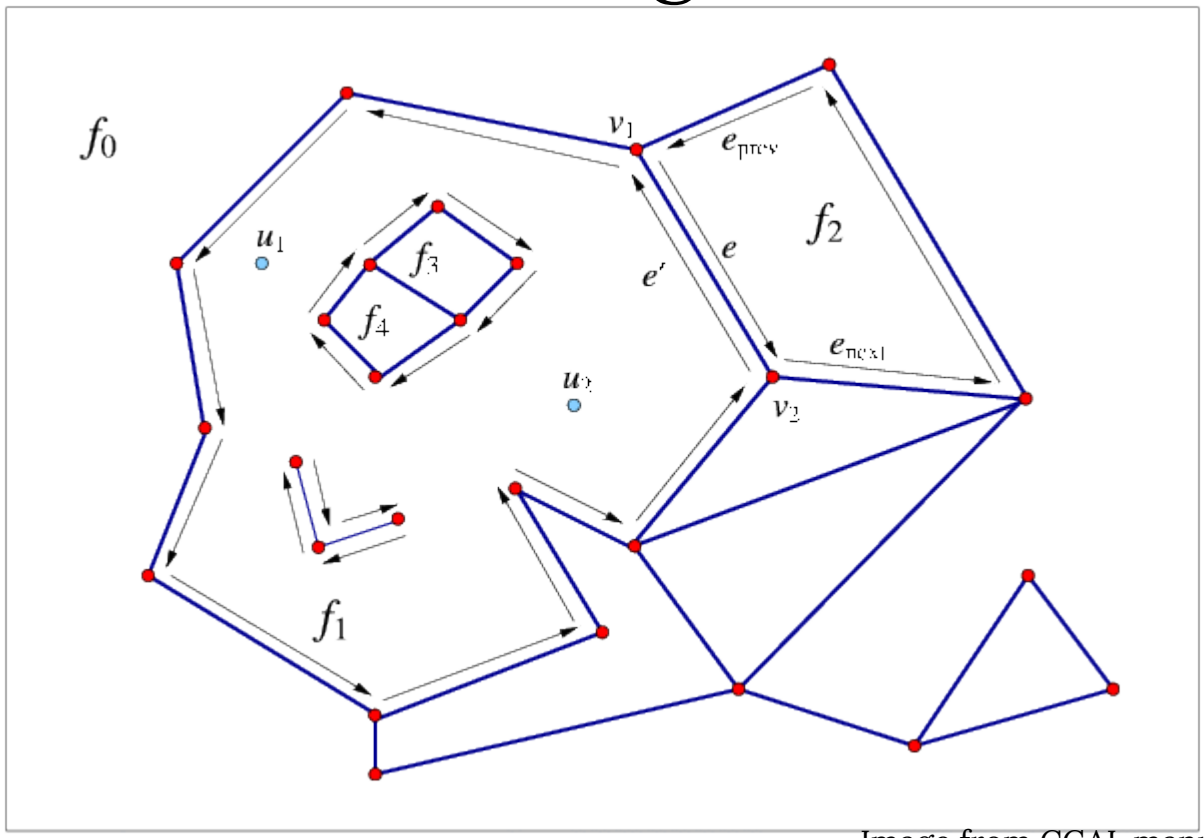
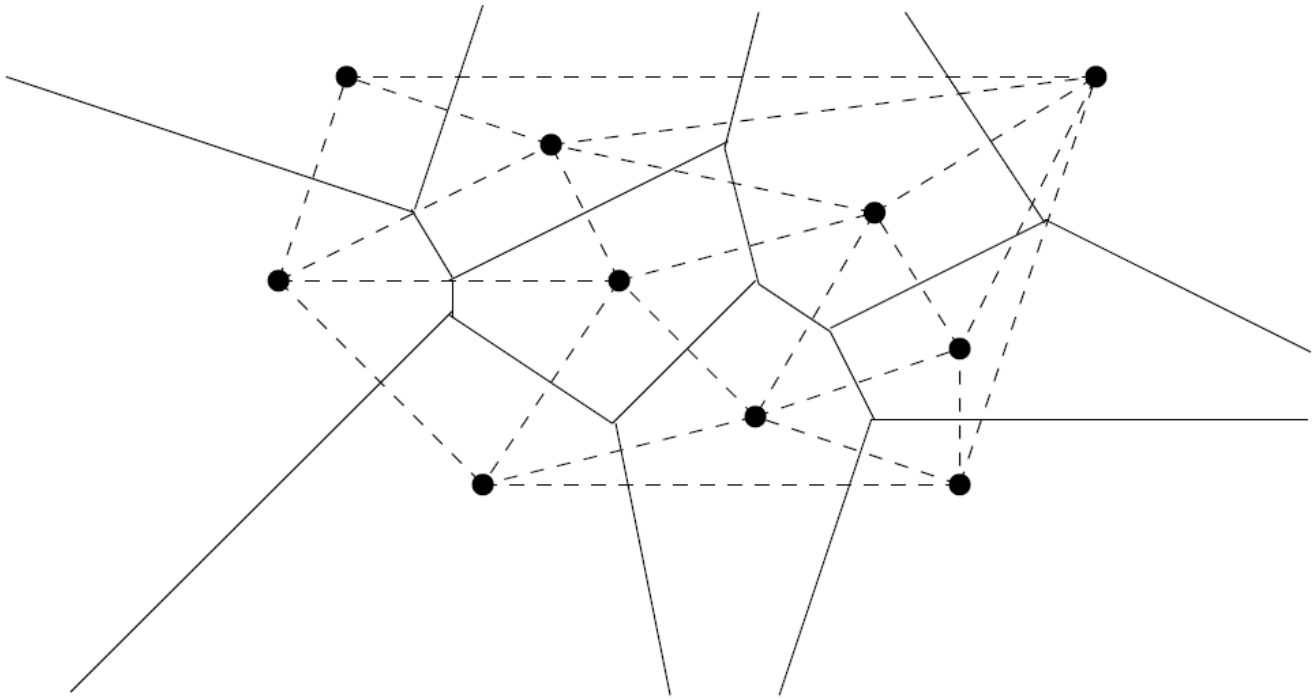


Image from CGAL manual

Main objects

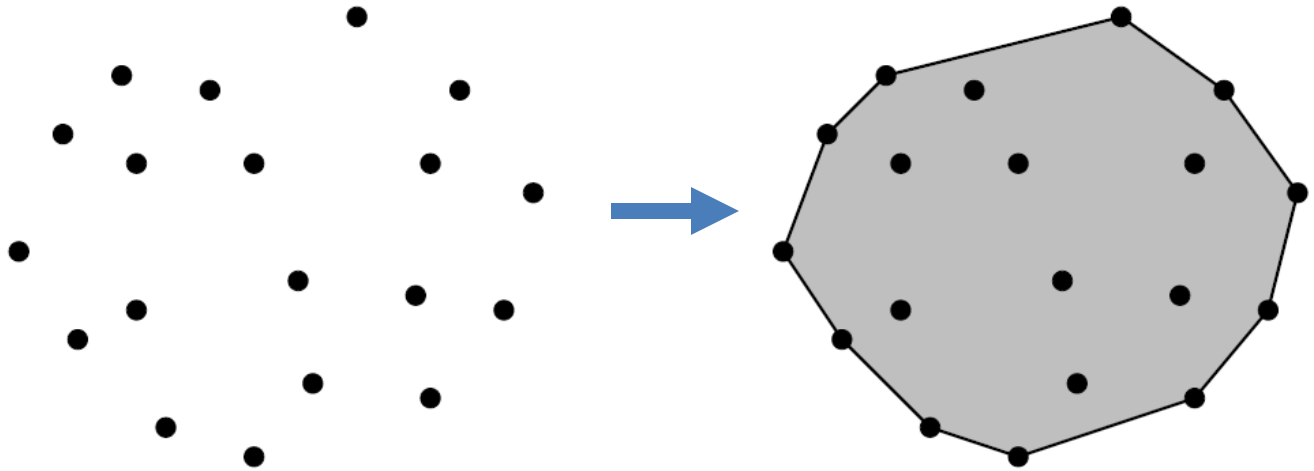
Delaunay triangulations and Voronoi diagrams



Main objects

Queries:

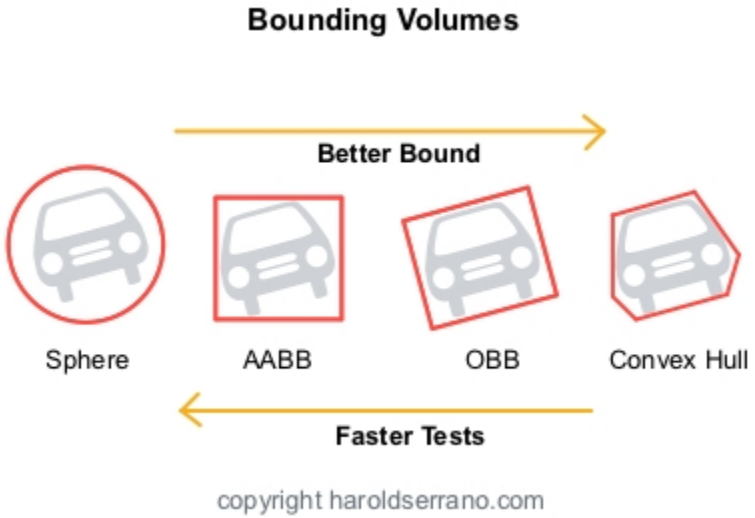
- Nearest neighbors?
- Range searches?



CONVEX HULLS

Why do we need convex hulls?

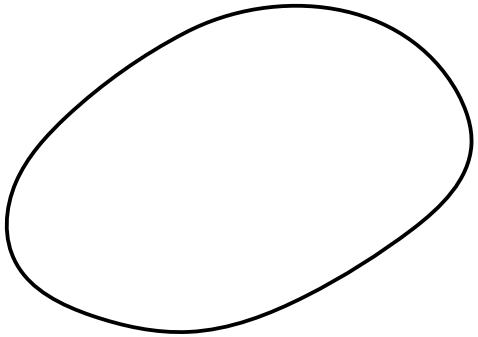
- Collision detecton



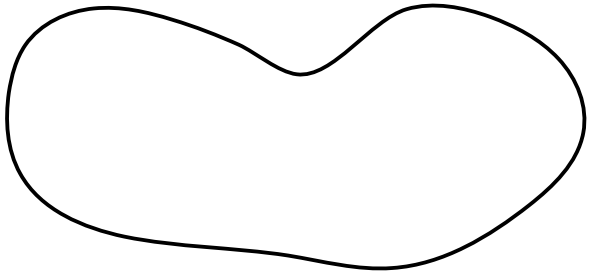
- Reconstructing convex objects from point clouds
- Farthest distance computation

Convexity?

- A set S is convex, if
  - for any two points $p, q \in S$, the line segment $\overline{pq} \in S$



Convex



Not convex

Convexity?

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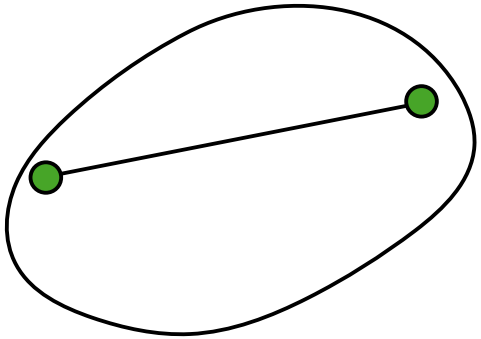
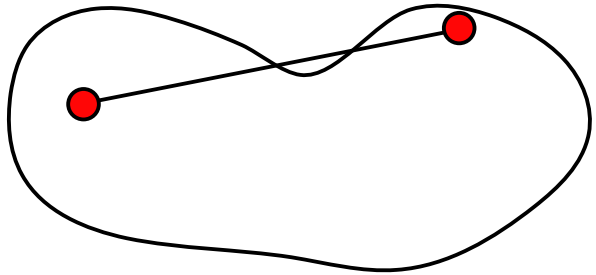


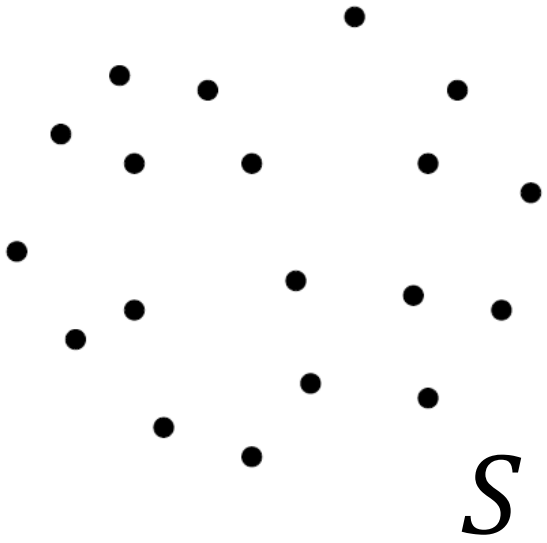
Convex

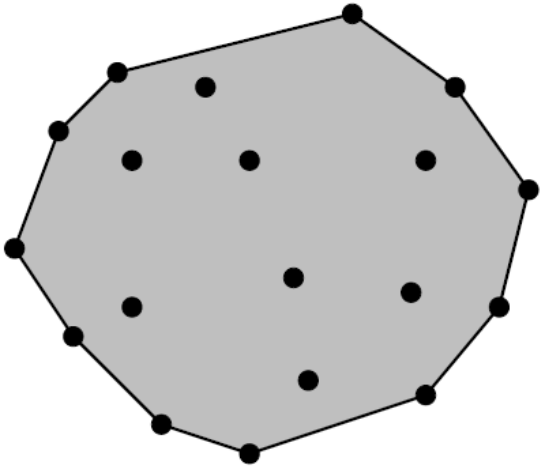


Not convex

Convex hull of a set of points S =

- the smallest convex set that contains S

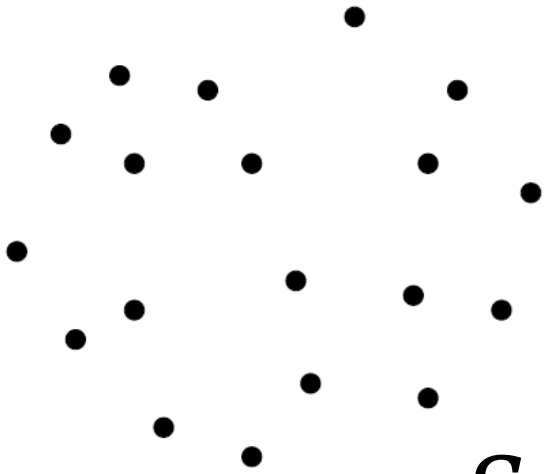




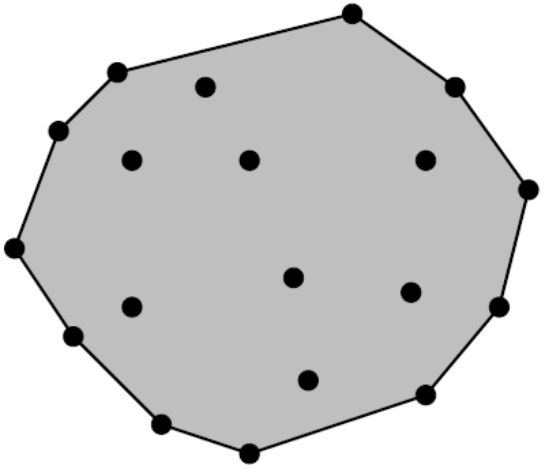
Convex hull of a set of points S =

- the smallest convex set that contains S
- intersection of all convex sets containing S

Equivalent definitions
(not obvious, see Carathéodory's theorem)



S

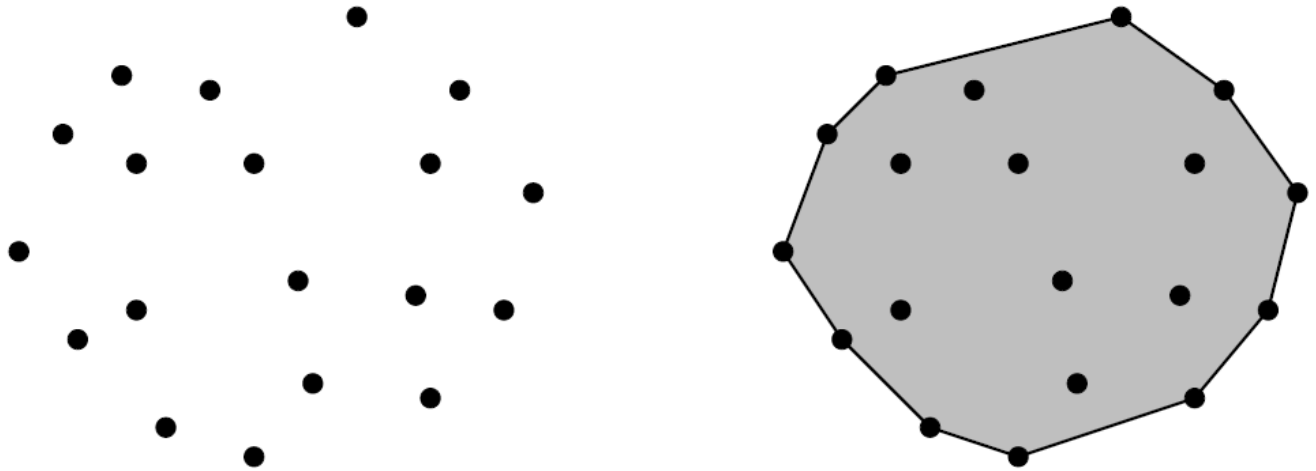


CH: Representation

- A sequence of points!

Simplest algorithm

- Ideas?



Simplest algorithm

- Ideas?
- Try every possible $\overline{pq} \in S$, test if all the other points lie on one side
- Complexity?

Better: Gift Wrapping

- Start with some extreme point p_0
  - (e.g. leftmost)
  - It belongs to CH
- Choose the next point so that $p_{i+1}p_i$ has all other points on the right
  - Sort by angle, choose minimum
  - $O(n)$
- Repeat

Gift Wrapping

- Start with some extreme point p_0
- Choose p_i, s.t. $p_{i+1}p_i$ has all other points on the right
- Repeat

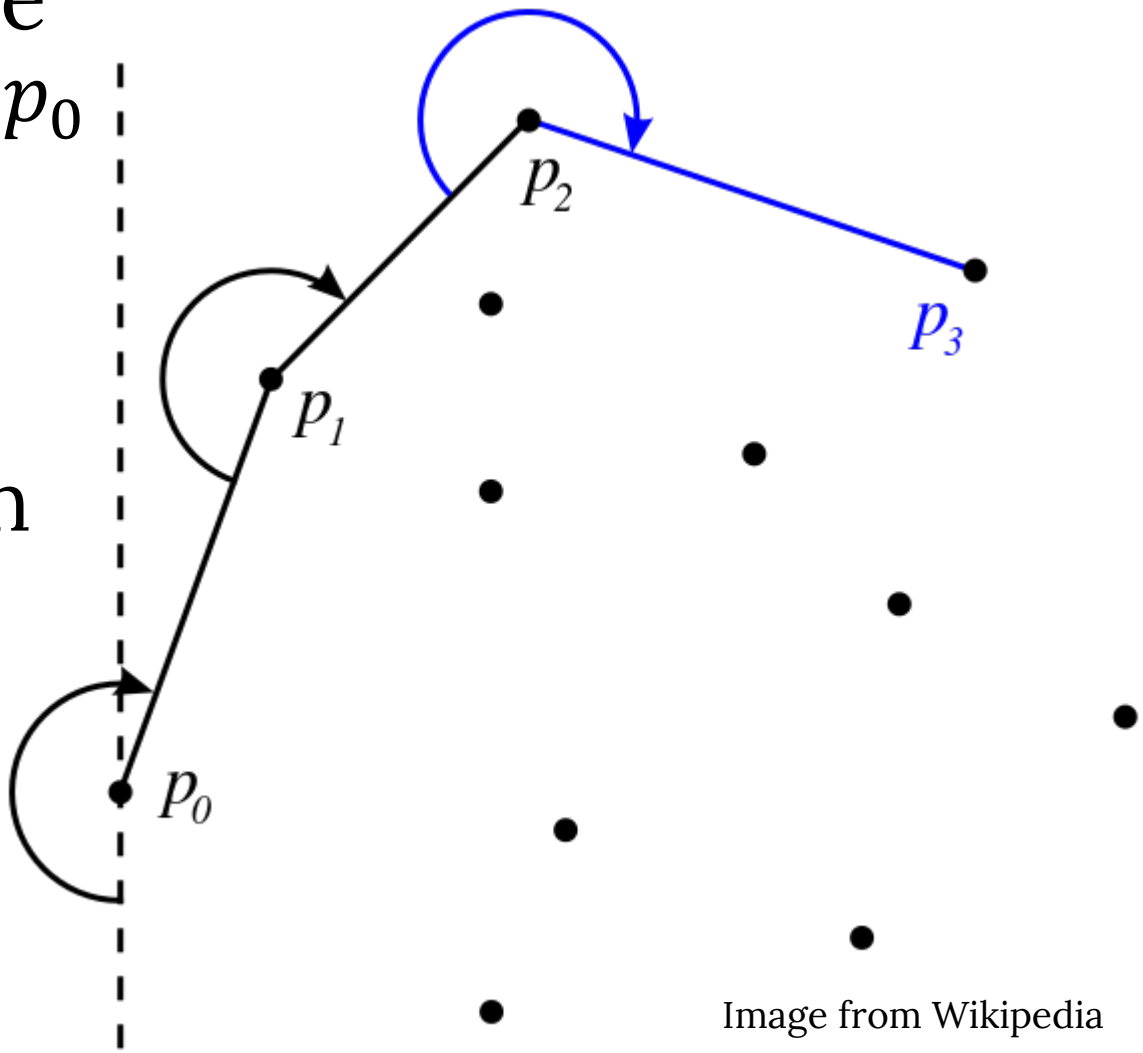


Image from Wikipedia

Gift Wrapping

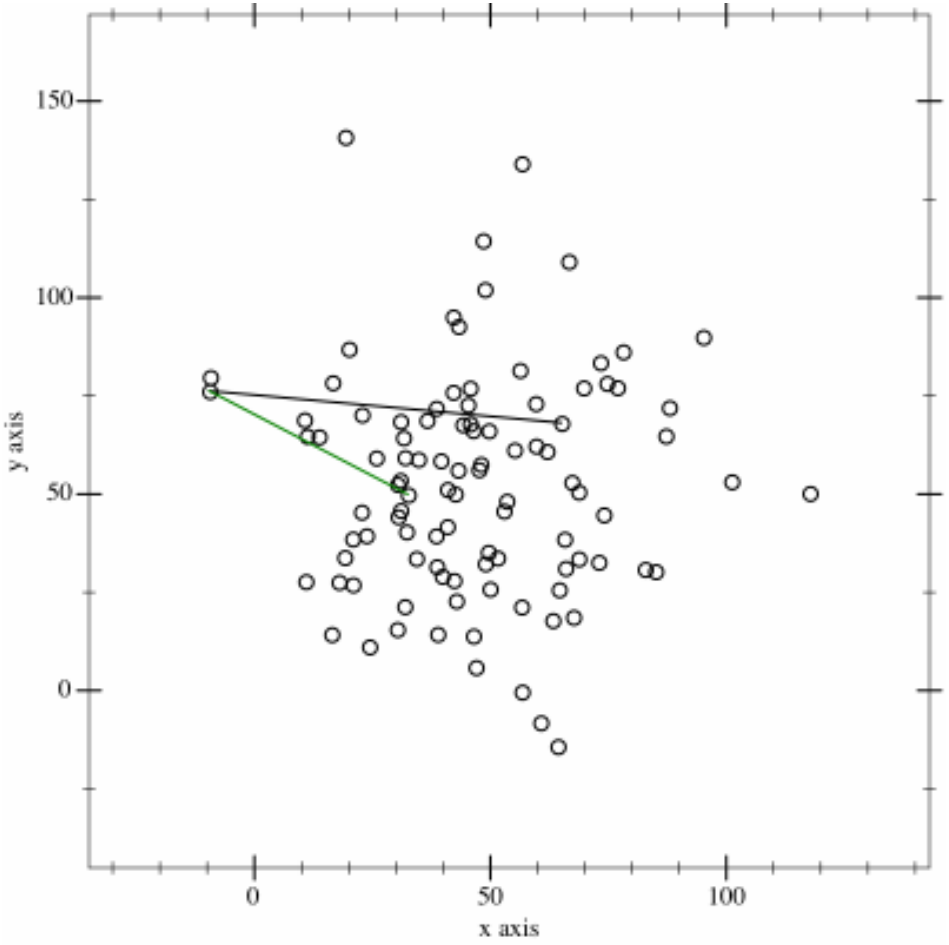


Image from Wikipedia

Output-sensitive complexity

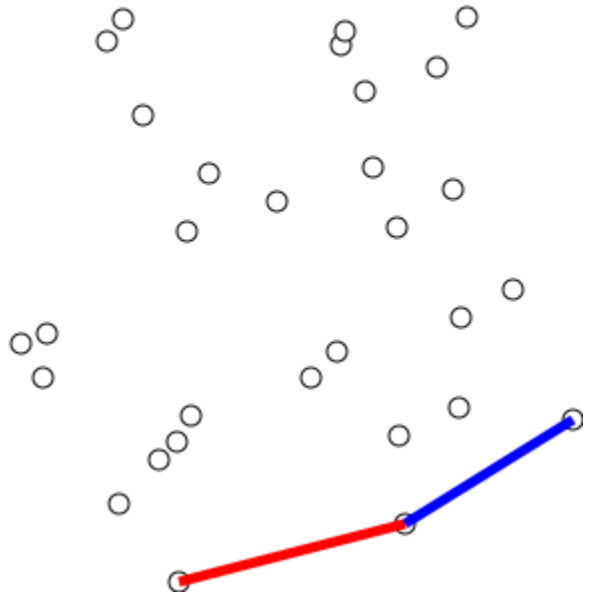
$$O(nh)$$

h - number of vertices on the CH

Graham's Scan

- Lower hull:
- Store CH vertices in a stack: $(\ldots, H_2, H_1)$
- Sort points by angle
- Take new point p, check if the (p, H_1, H_2) is counterclockwise
  - Yes → push!
  - No → pop!

Graham's Scan



Graham's Scan

- Complexity?

Divide-and-Conquer CH

- Sort by x
- Split by the median point
- Recursively find CH for left and right parts
- Merge

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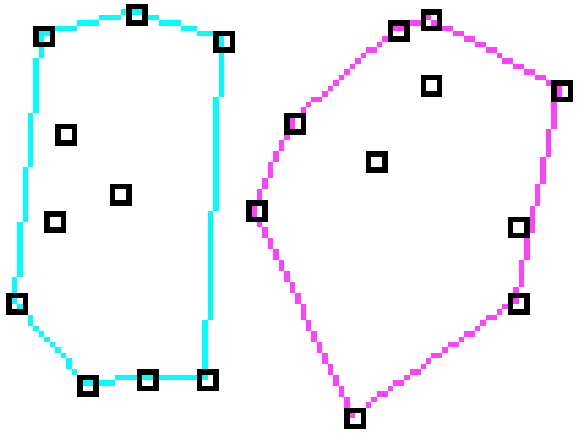


Divide-and-Conquer CH

- Sort by x: $O(n \log n)$
- Split by the median point
- Recursively find CH for left and right parts
- Merge
  - Find common tangent line, $O(n)$

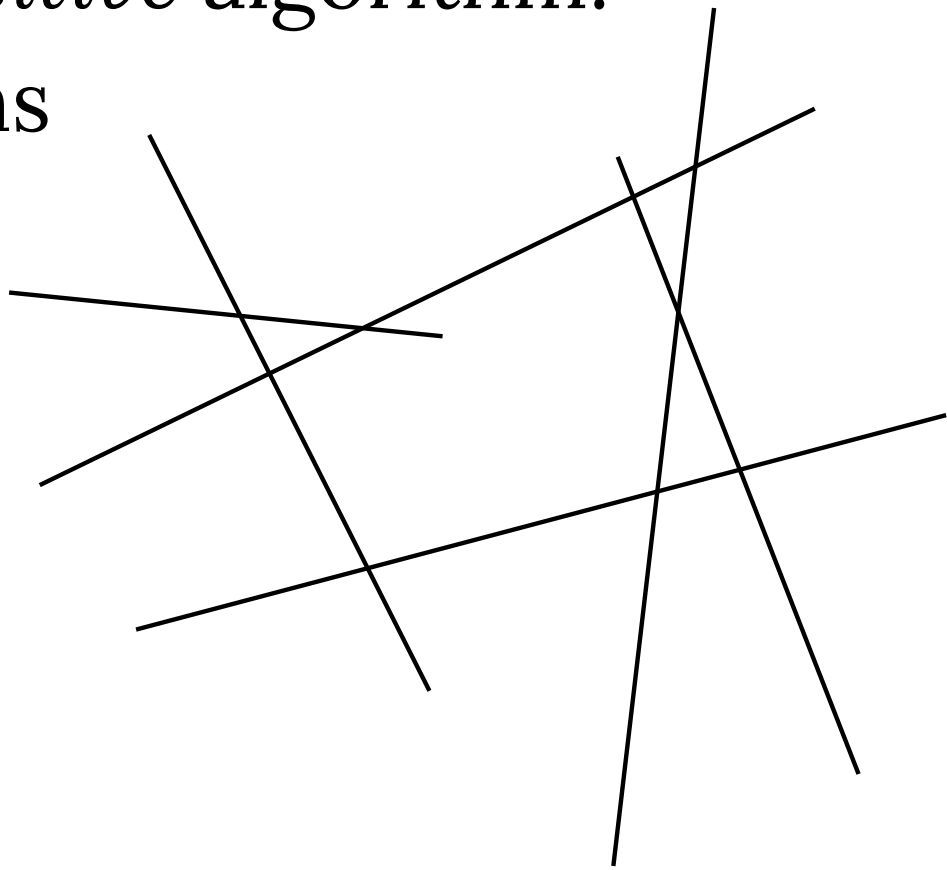
Today

- ✓ Intro
- ✓ Orientation and convex hulls
- **Line segment intersection**
- Polygons and triangulations
- Halfplane intersection and LP
- Voronoi diagrams
- Delaunay triangulations
- Point location

Credit: most of images and ideas from Dave Mount's lecture notes
http://www.cs.umd.edu/~mount/754/Lects/754lects.pdf
And David Kirkpatrick's lectures

Line Segment Intersection

- Max # of intersections: $O(n^2)$
- Need an *output sensitive* algorithm!
- I – # of intersections



Plane Sweep

- Sweep virtual line
- Stop at events
  - segment started/finished
  - intersection

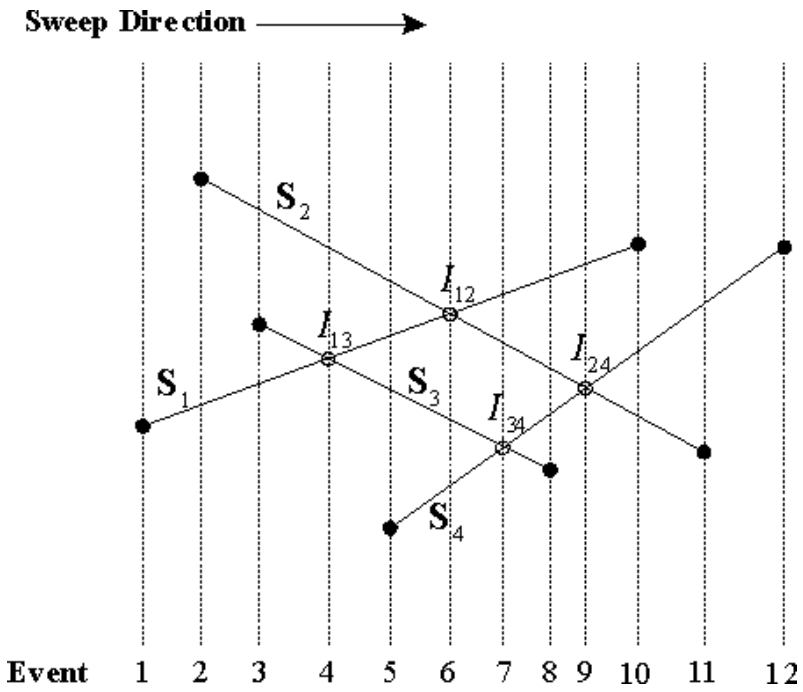
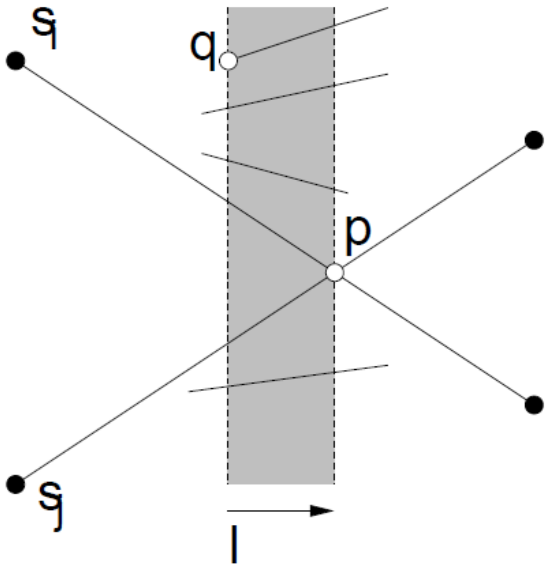


Image from geomalgorithms.com

Plane Sweep

- Presort the endpoints
- How to detect next intersection?
  - If two segments are adjacent along the sweep line,
  - Check if they intersect to the right of the sweep line



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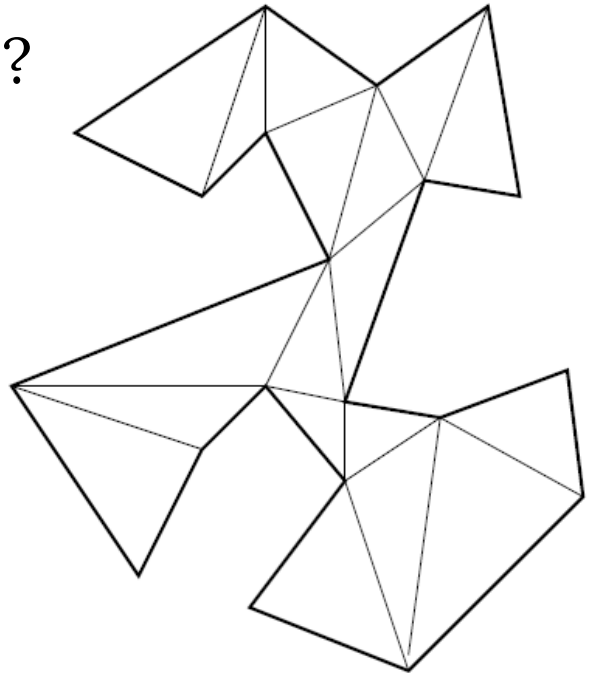
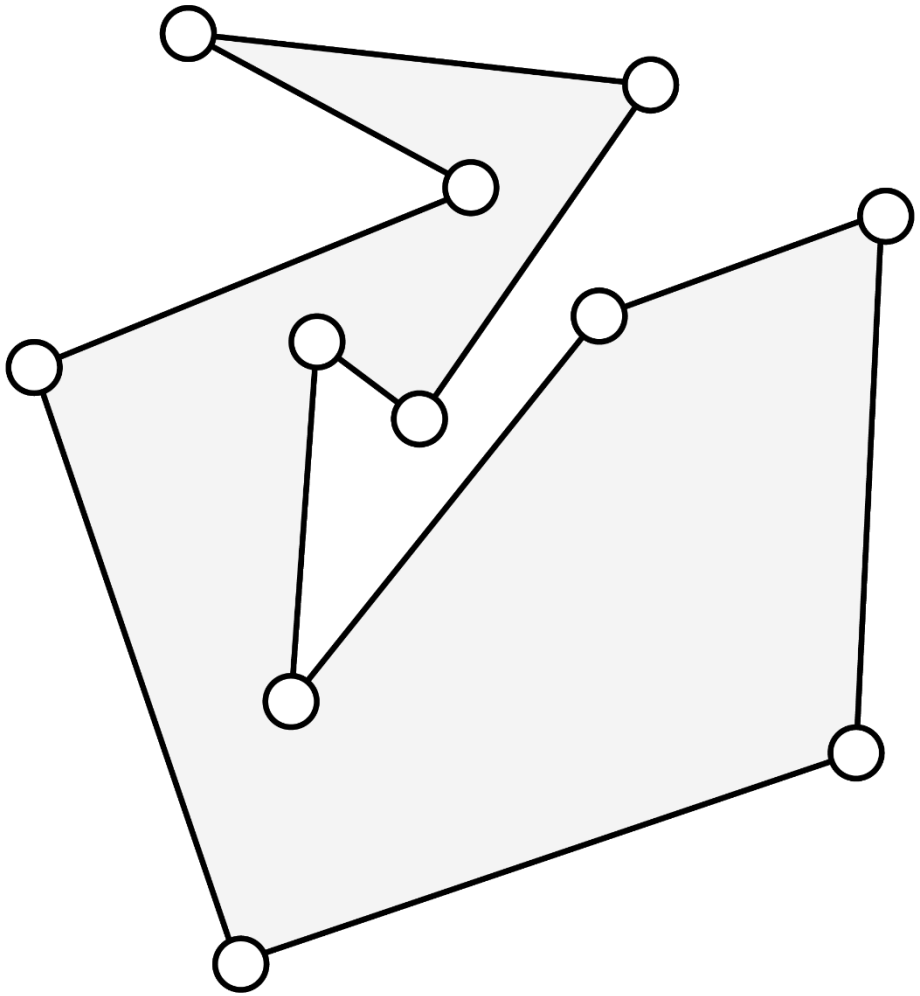
Polygons and triangulations

- Why?
  - Decomposition of complex shapes
    - How to compute area of a polygon?
  - Art gallery problem
    - How many cameras and where?



Art Gallery Problem



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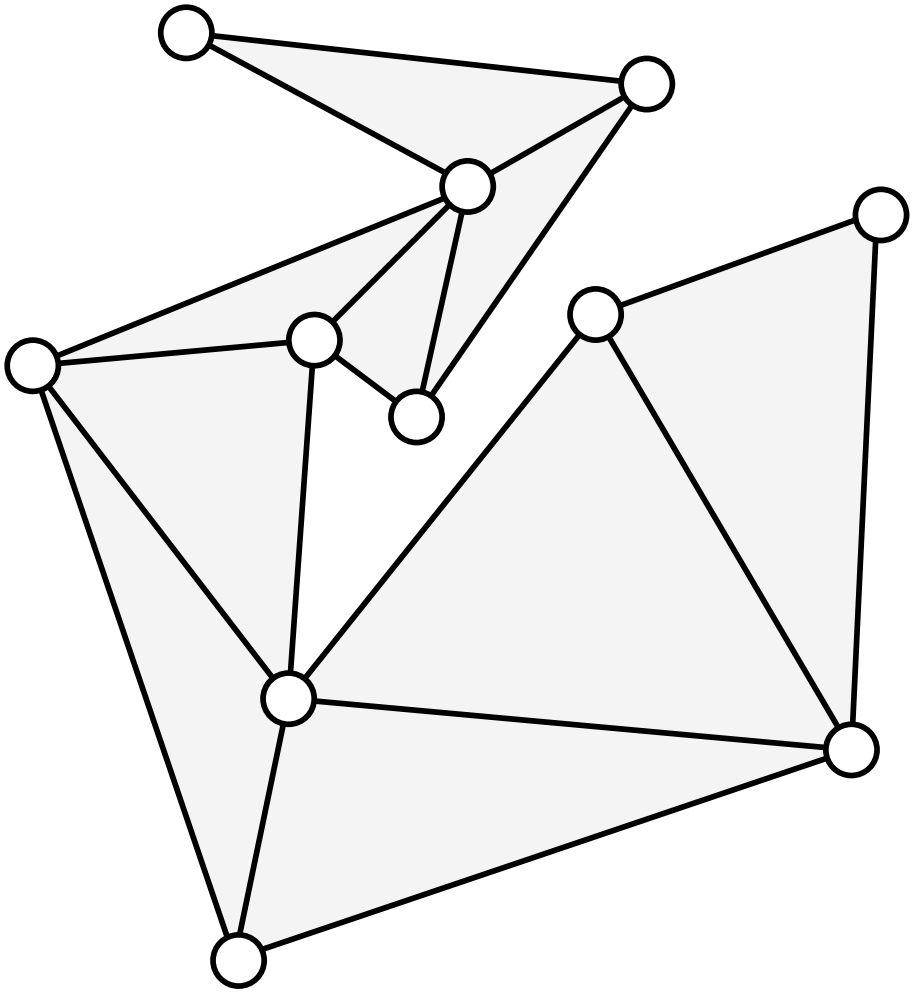

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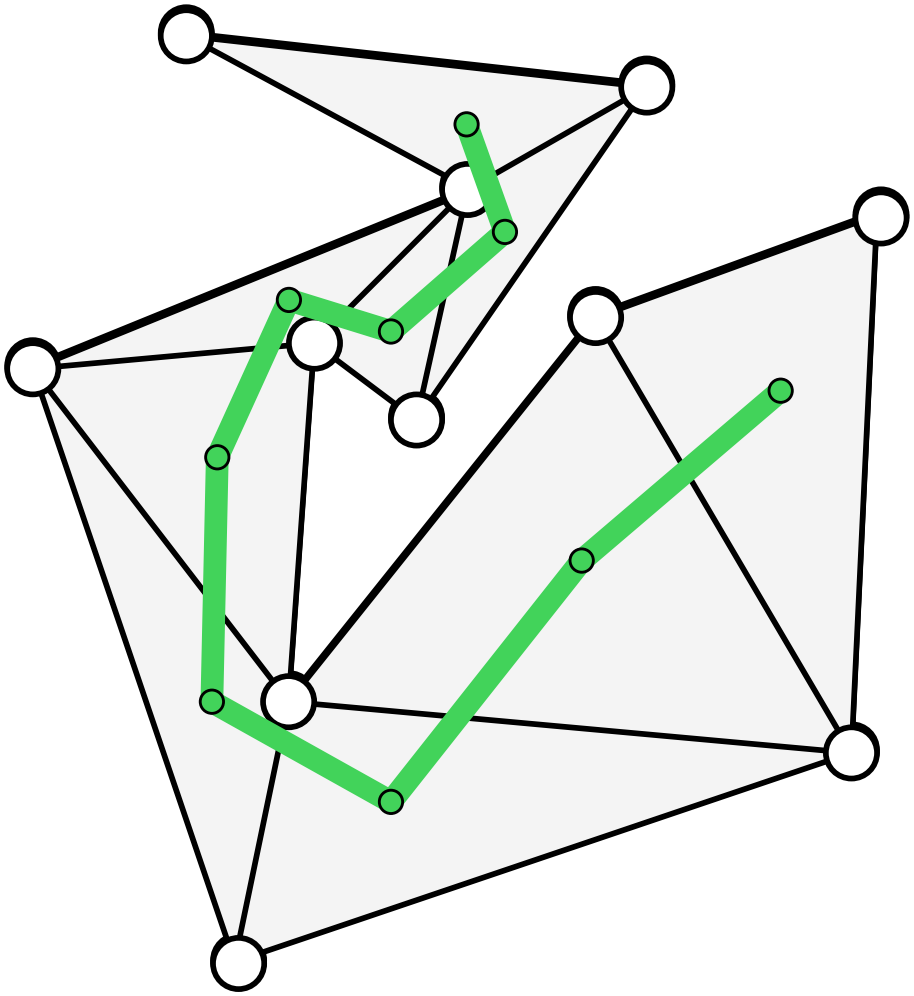

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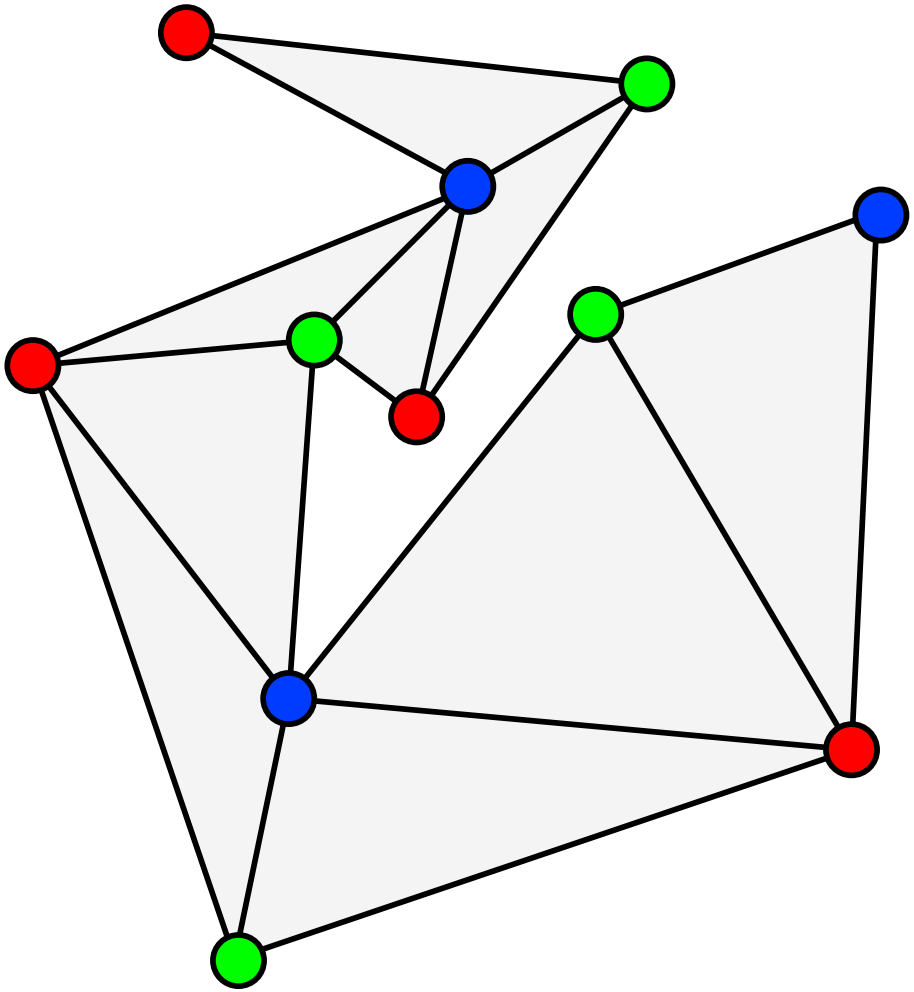

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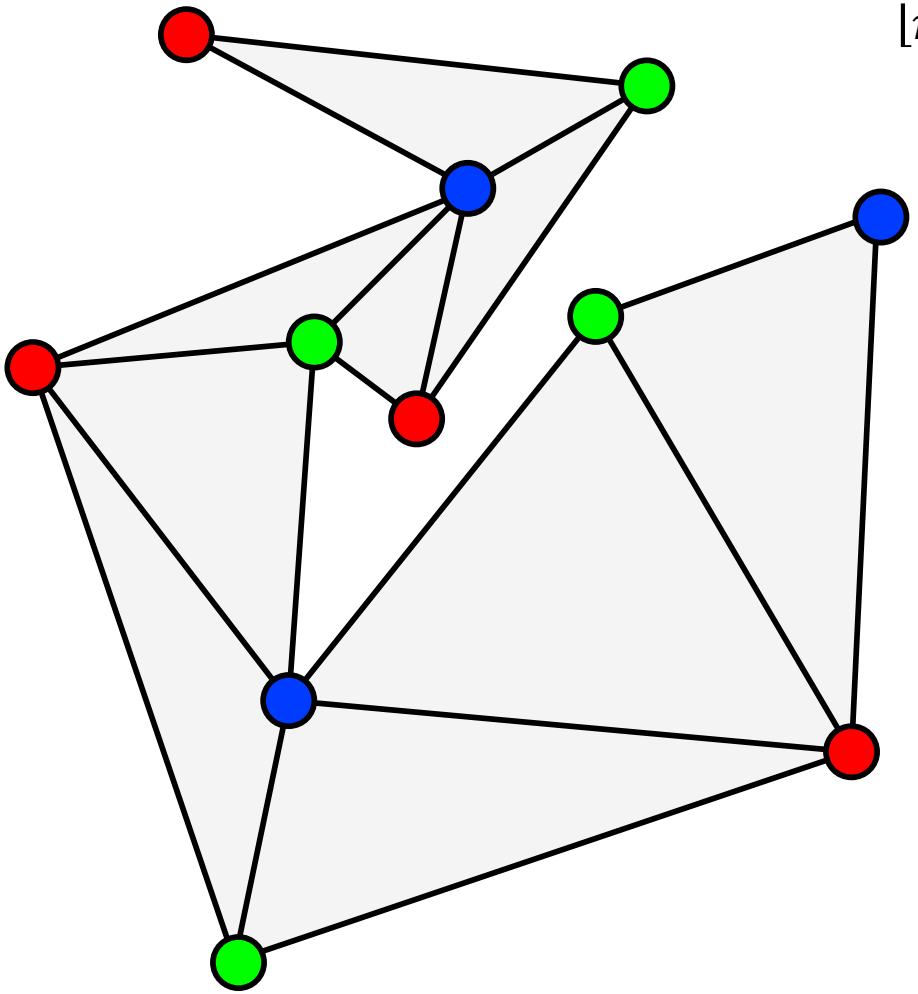
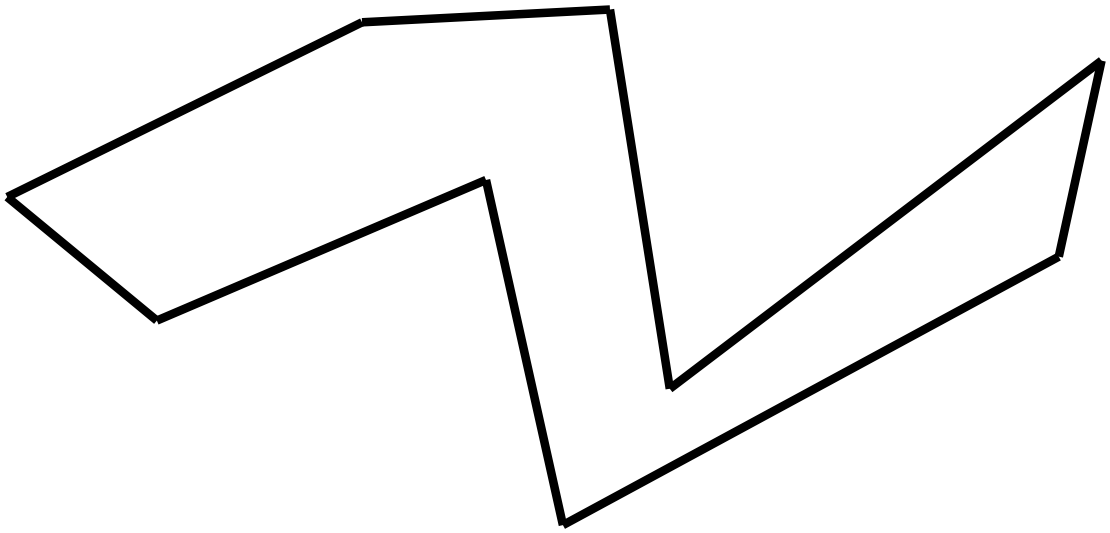


$\lfloor n/3 \rfloor$ cameras are enough!

Is it optimal?

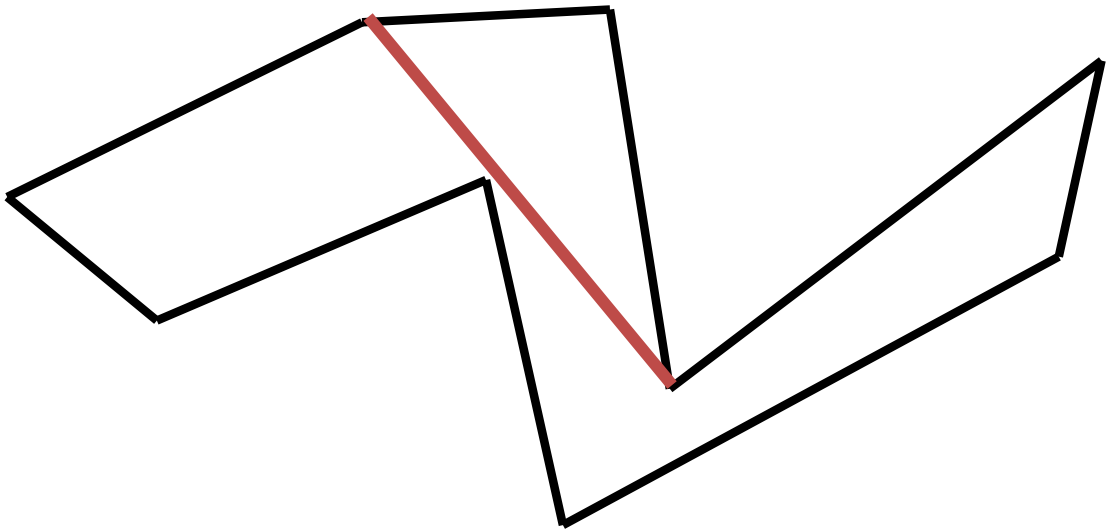
Triangulations

- How to find ***a*** triangulation?



Triangulations

Diagonal: line segment connecting two vertices completely within the polygon



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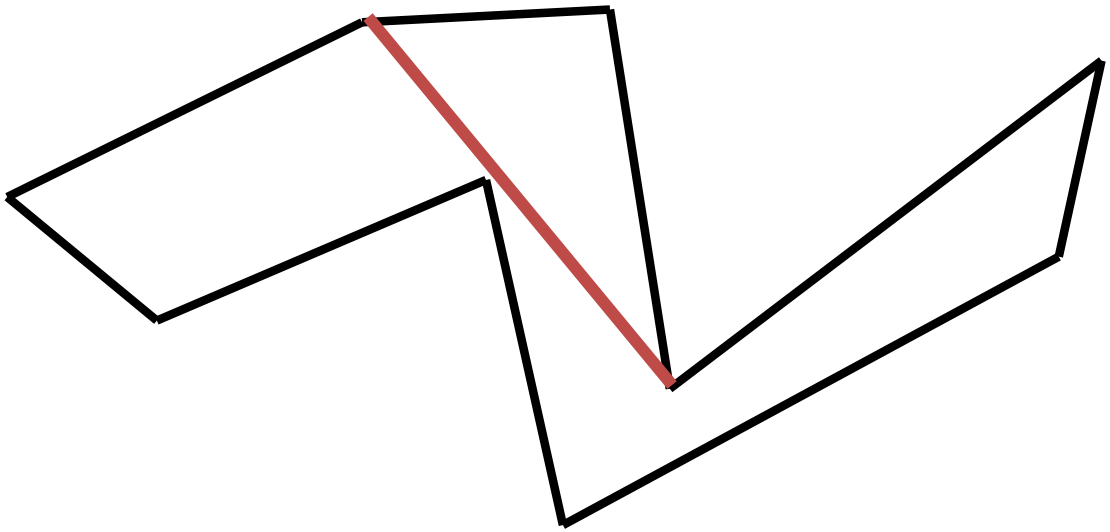
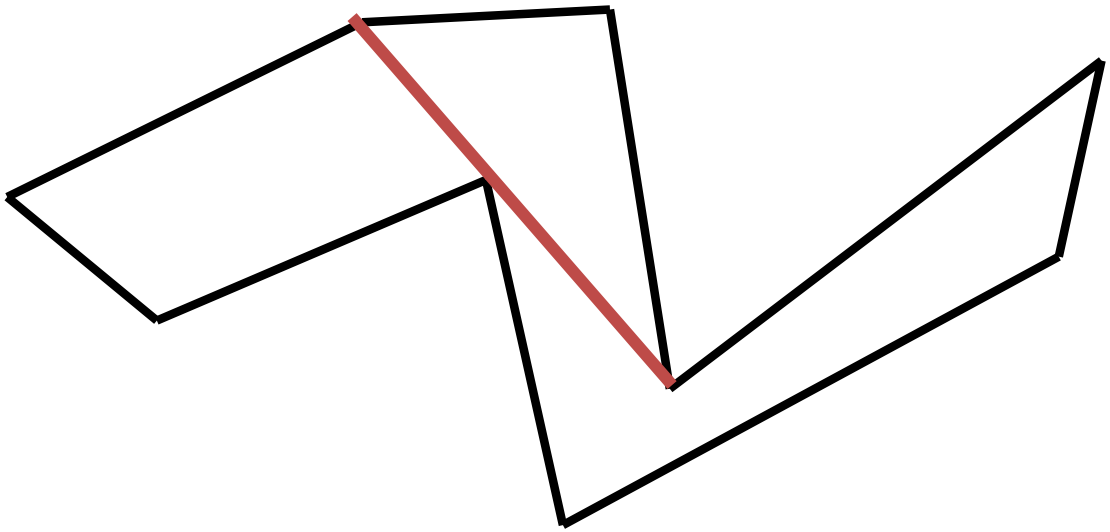
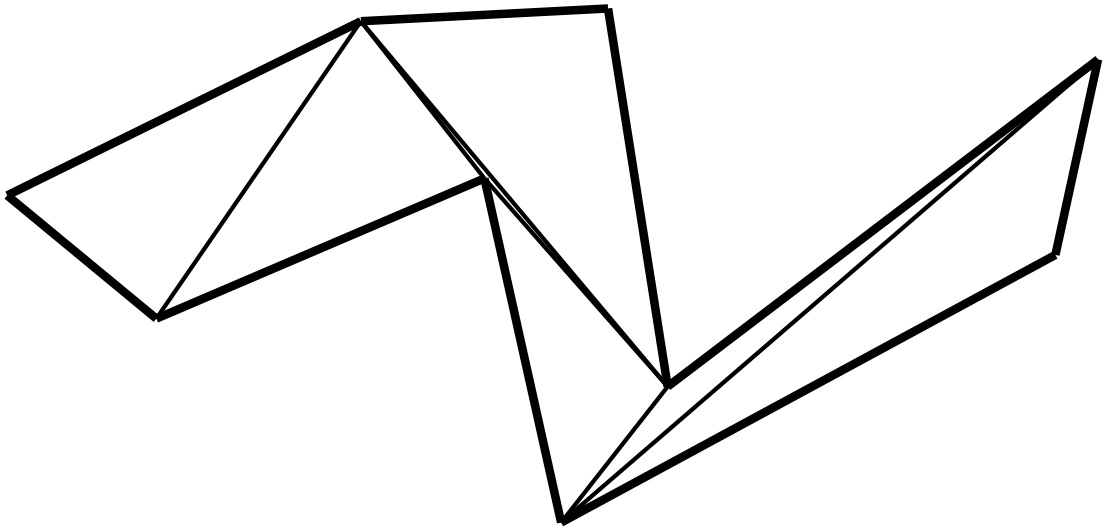


Every polygon has at least 1 diagonal!

Triangulations

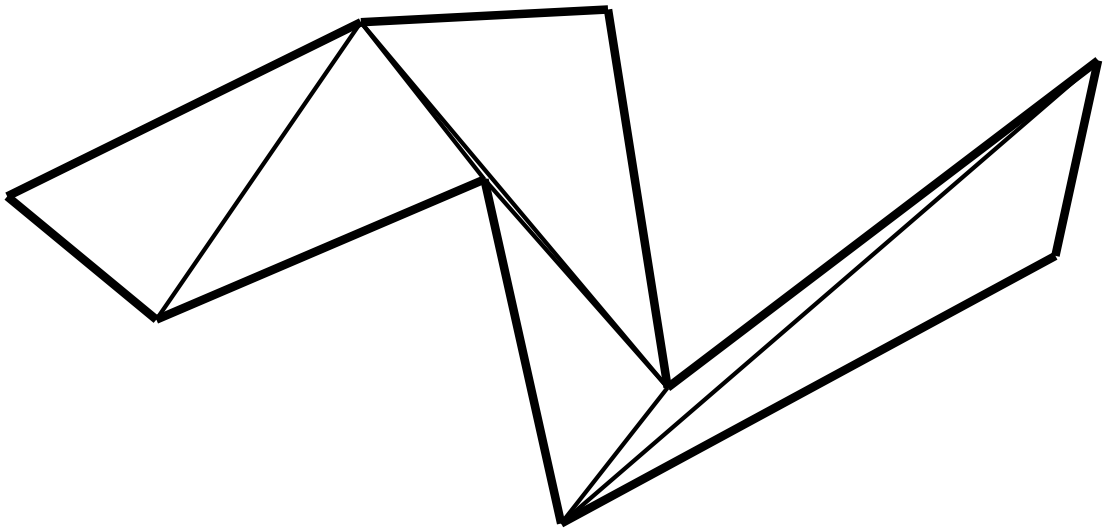


Triangulations



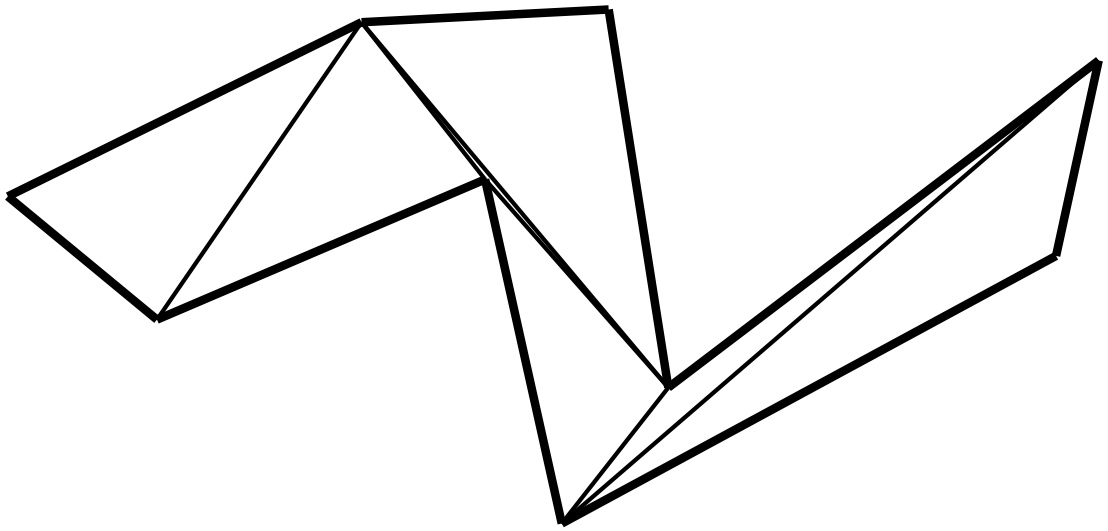
Triangulations

- Algorithm: add diagonal, repeat for the two new polygons



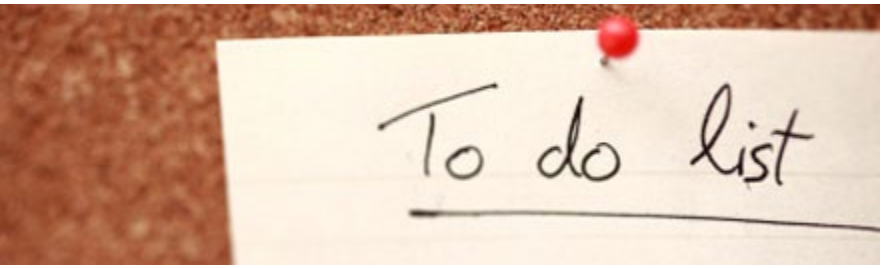
Triangulations

- Algorithm: add diagonal, repeat for the two new polygons
  - Bottleneck?
  - Complexity?



A better algorithm

- $O(n \log n)$
- Input: cyclic list of vertices
  - Presorted!



https://www.smead.com/hot-topics/organized-to-do-list-1023.asp

1. Learn how to triangulate monotone polygons
2. Subdivide arbitrary polygon into monotone ones
3. Profit!

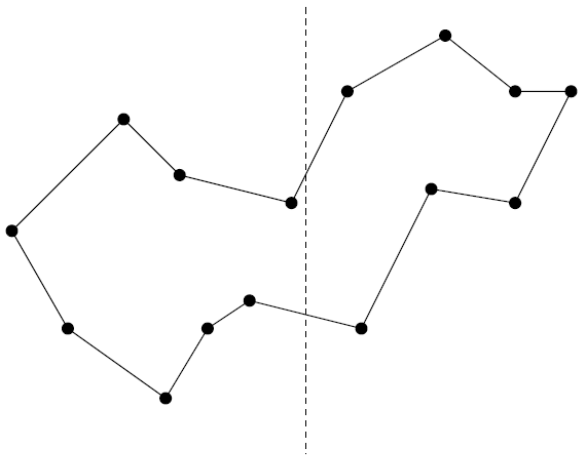
Monotone polygons

Polygonal chain is *x-monotone* ⇔

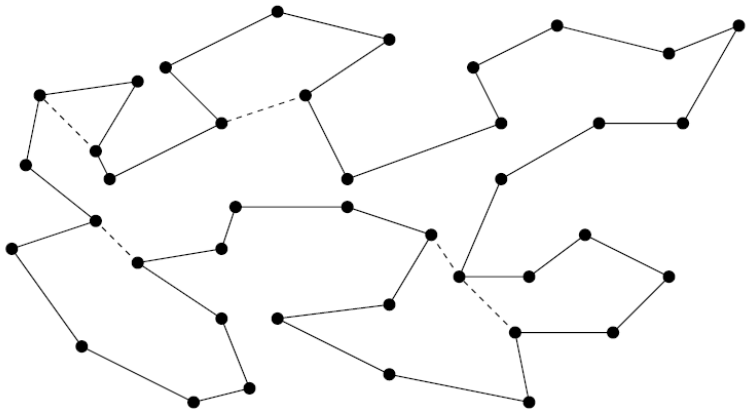
every vertical line intersects chain in at most 1 point

Polygon is *x-monotone* ⇔

can be split into two x-monotone chains



x-monotone polygon

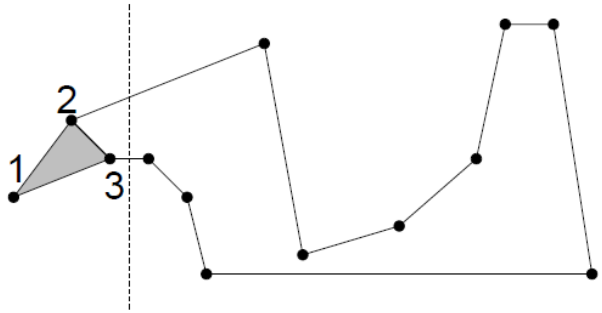


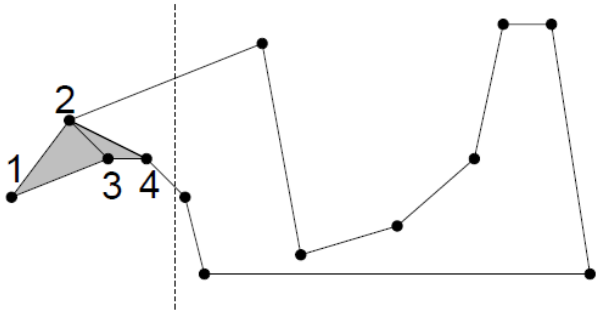
Monotone decomposition

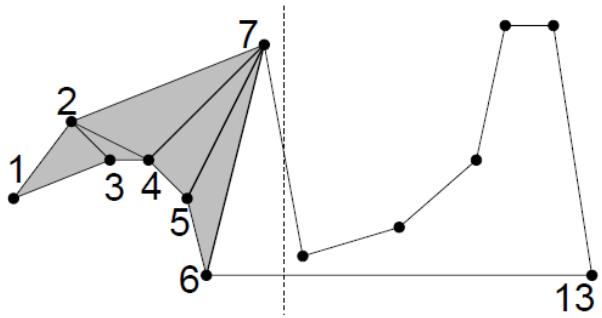
Triangulating monotone polygons

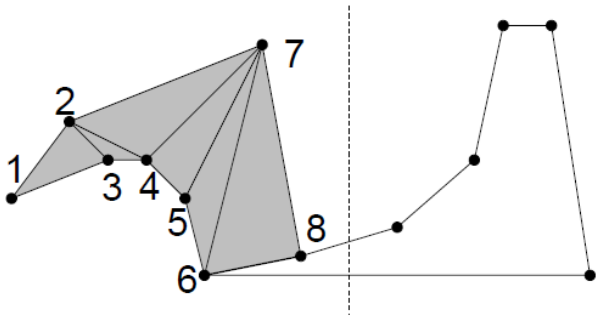
- Sort vertices left to right
  - Doesn't require actual sorting (why?)
  - Requires $O(n)$ time
- Line sweep
  - Triangulate everything to the left of the sweep line
  - Discard the triangulated part
  - Testing diagonals is now constant-time
    - Why?

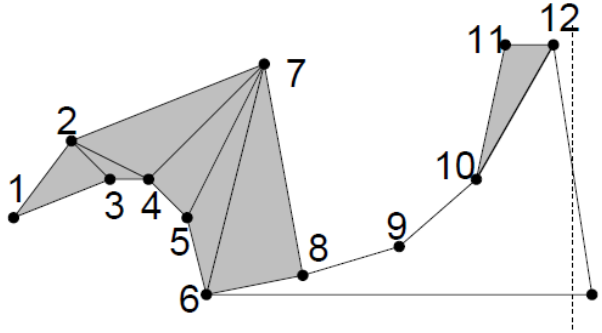
Triangulating monotone polygons

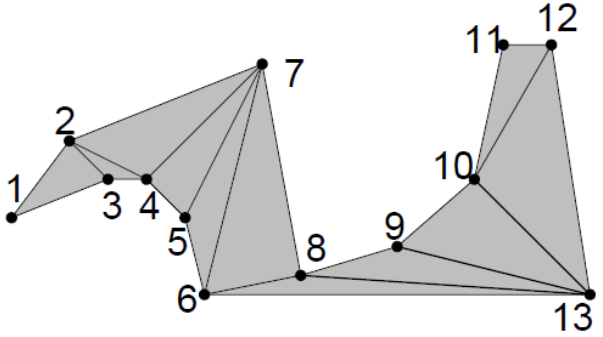








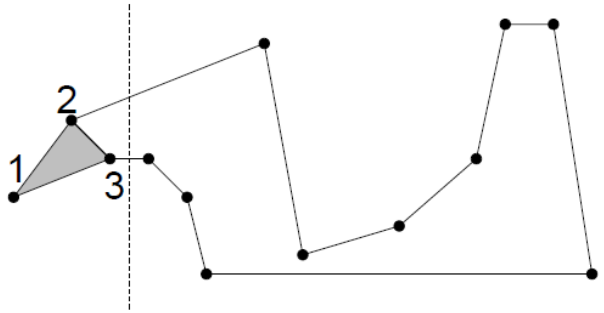


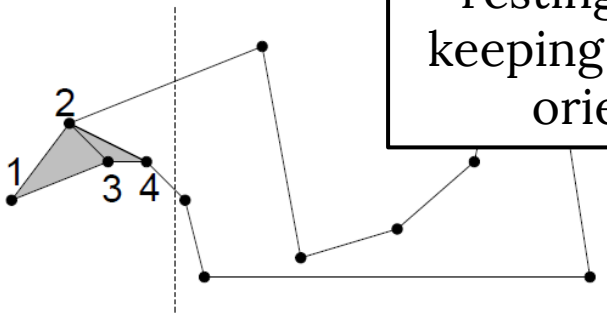


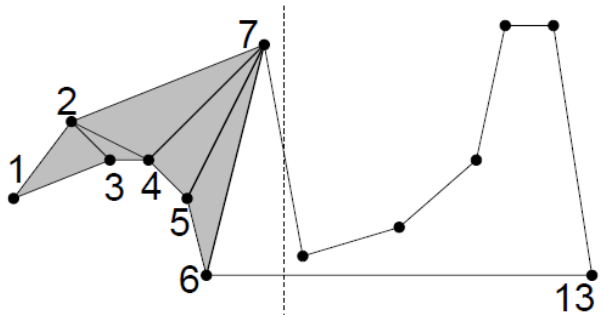
$O(n)$

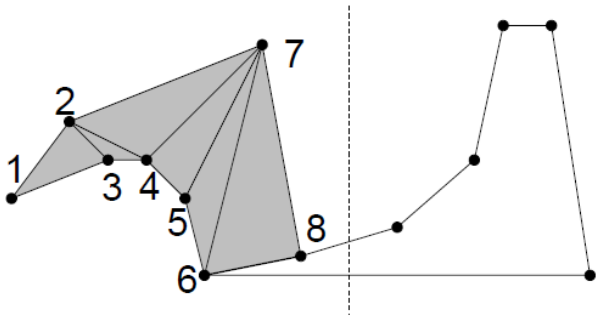
Triangulating monotone polygons

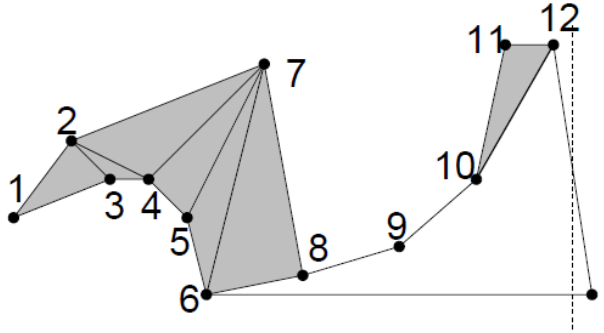
Testing diagonals = keeping track of edge orientations

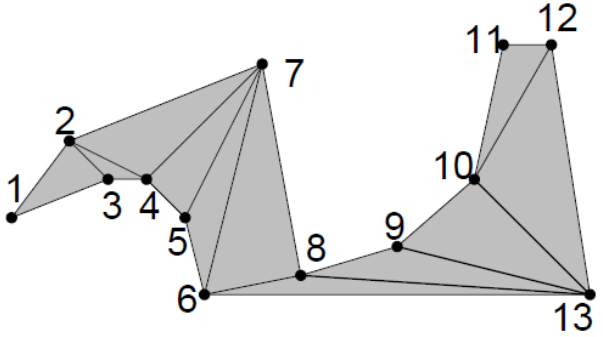












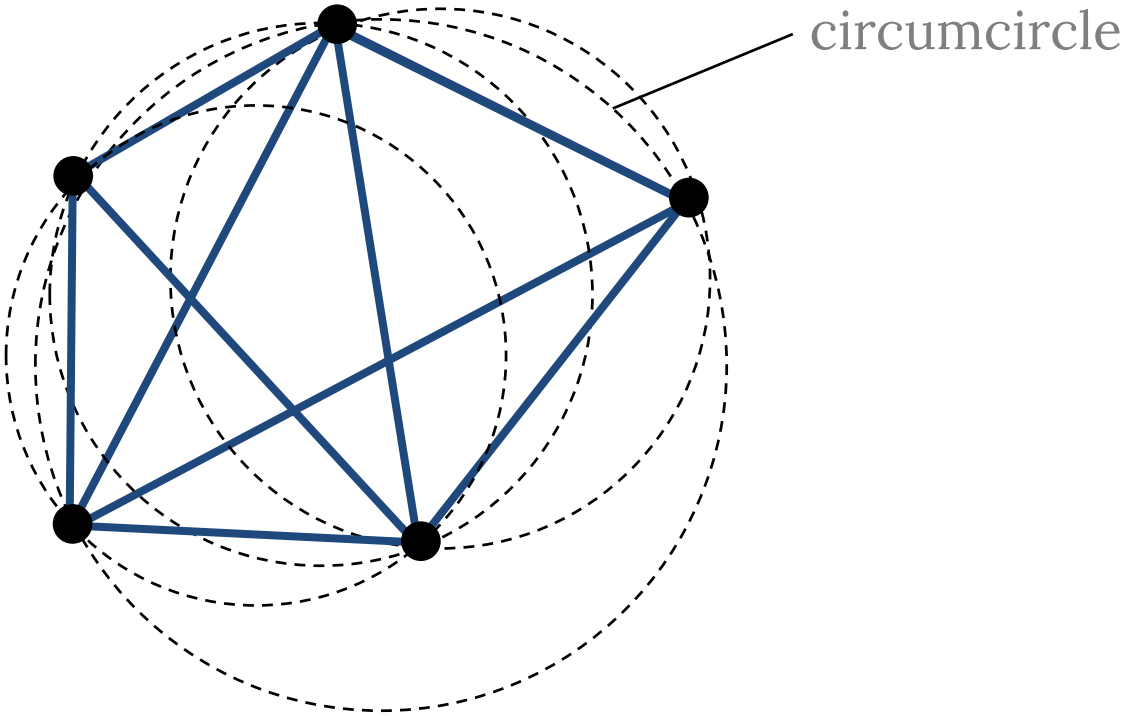
$O(n)$

Today

- ✓ Intro
- ✓ Orientation and convex hulls
- ✓ Line segment intersection
- ✓ Polygons and triangulations
- **Voronoi diagrams**
- **Delaunay triangulations**

Credit: most of images and ideas from Dave Mount's lecture notes
http://www.cs.umd.edu/~mount/754/Lects/754lects.pdf
And David Kirkpatrick's lectures

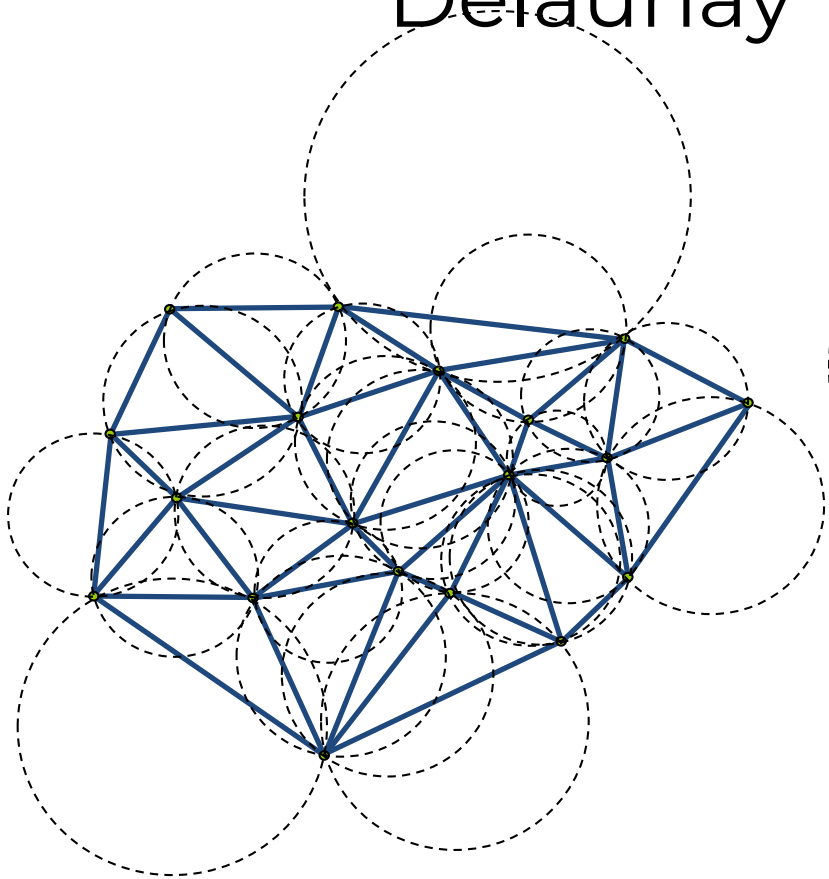
Triangulation Quality: Delaunay Criterion

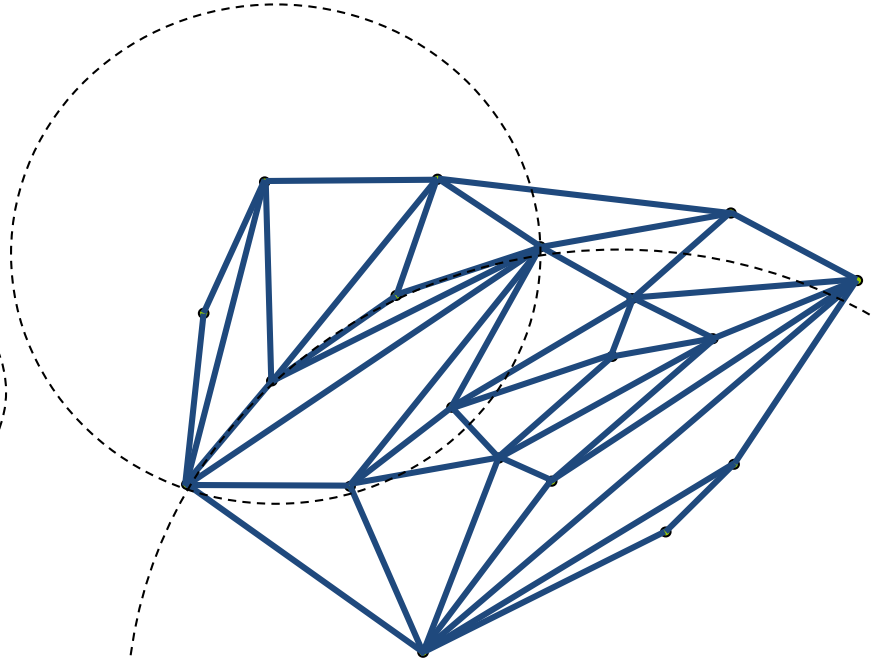


Empty Circle Property (definition of Delaunay Triangulation)

No other vertex is contained within the circumcircle of any triangle

Delaunay Triangulation



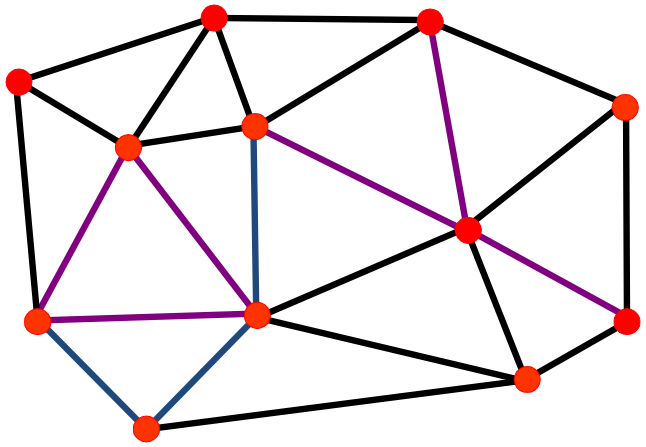


Non-Delaunay
Triangulation

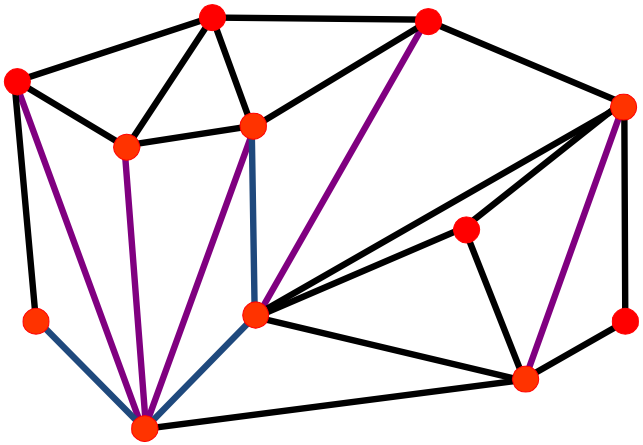
Delaunay Triangulation

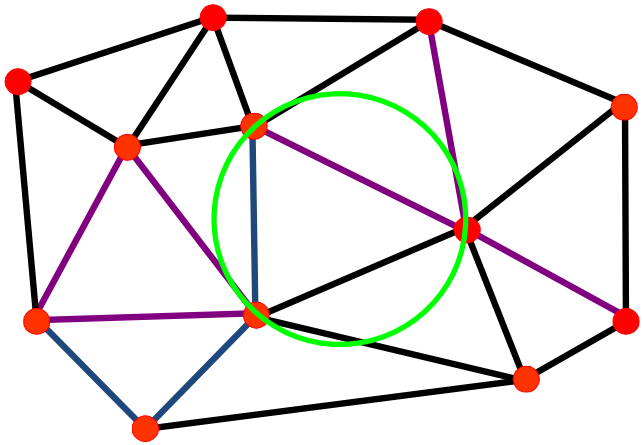
✓ Exists for any set of vertices

✓ Is **unique** (up to degenerate cases)



FUN FACTS ABOUT DT

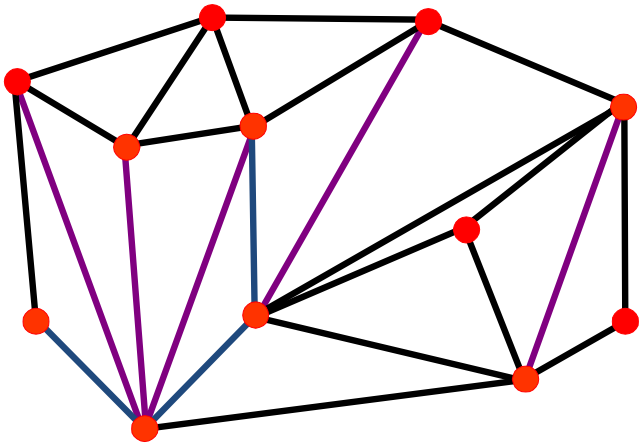


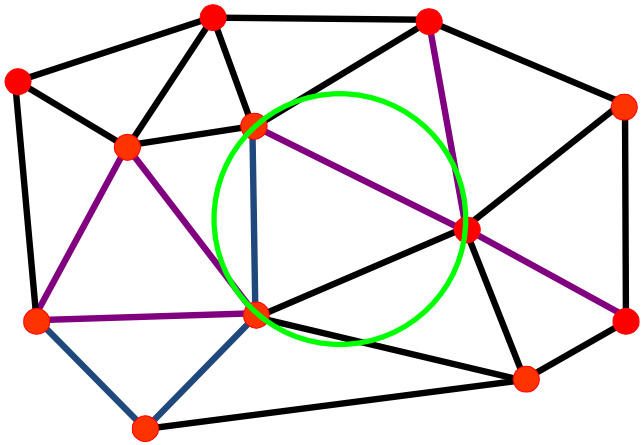


Theorem.
The exterior polygon of the DT is the convex hull.

DT is **the best triangulation** you can get in 2D.

THE BEST *HOW*?





Take all angles, sort = Sequence of numbers

$(9°, 11°, 12°, 15°, \ldots)$

DT's sequence is **the last lexicographically**

How to test if

- To test – enough to check pairs of triangles sharing common edge

Edge flipping

Start with any triangulation

1. Find a non-Delaunay edge
2. Flip edge
3. Repeat

Converges to Delaunay

Slow: $O(n^2)$

Vertex Insertion

Start with Delaunay mesh covering domain

- Typically 2 triangle bounding box

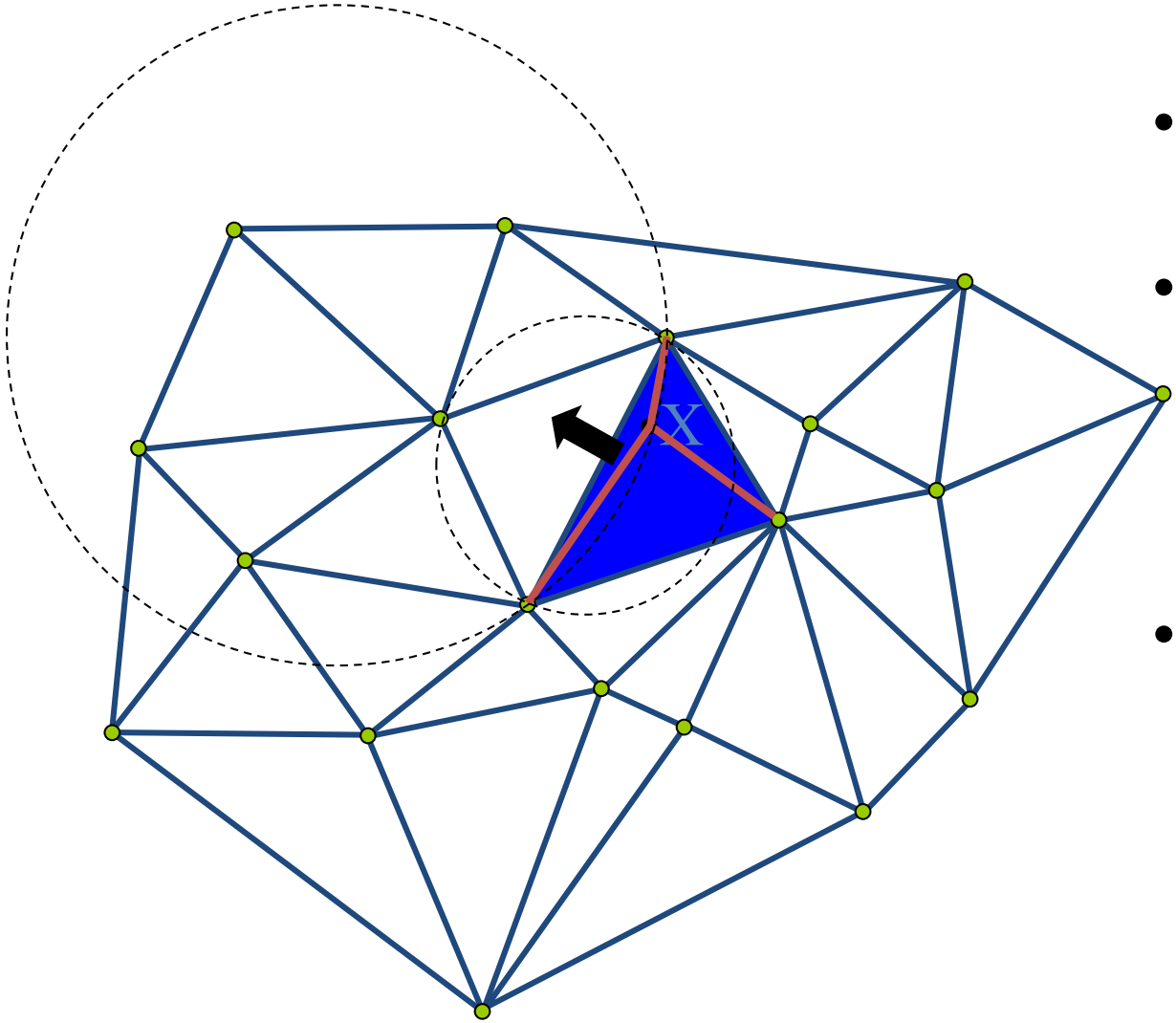
Insert one vertex at a time

- Add vertex to mesh
- Flip edges locally to maintain Delaunay property

Boundary recovery

Expected time $O(n \log n)$

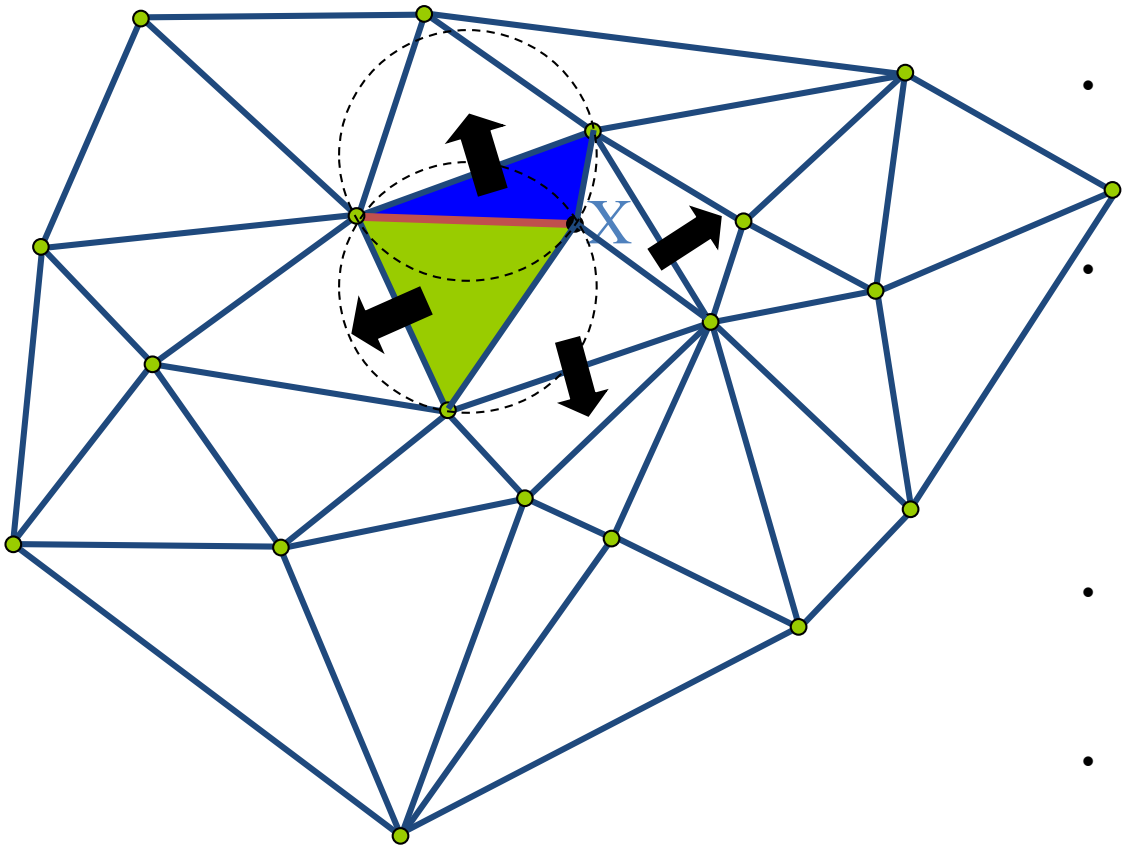
Vertex Insertion



- Locate triangle containing X
- Subdivide triangle

- Recursively check adjoining triangles to ensure empty-circle property

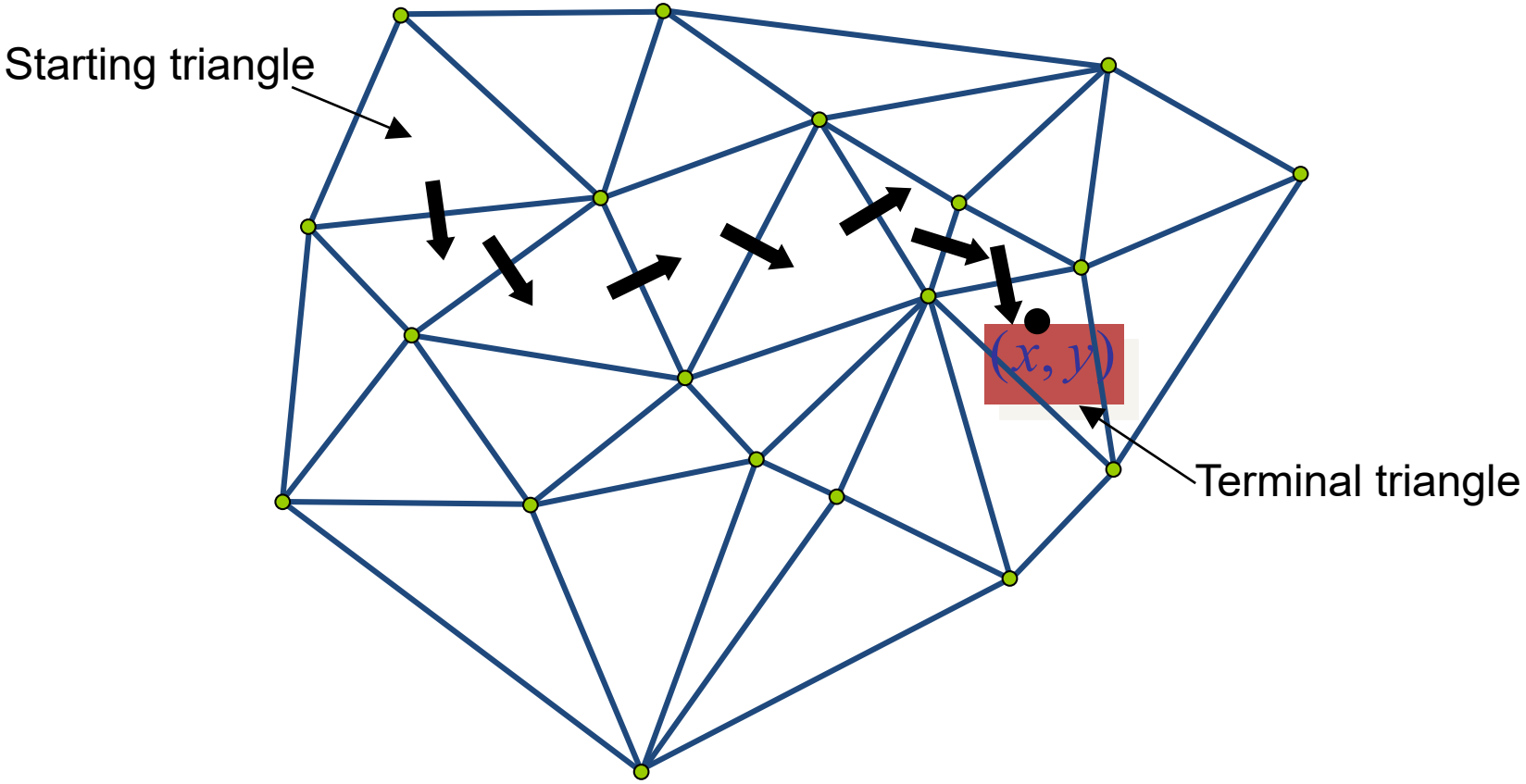
Vertex Insertion



- Locate triangle containing X
- Subdivide triangle
- Recursively check adjoining triangles to ensure empty-circle property
- Swap diagonal if needed
- Typically very small number of swaps

Find Triangle - Efficiency

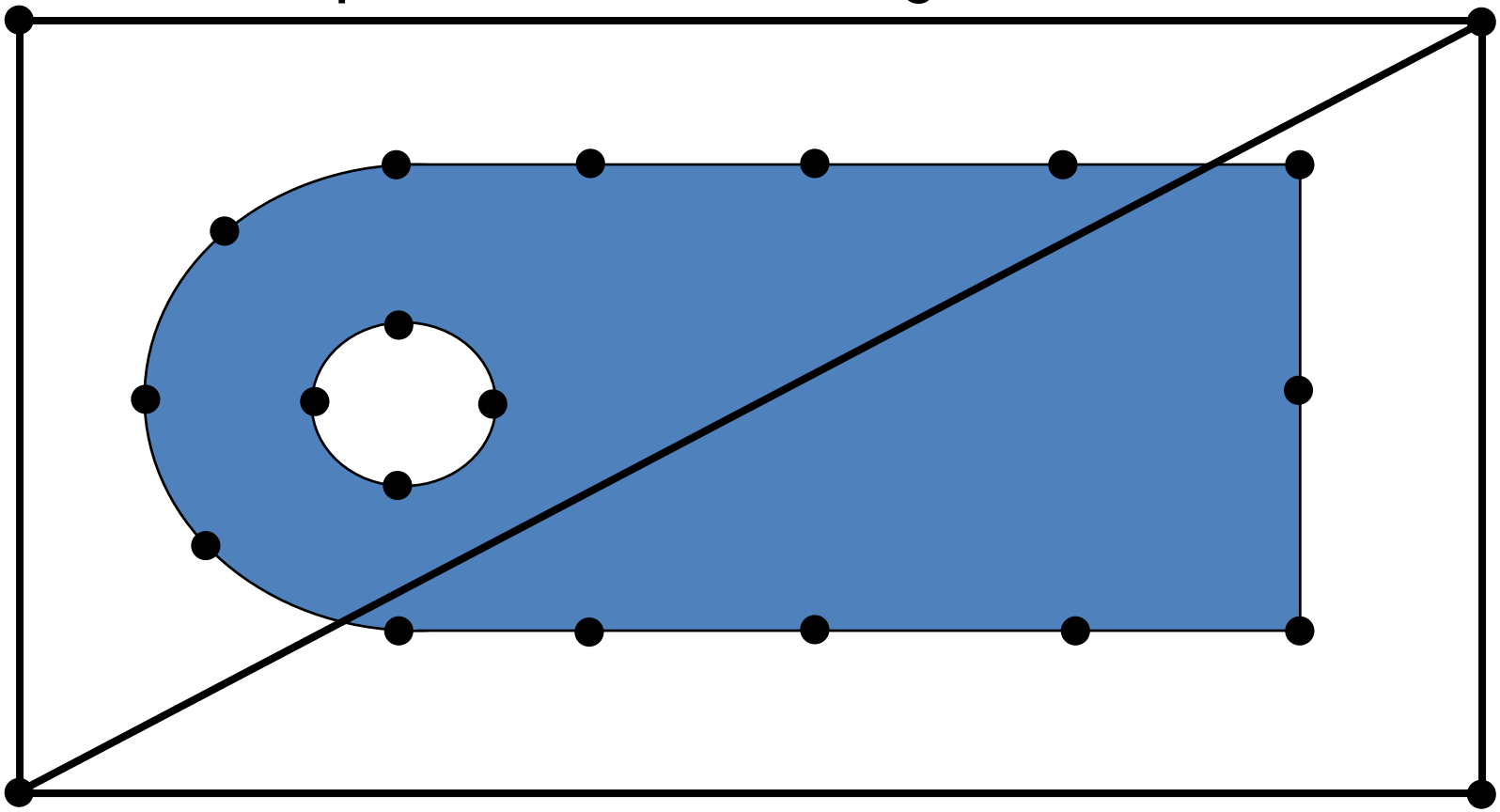
- Use Barycentric Coordinates
  - Test inside triangle
  - If outside - outside which edge?



Vertex Insertion

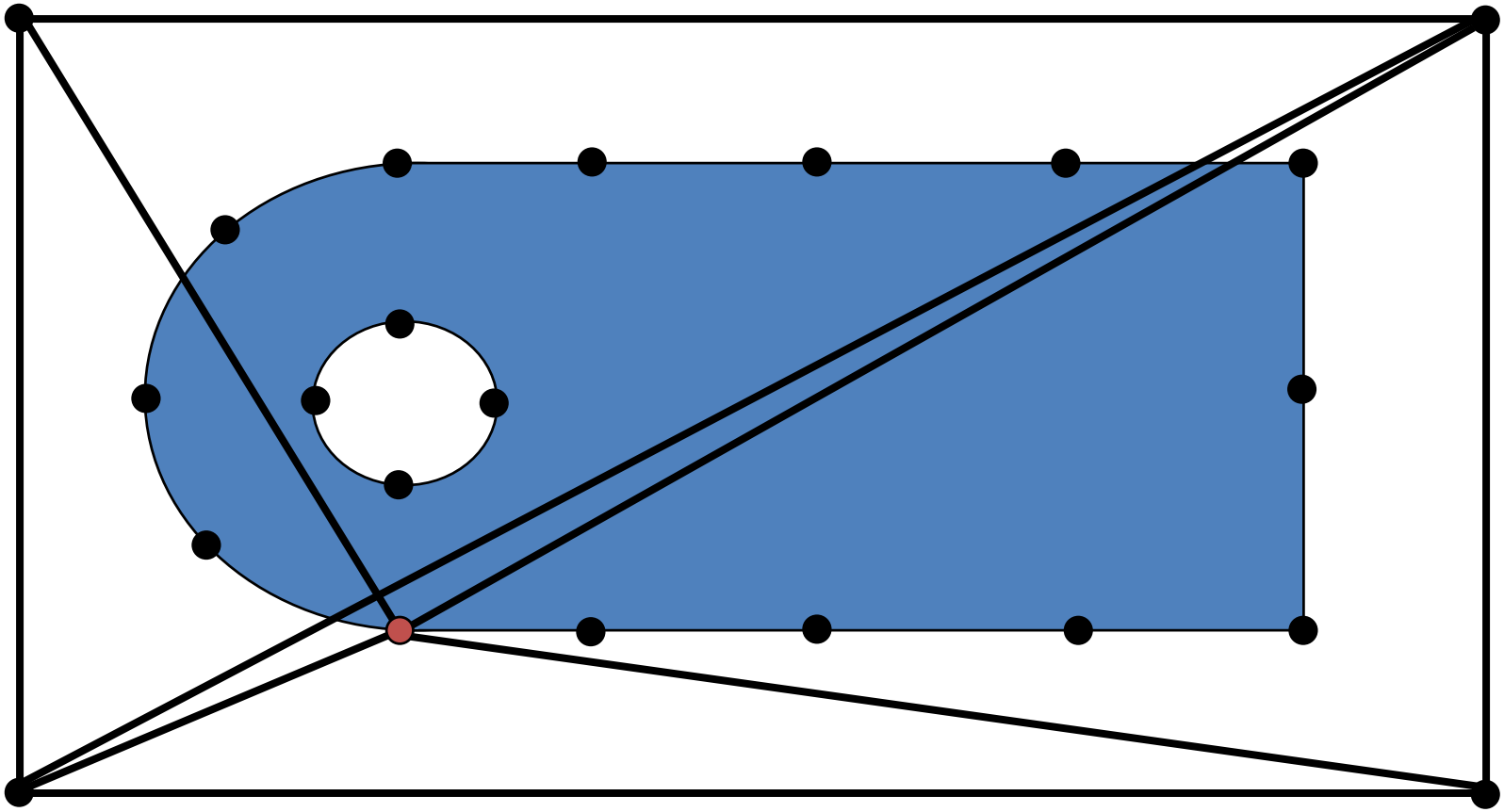
- Start with Delaunay mesh covering domain
  - Typically 2 triangle bounding box
- Insert one vertex at a time
  - Add vertex to mesh (locate triangle to split)
  - Flip edges (locally) to preserve Delaunay property
- Boundary recovery

Example: Boundary Insertion



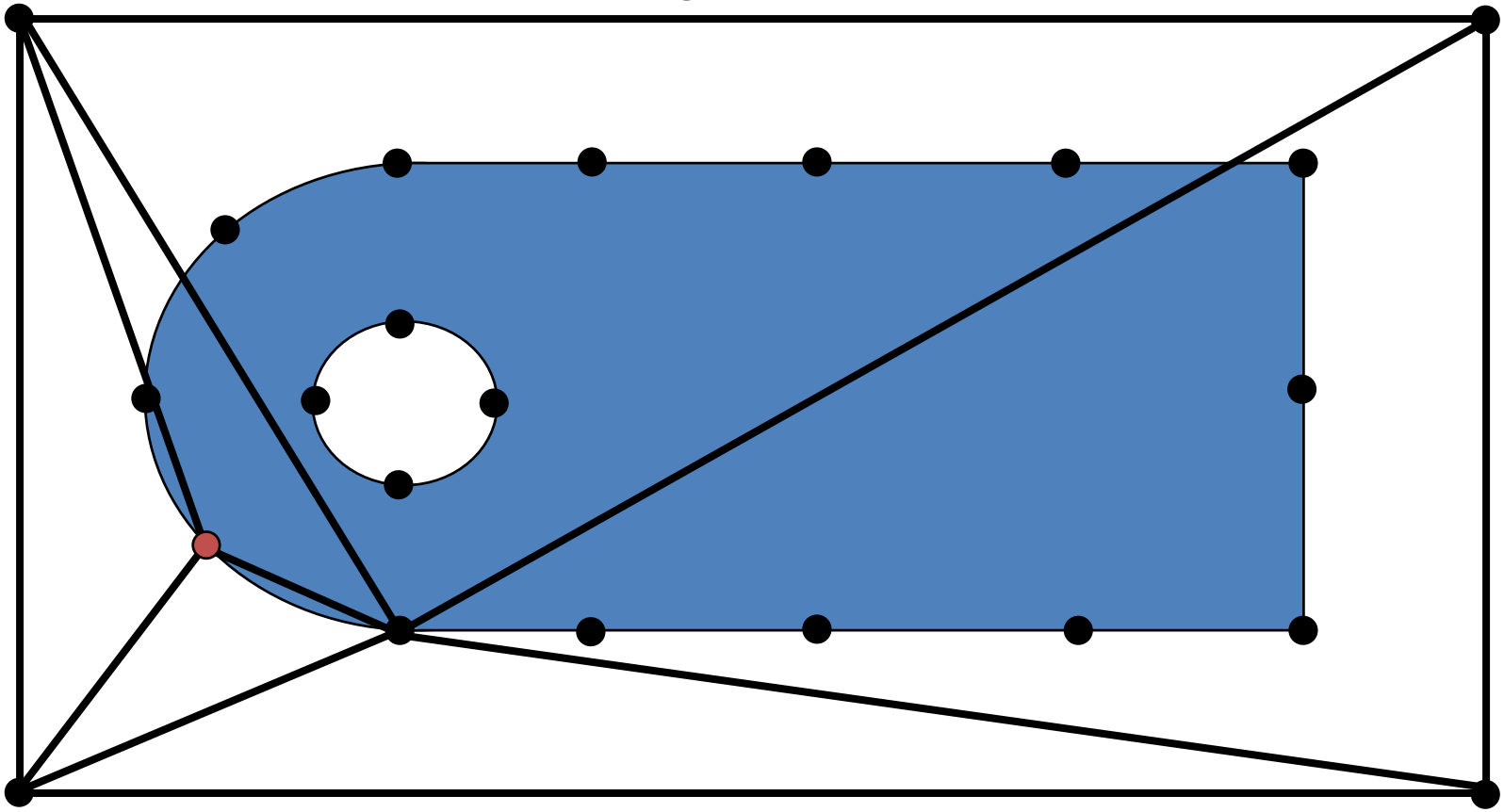
- Create bounding triangles

Boundary Insertion



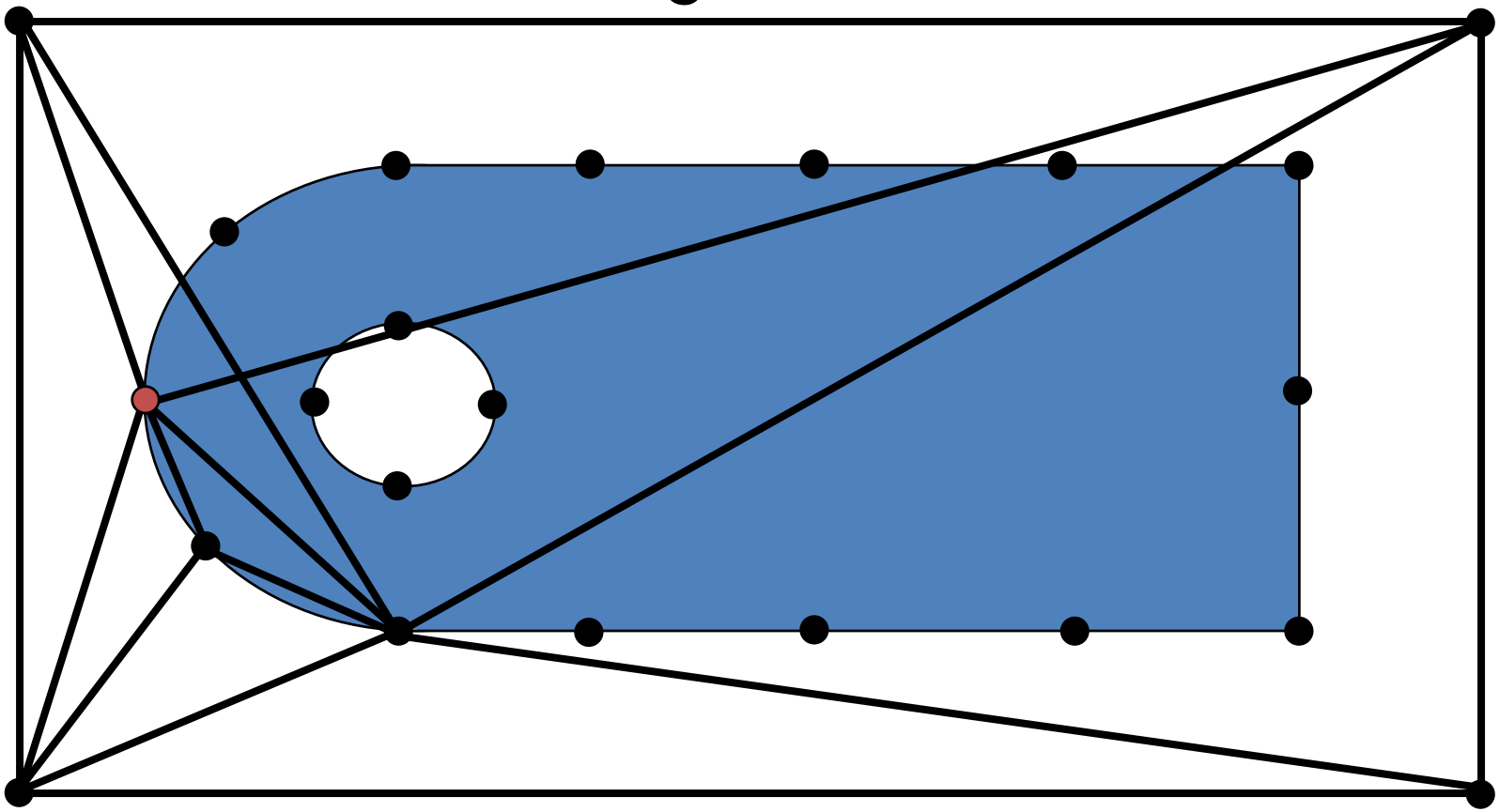
- Insert vertices using Delaunay method

Boundary Insertion



- Insert vertices using Delaunay method

Boundary Insertion



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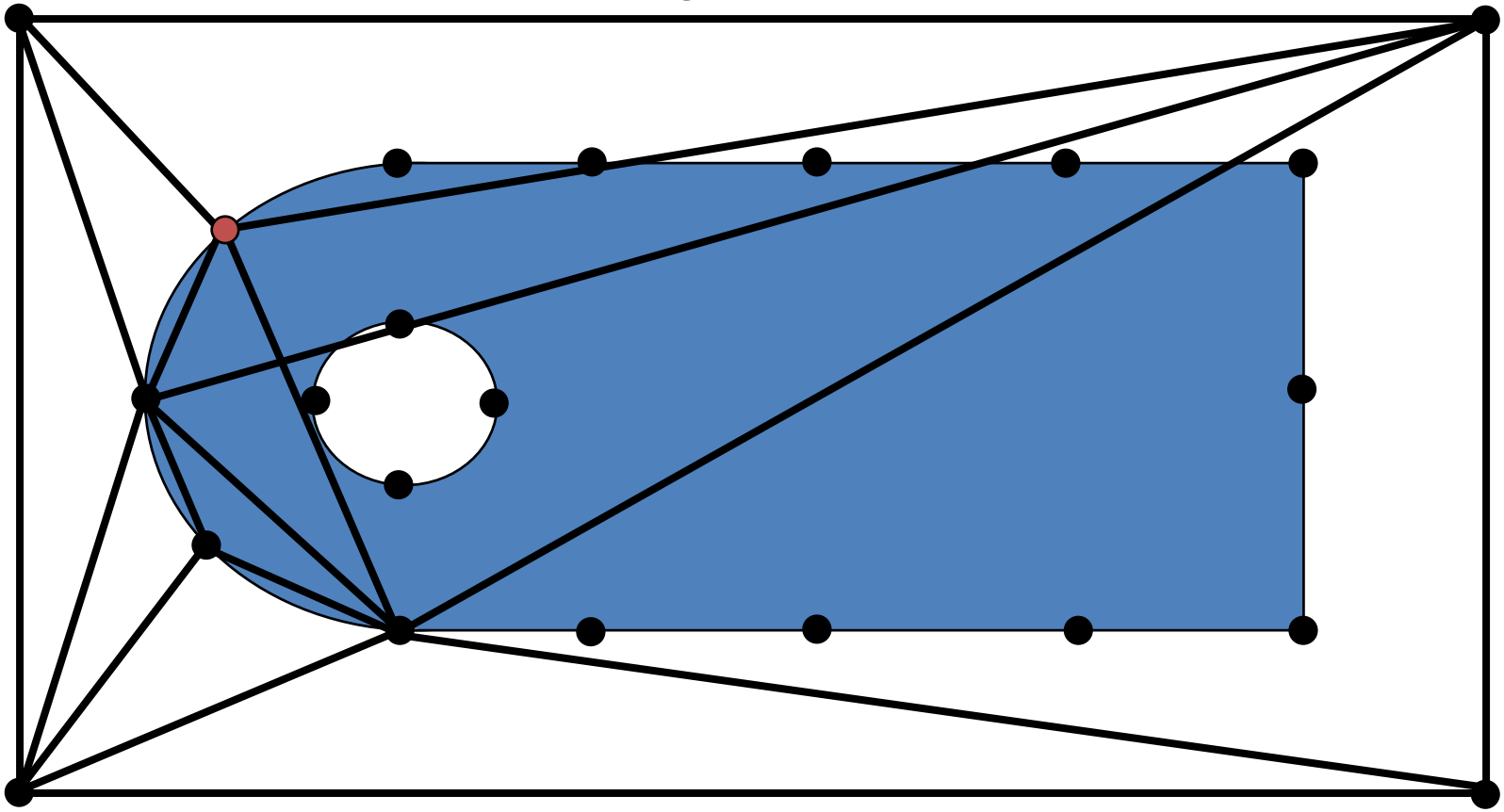
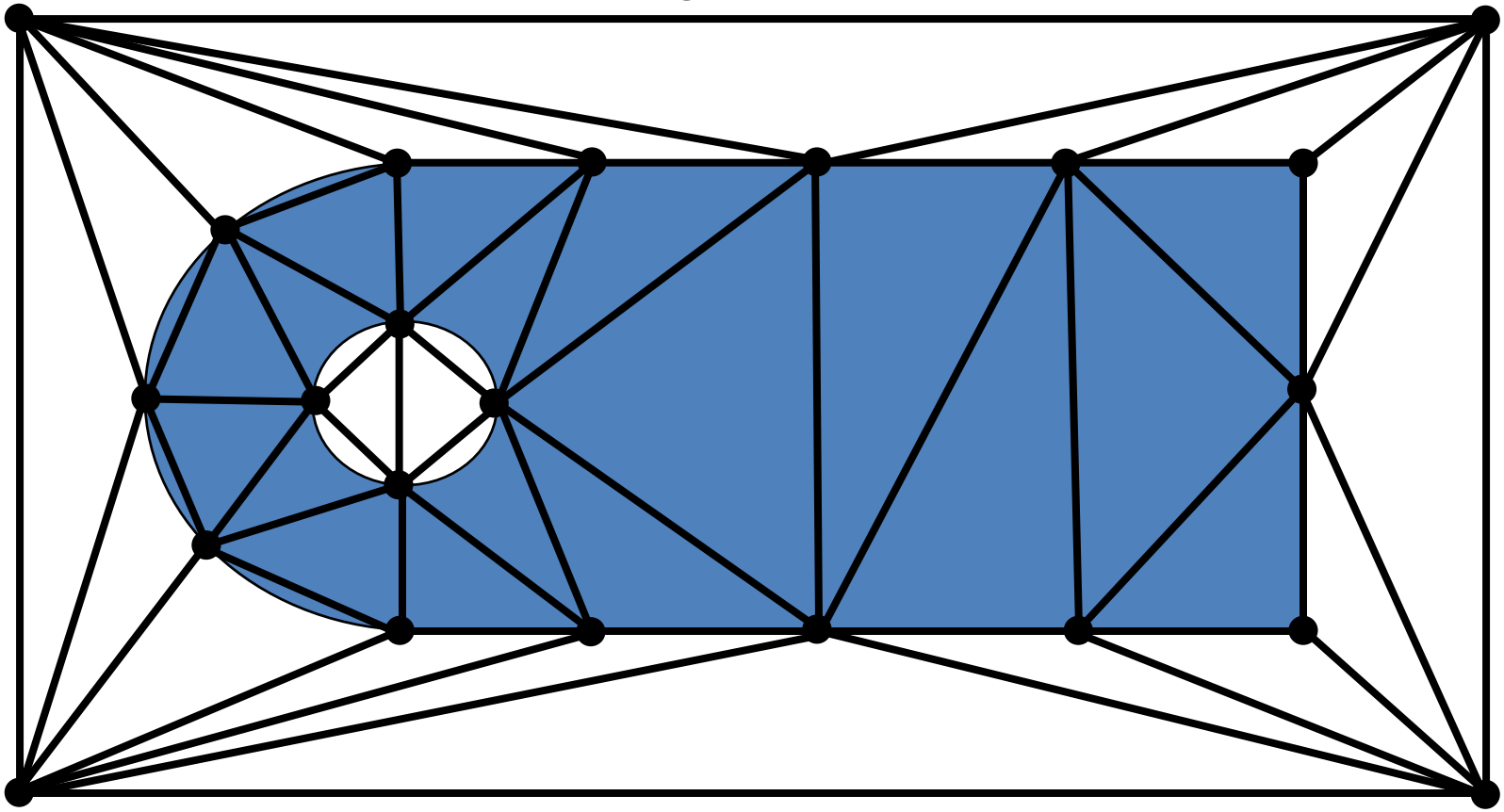
Boundary Insertion



- Insert vertices using Delaunay method

Boundary Insertion

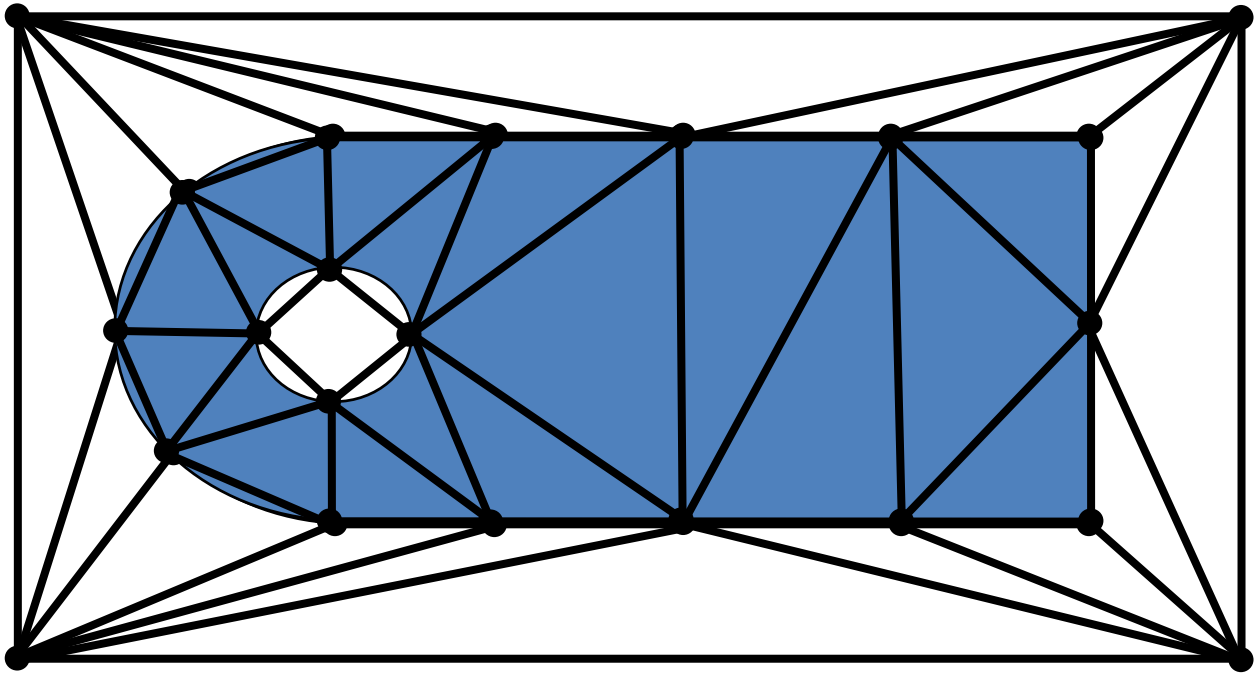


- Insert vertices using Delaunay method

Vertex Insertion

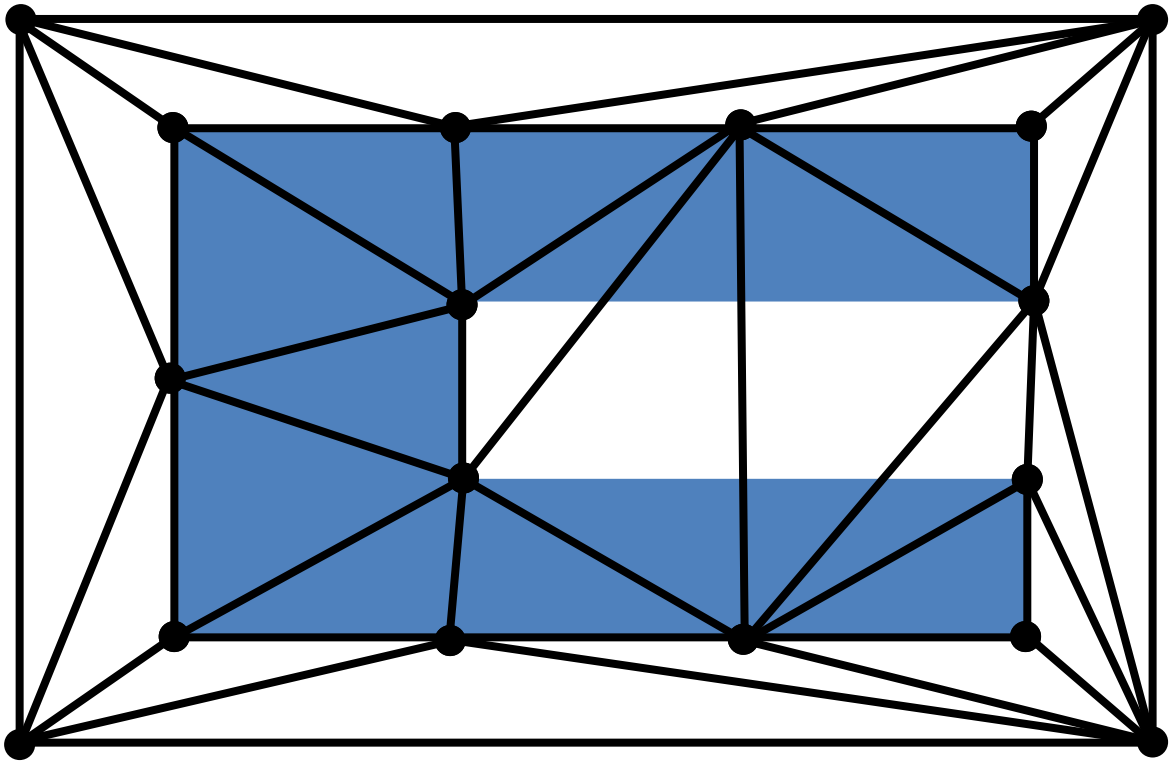
- Start with Delaunay mesh covering domain
  - Typically 2 triangle bounding box
- Insert one vertex at a time
  - Add vertex to mesh (locate triangle to split)
  - Flip edges (locally) to preserve Delaunay property
- Boundary recovery

Boundary Recovery



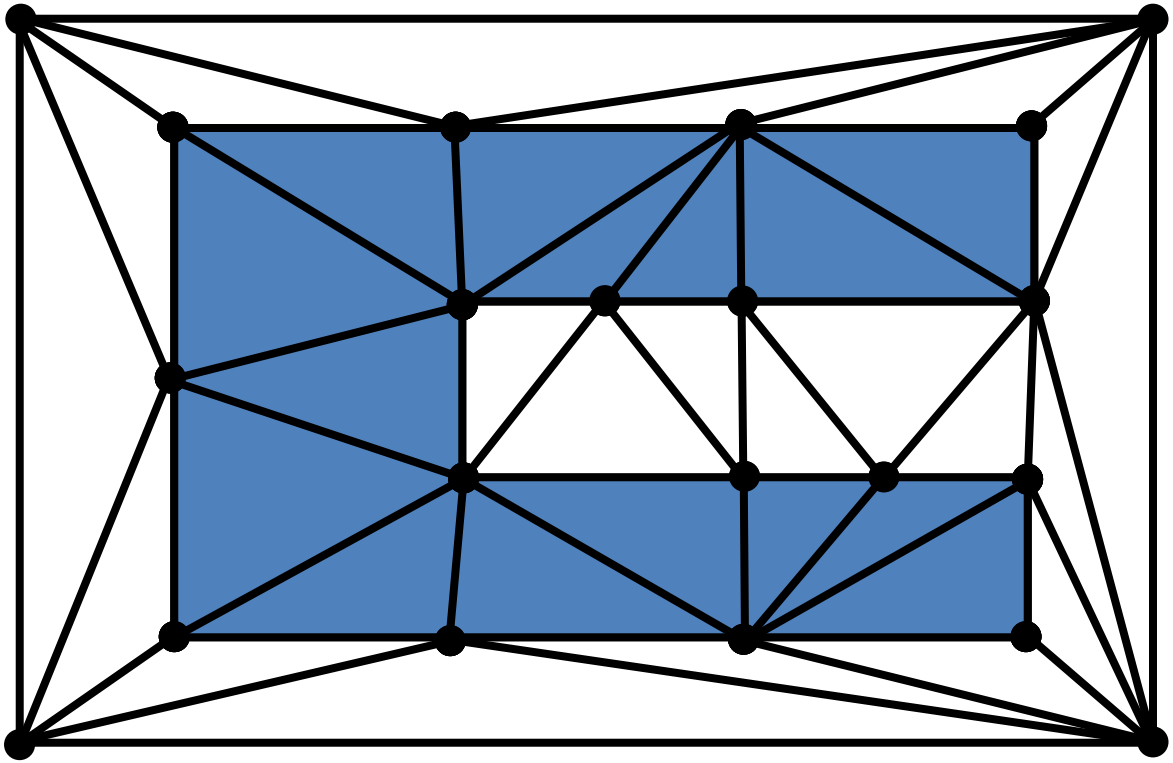
- Delete outside triangles (if can)
  - Delaunay triangulation does not have to obey polygon boundary

Boundary Recovery



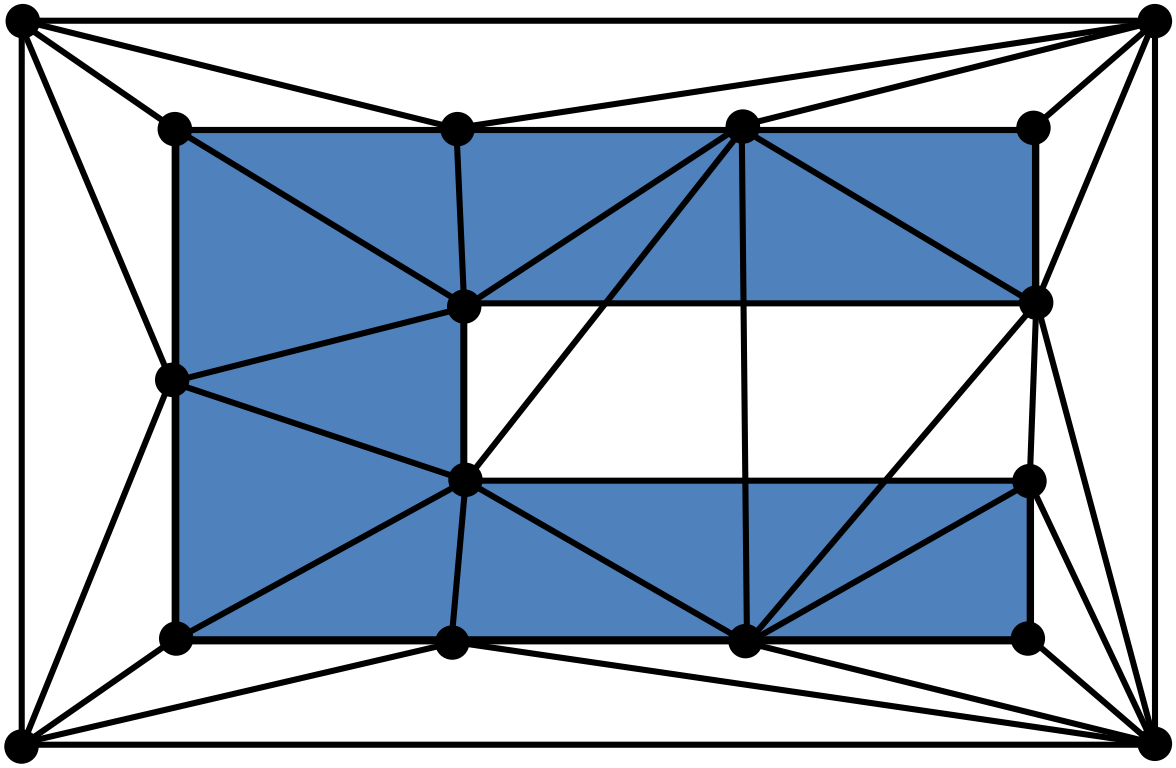
- Delaunay triangulation does not always obey polygon boundary

Boundary Recovery



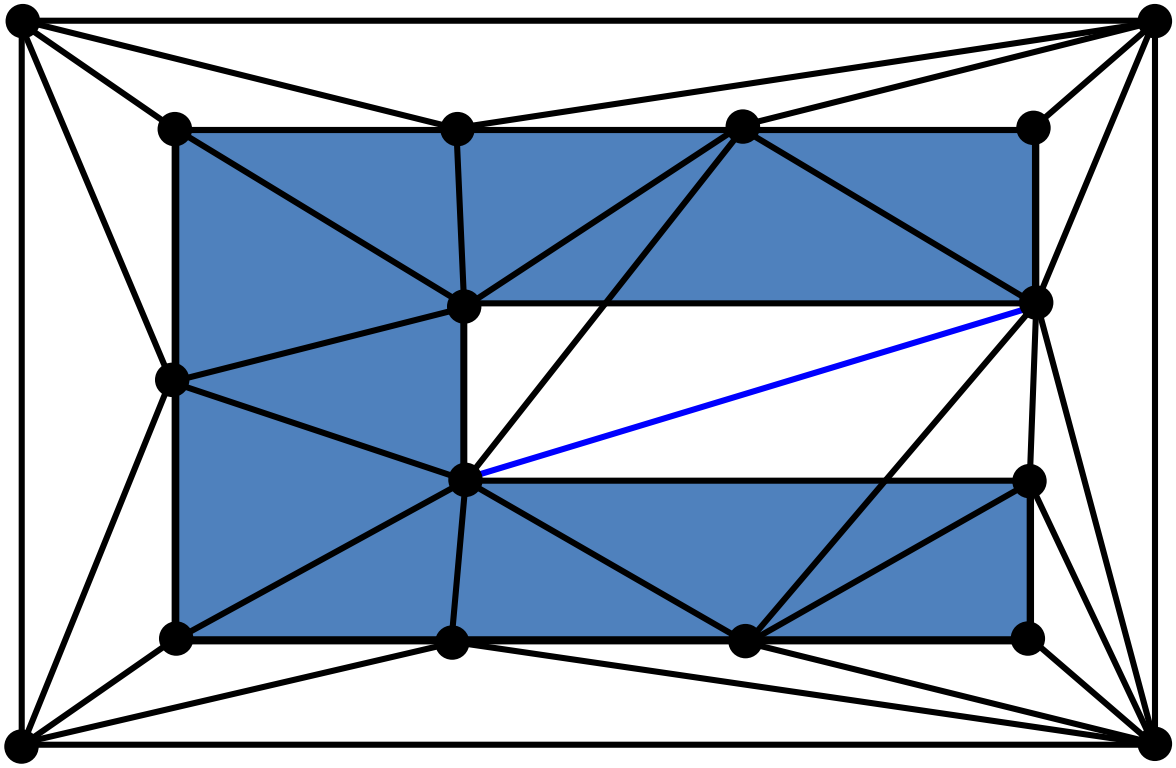
- Boundary Conforming Solution
  - Add vertices at intersections
  - Repeat if necessary

Boundary Recovery - Constrained



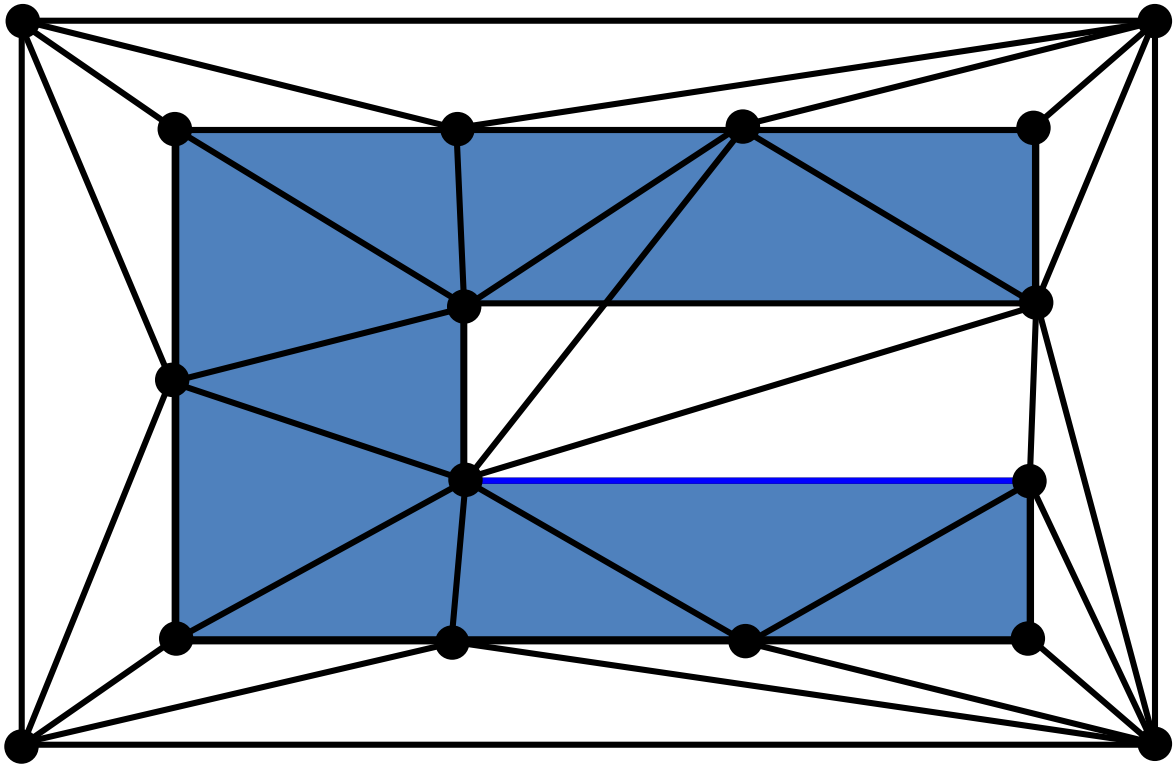
- Not always can add boundary vertices (shared edges)

Boundary Recovery - Constrained



- Swap edges between adjacent pairs of triangles
- Repeat till recover the boundary

Boundary Recovery - Constrained



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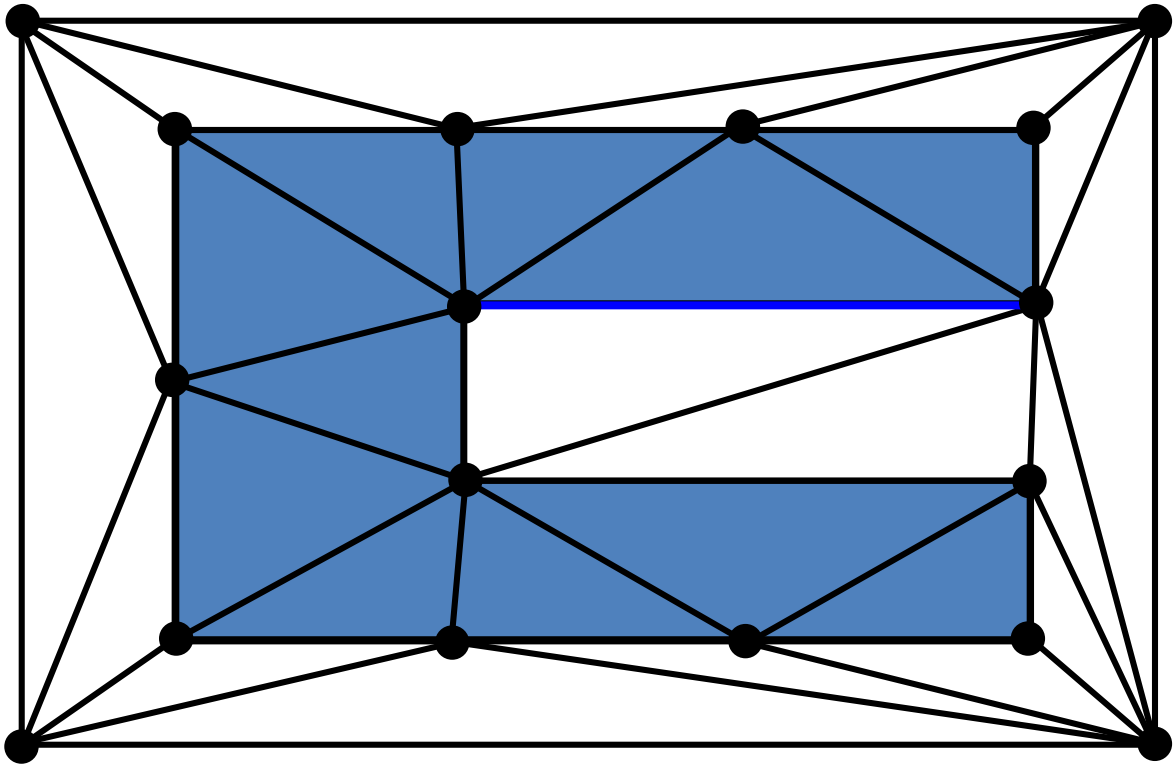
Boundary Recovery - Constrained



- Swap edges between adjacent pairs of triangles
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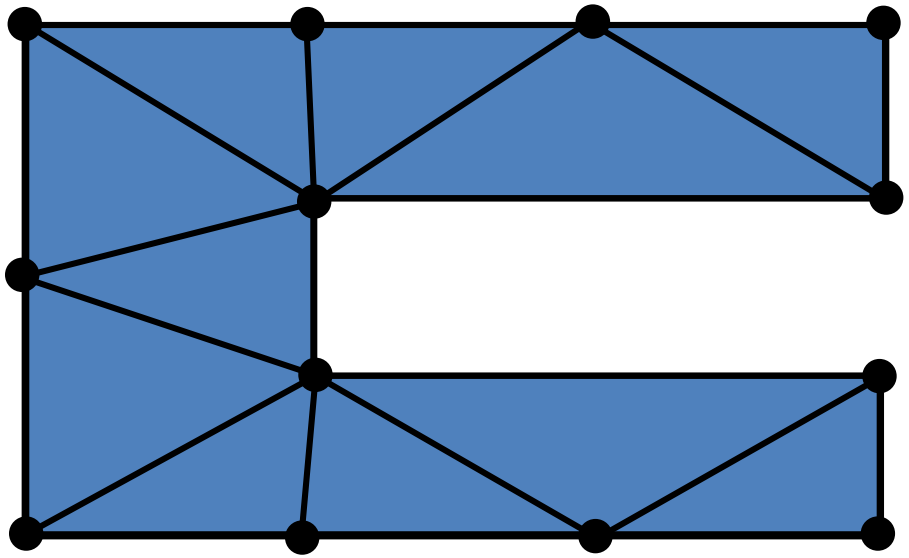
Boundary Recovery - Constrained



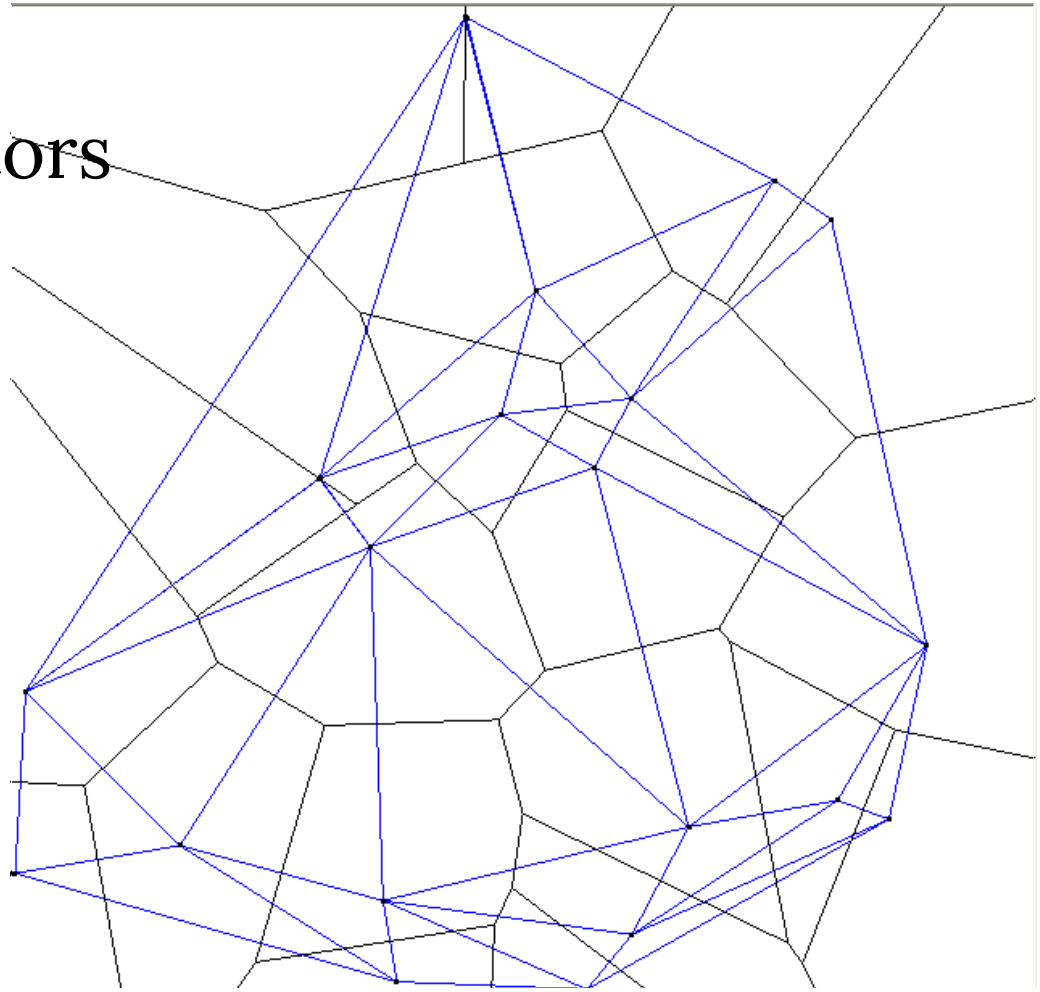
Not actually Delaunay

Voronoi Diagram

Dual to Delaunay Triangulation

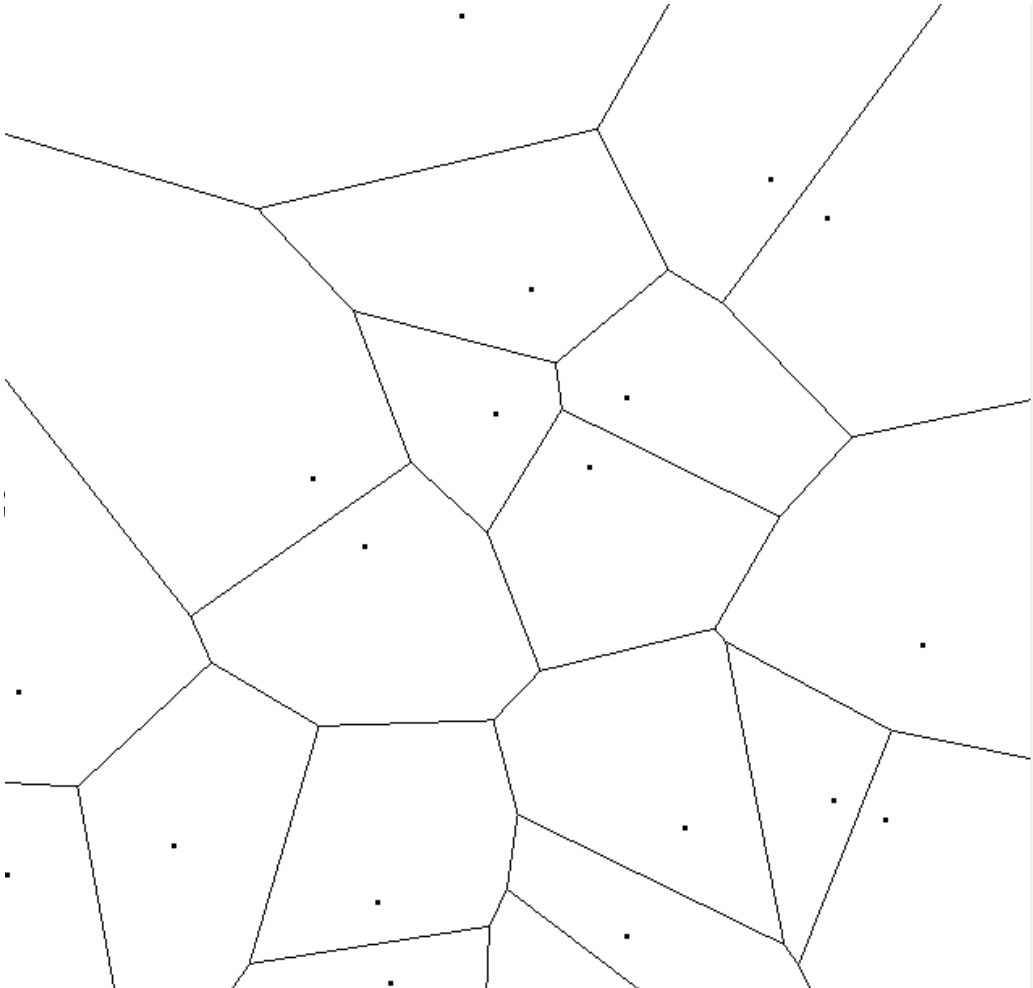
- Vertices → faces
- Voronoi edges =
- perpendicular bisectors

of Delaunay edges



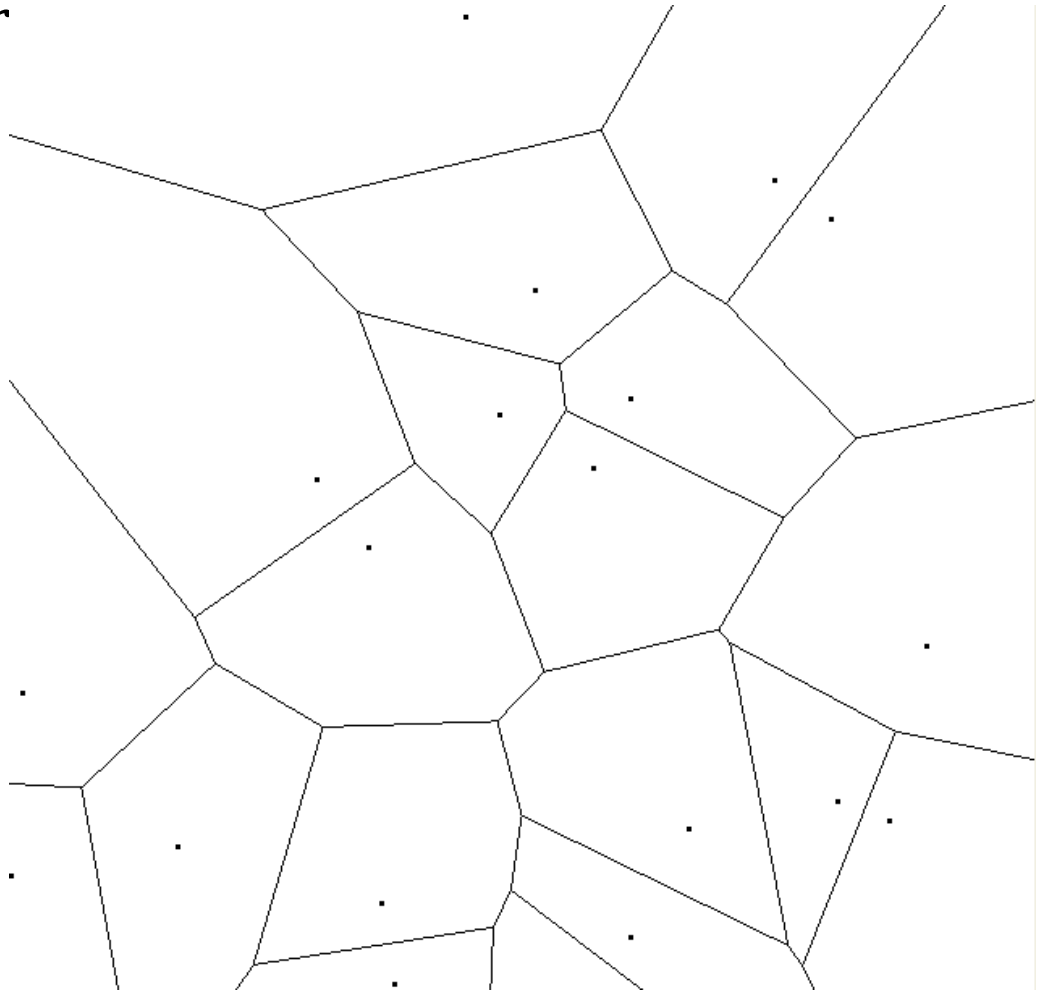
Sampling Using Centroidal Voronoi Diagrams

- *Voronoi Diagram for given set of vertices:* union of all locations at equal distance from two or more vertices
- Consists of
  - straight lines
    - vertex bisectors
  - vertices
    - bisector intersection



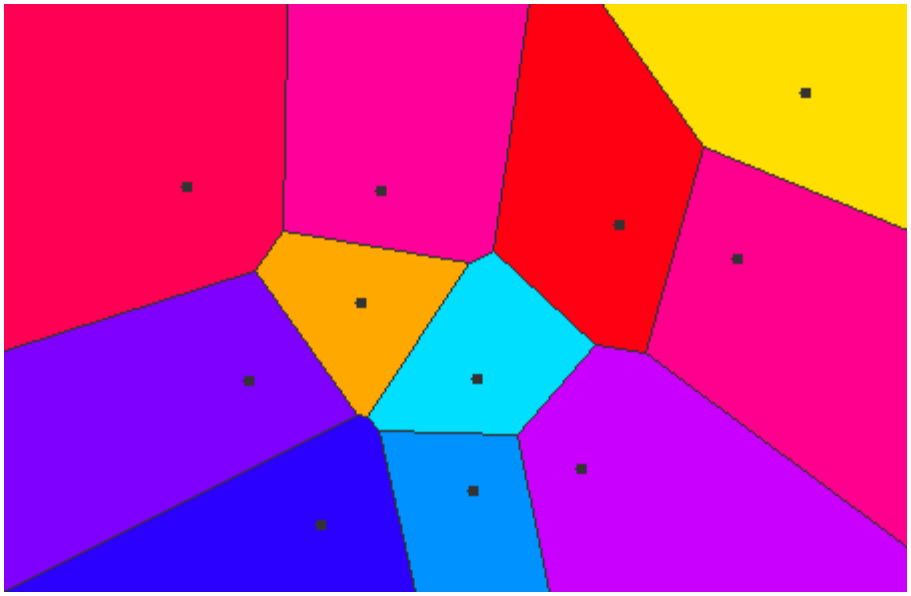
Voronoi Diagram

- Diagram partitions space into regions “closer” to one vertex than other
- Can define using distance function



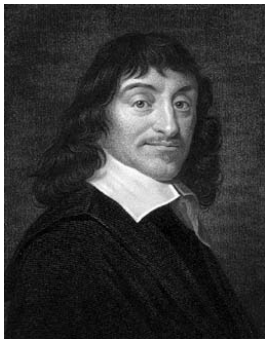
Voronoi Diagrams

Georgy Voronoi
Георгий Феодосьевич Вороной
1868-1908

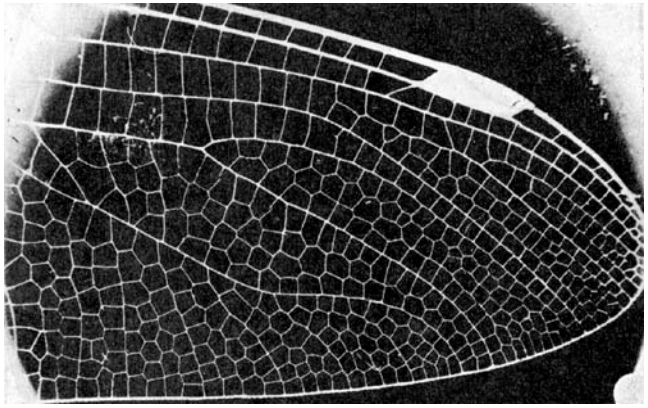


Slides from Prof. Joseph S.B. Mitchell's AMS 345 course materials

Historical Origins and Diagrams in Nature

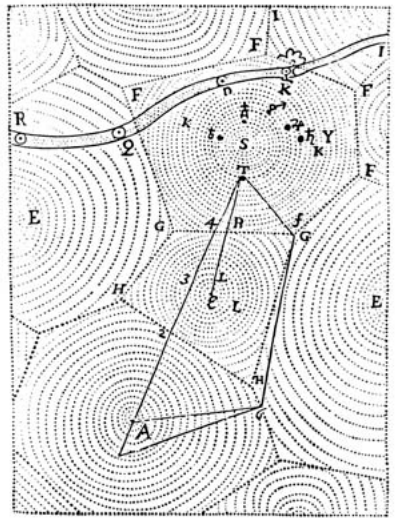


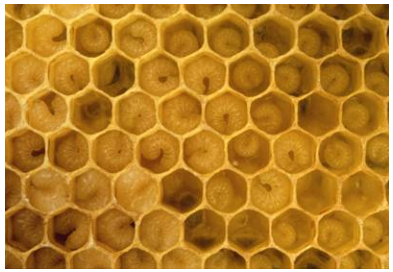
René Descartes
1596-1650
1644: Gravitational Influence of stars



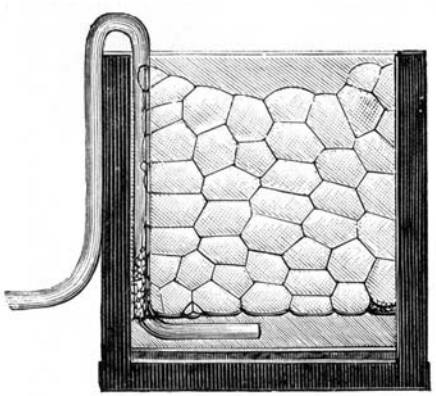
Dragonfly wing

Giraffe pigmentation





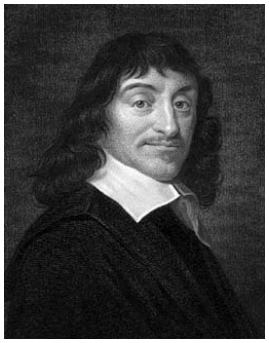
Honeycomb



Constrained soap bubbles

Slides from Prof. Joseph S.B. Mitchell's AMS 345 course materials

Historical Origins and Diagrams in Nature



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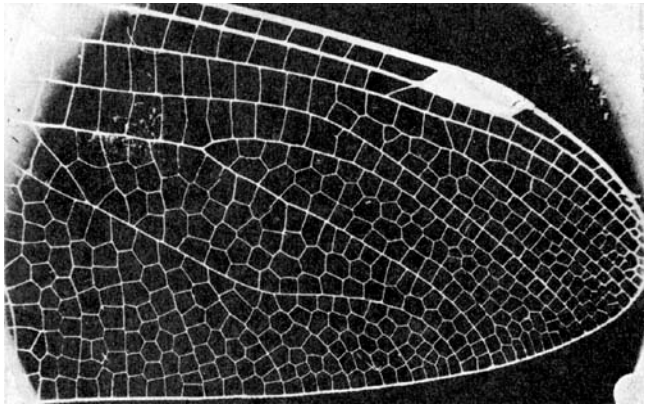
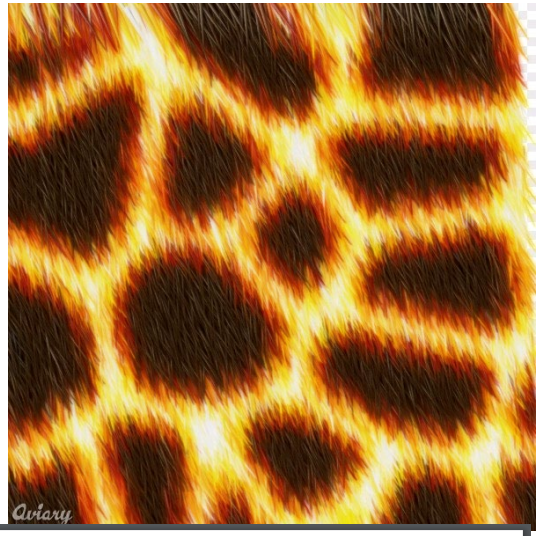
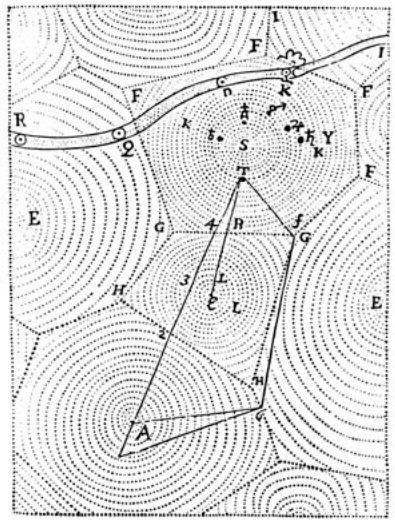
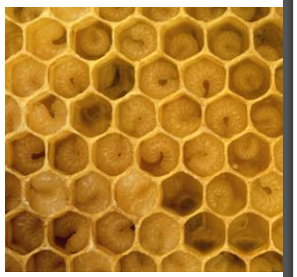


Dragonfly wing







Honeycom

PHYSICAL REVIEW E VOLUME 55, NUMBER 6 JUNE 1997

Geometrical consequences of foam equilibrium

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Höchstleistungsrechenzentrum, Forschungszentrum Jülich, D-52425 Jülich, Germany
(Received 6 January 1997)

Equilibrium conditions impose nontrivial geometrical constraints on the configurations that a two-dimensional foam can attain. In the first place, the three centers of the films that converge to a vertex have to be on a line, i.e., all vertices are *aligned*. Moreover, an equilibrated foam must admit a *reciprocal figure*. This means that it must be possible to find a set of points P_i on the plane, one per bubble, such that the segments $\overline{P_iP_j}$ are normal to the corresponding foam films. It is furthermore shown that these constraints are equivalent to the requirement that the foam be a sectional multiplicative Voronoï partition (SMVP). A SMVP is a cut with a two-dimensional plane of a three-dimensional multiplicative Voronoï partition. Thus, given an arbitrary equilibrated foam, we can always find pointlike sources (one per bubble) in three dimensions that reproduce this foam as a generalized Voronoï partition. These sources are the only degrees of freedom that we need in oder to describe the foam fully. [S1063-651X(97)14405-1]

PACS number(s): 82.70.Rr, 02.40.Sf, 02.40.Dr

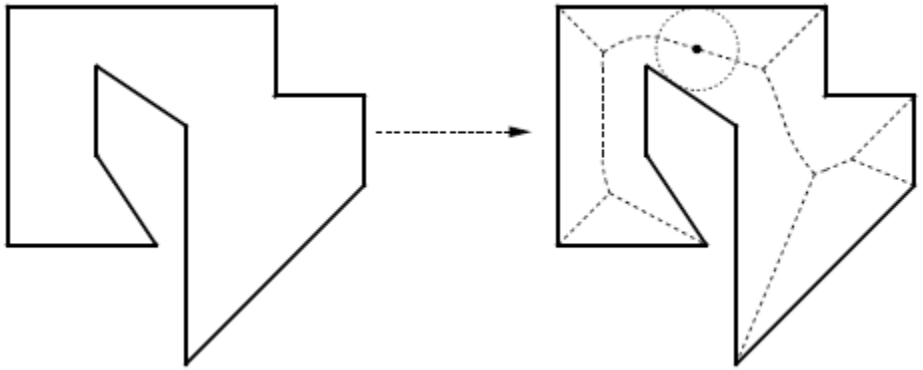
I. INTRODUCTION

Cellular structures [1–4] appear in a wide range of natural phenomena, and have puzzled and fascinated scientists for decades [5]. They can be generally described as packings of

On the other hand, an efficient method for the numerical simulation of ideal foams would also be highly desirable, and we propose that such a method could be obtained by exploiting the equivalence between foams and VP's reported in this work. More precisely, we establish the following *cor-*

Voronoi Applications

- Voronoi + point location search: nearest neighbor queries
- Facility location: Largest empty disk (centered at a Voronoi vertex)
- Shape description/approximation: medial axis

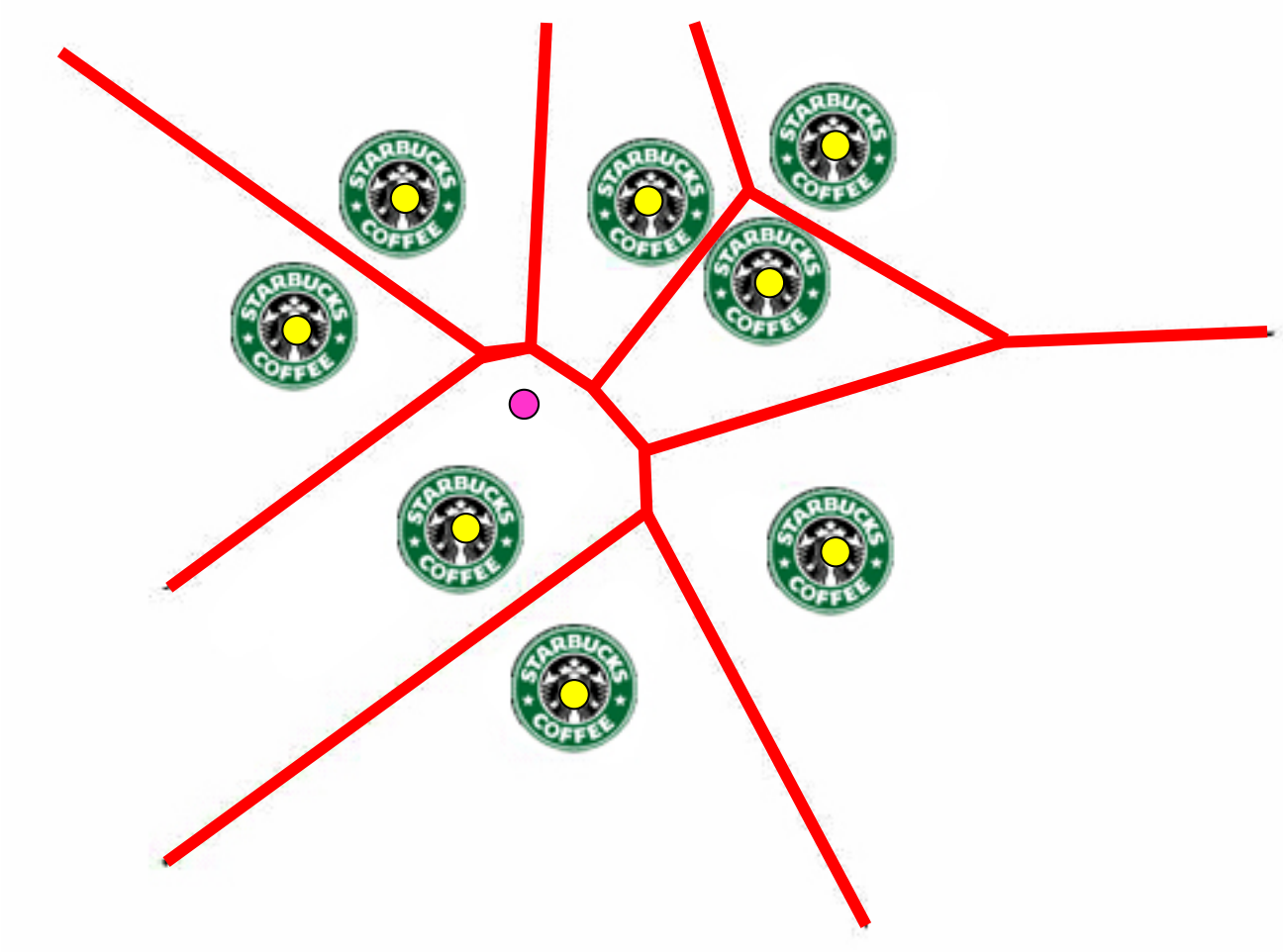


Slides from Prof. Joseph S.B. Mitchell's AMS 345 course materials

Starbucks Post Office Problem

Query point

Post offices



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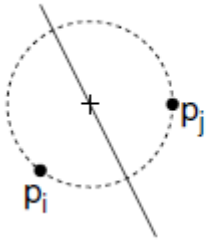
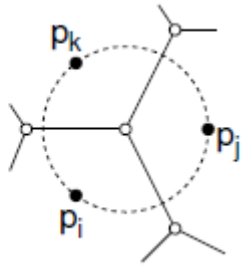
Voronoi Diagram

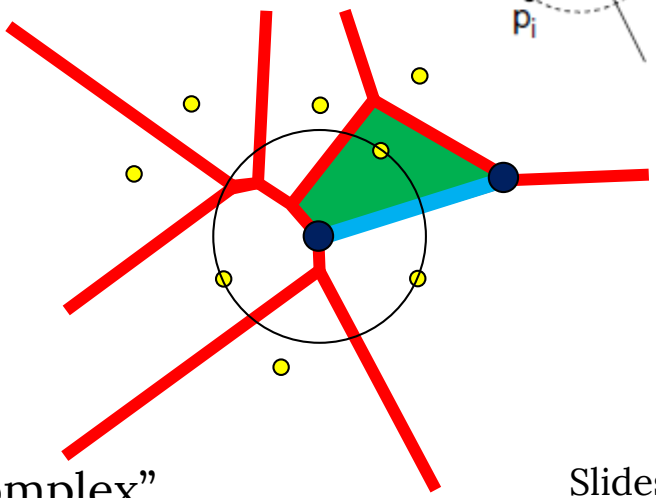
Partition the plane into cells:

$$\mathcal{V}(p_i) = \{q \mid \|p_i q\| < \|p_j q\|, \forall j \neq i\}$$

Voronoi cell of p_i is open, convex







"cell complex"

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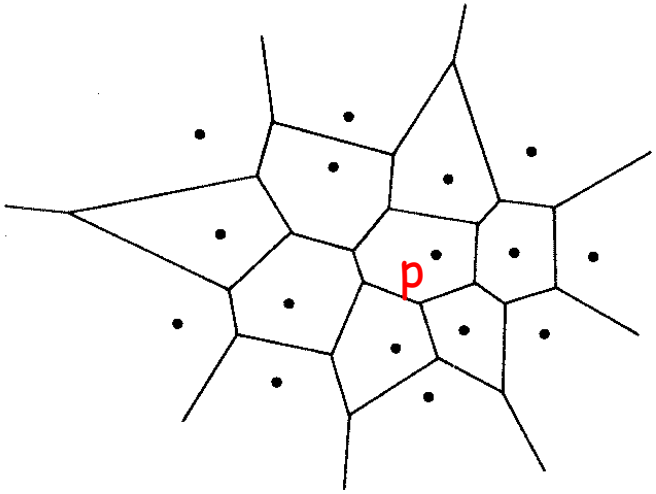
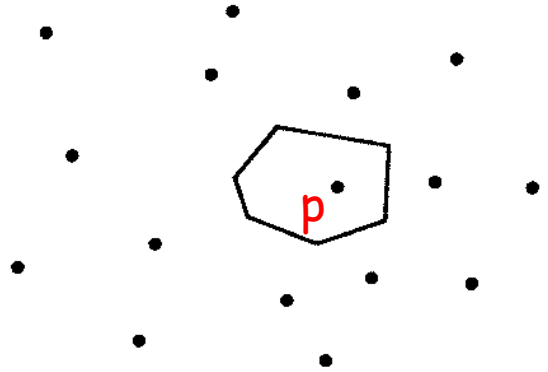
Example





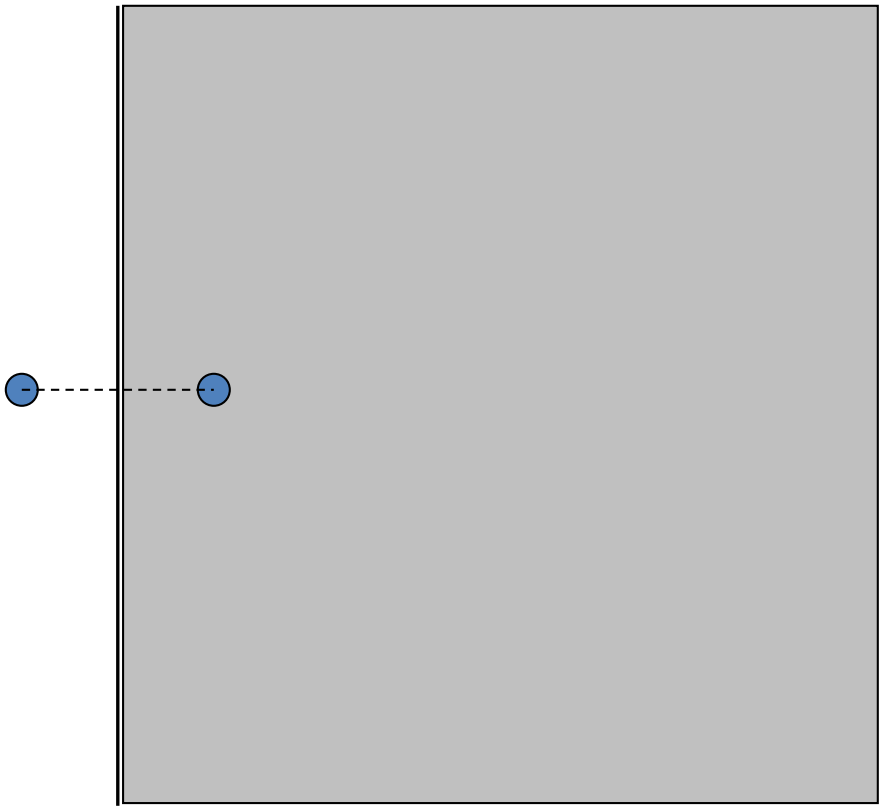
Voronoi cell of p

Constructing Voronoi Diagrams

Given a half plane intersection algorithm...

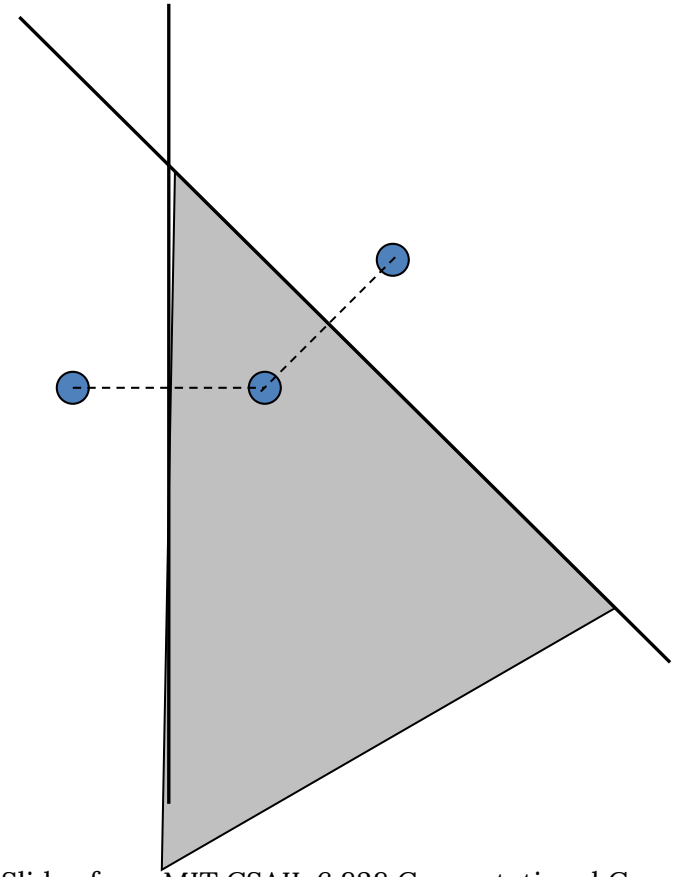
Constructing Voronoi Diagrams

Given a half plane intersection algorithm...



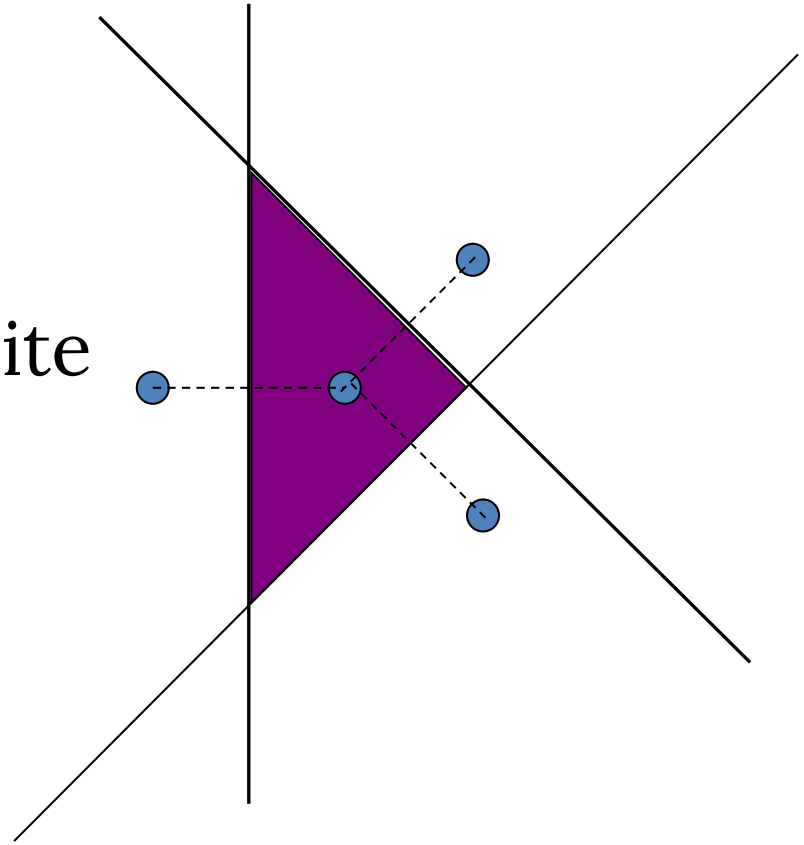
Constructing Voronoi Diagrams

Given a half plane intersection algorithm...



Constructing Voronoi Diagrams

Given a half plane intersection algorithm...



Repeat for each site

Running Time:
$O(n^2 \log n)$

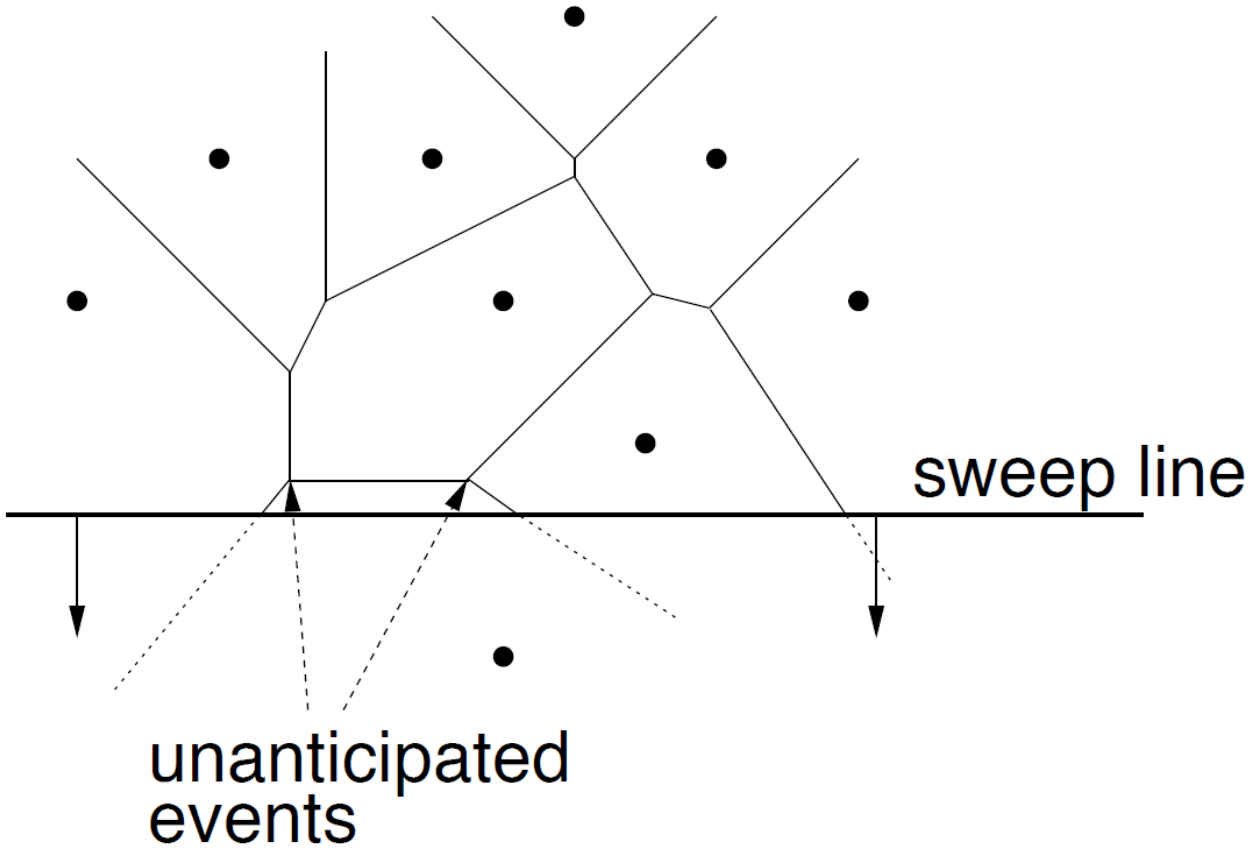
Constructing Voronoi Diagrams

- Half plane intersection O($n^2 \log n$)
- Fortune's Algorithm
  - Sweep line algorithm
    - Voronoi diagram constructed as horizontal line sweeps the set of sites from top to bottom
    - Incremental construction → maintains portion of diagram which doesn't change as we sweep down

Slides from MIT CSAIL 6.838 Computational Geometry, 2001, taught by Allen Miu

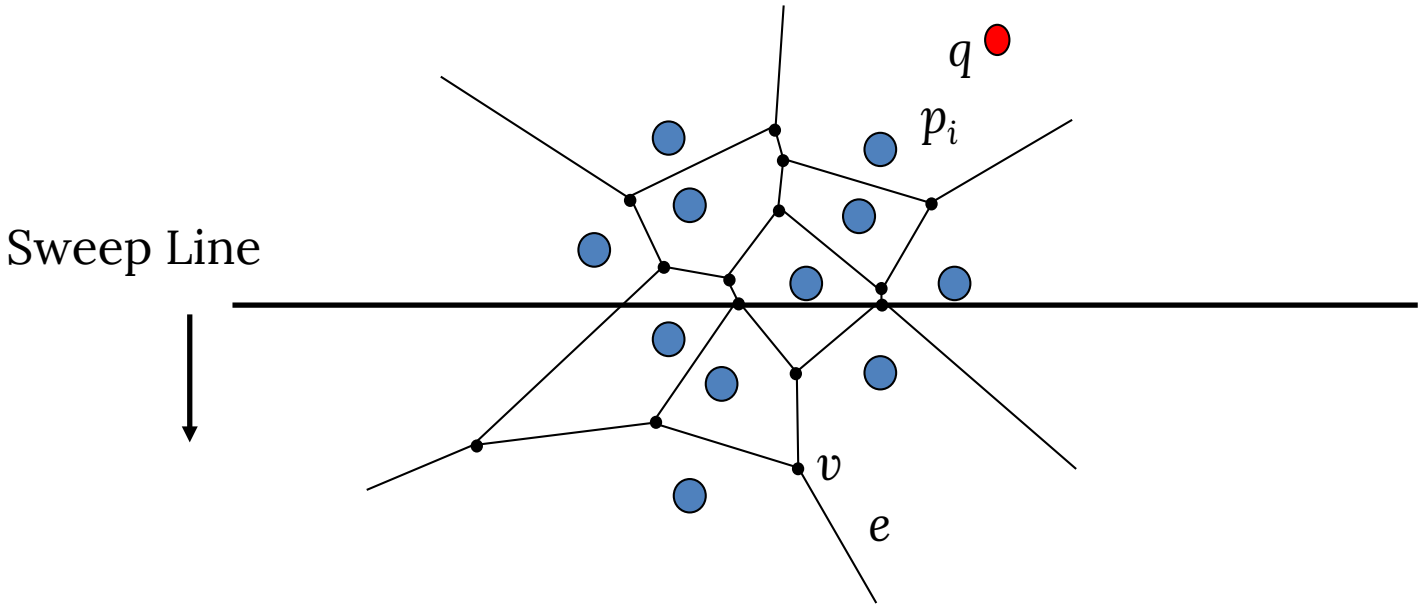
Constructing Voronoi Diagrams

Idea: line sweep?



Constructing Voronoi Diagrams

What is the invariant we are looking for?

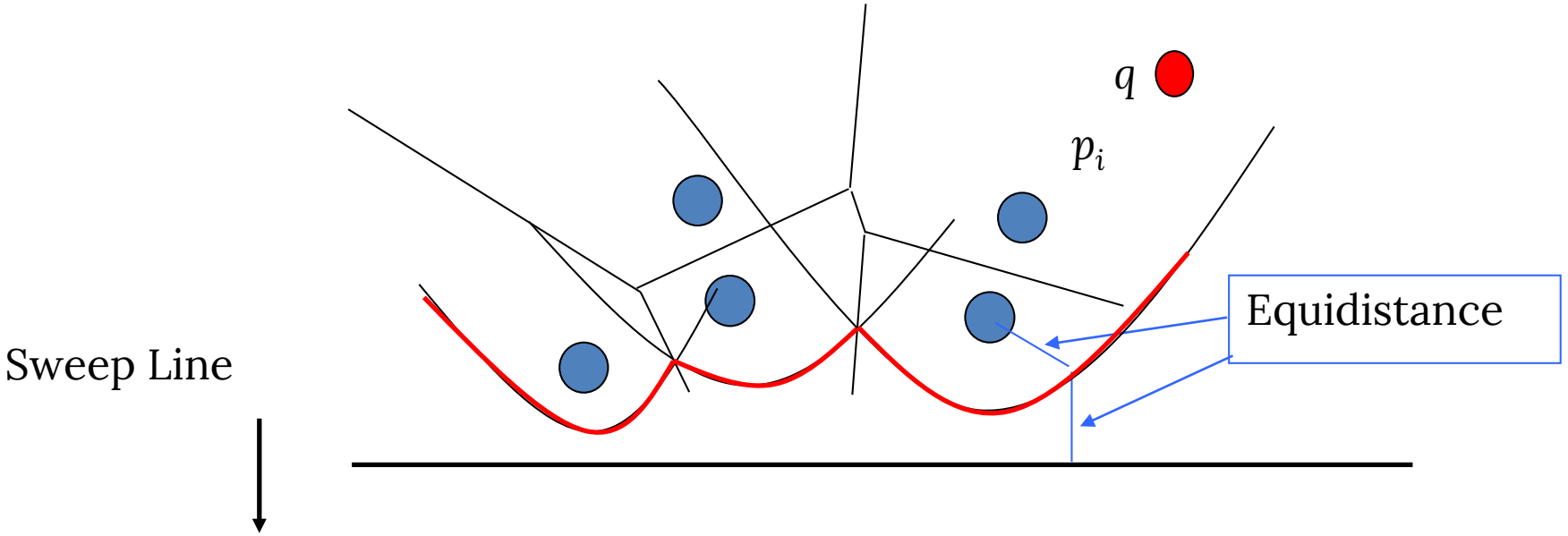


Maintain a representation of the locus of points q that are closer to some site p_i *above* the sweep line than to the line itself (and thus to any site below the line).

Slides from MIT CSAIL 6.838 Computational Geometry, 2001, taught by Allen Miu

Constructing Voronoi Diagrams

Which points are closer to a site above the sweep line than to the sweep line itself?

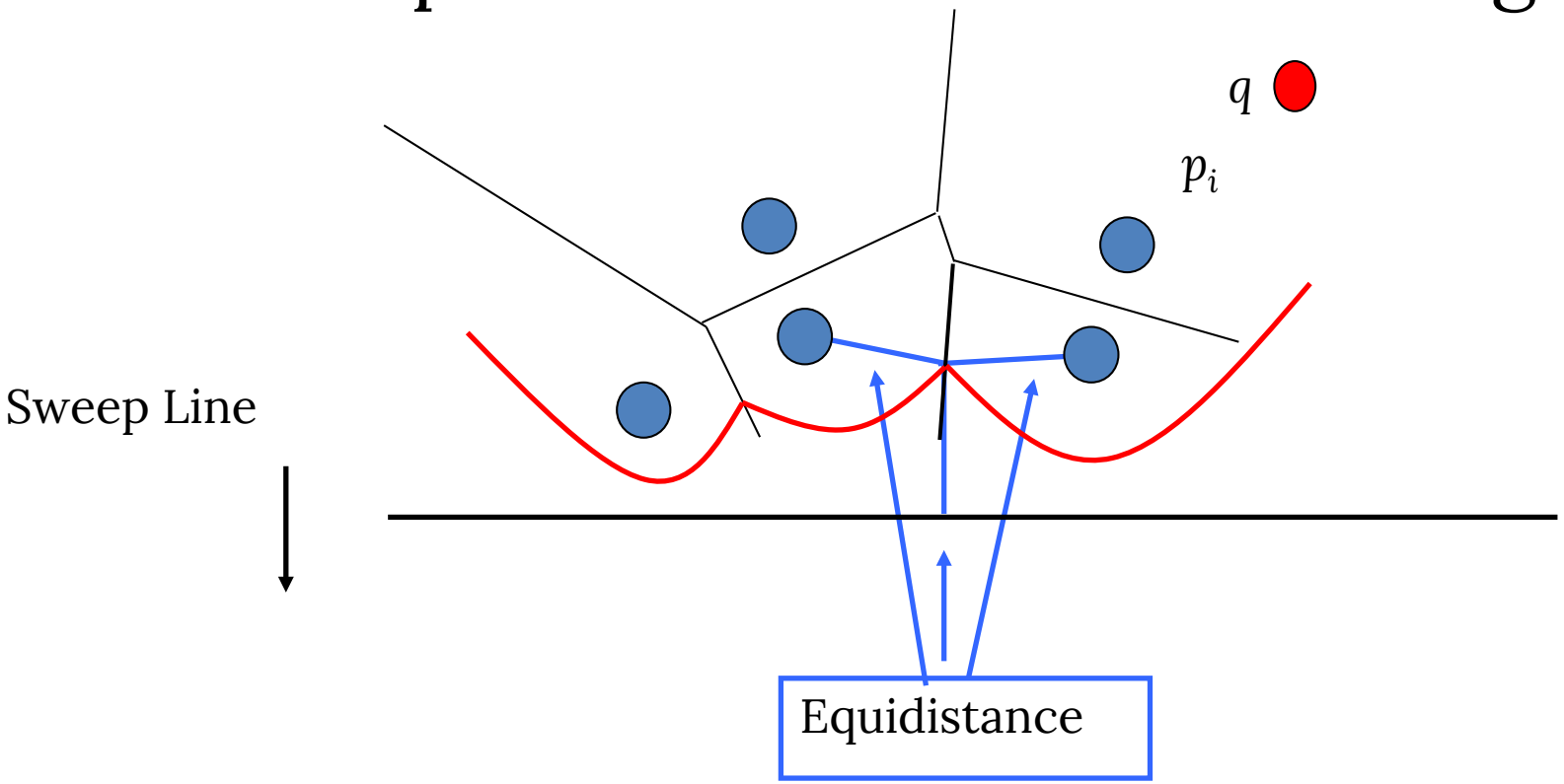


Beach line: lower envelope of all parabolas

Slides from MIT CSAIL 6.838 Computational Geometry, 2001, taught by Allen Miu

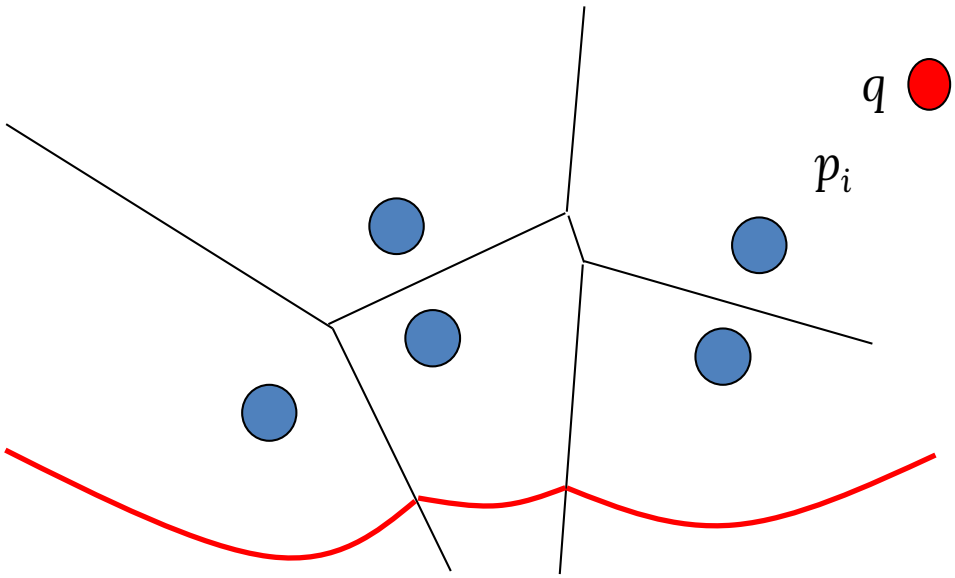
Constructing Voronoi Diagrams

Break points trace out Voronoi edges.

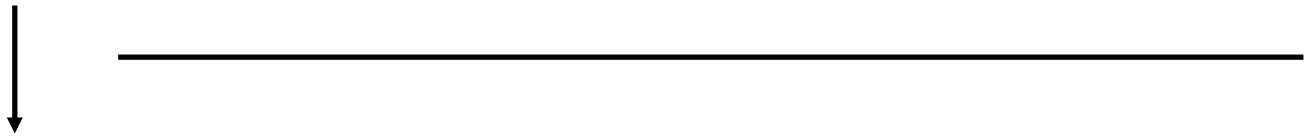


Constructing Voronoi Diagrams

Arcs flatten out as sweep line moves down.

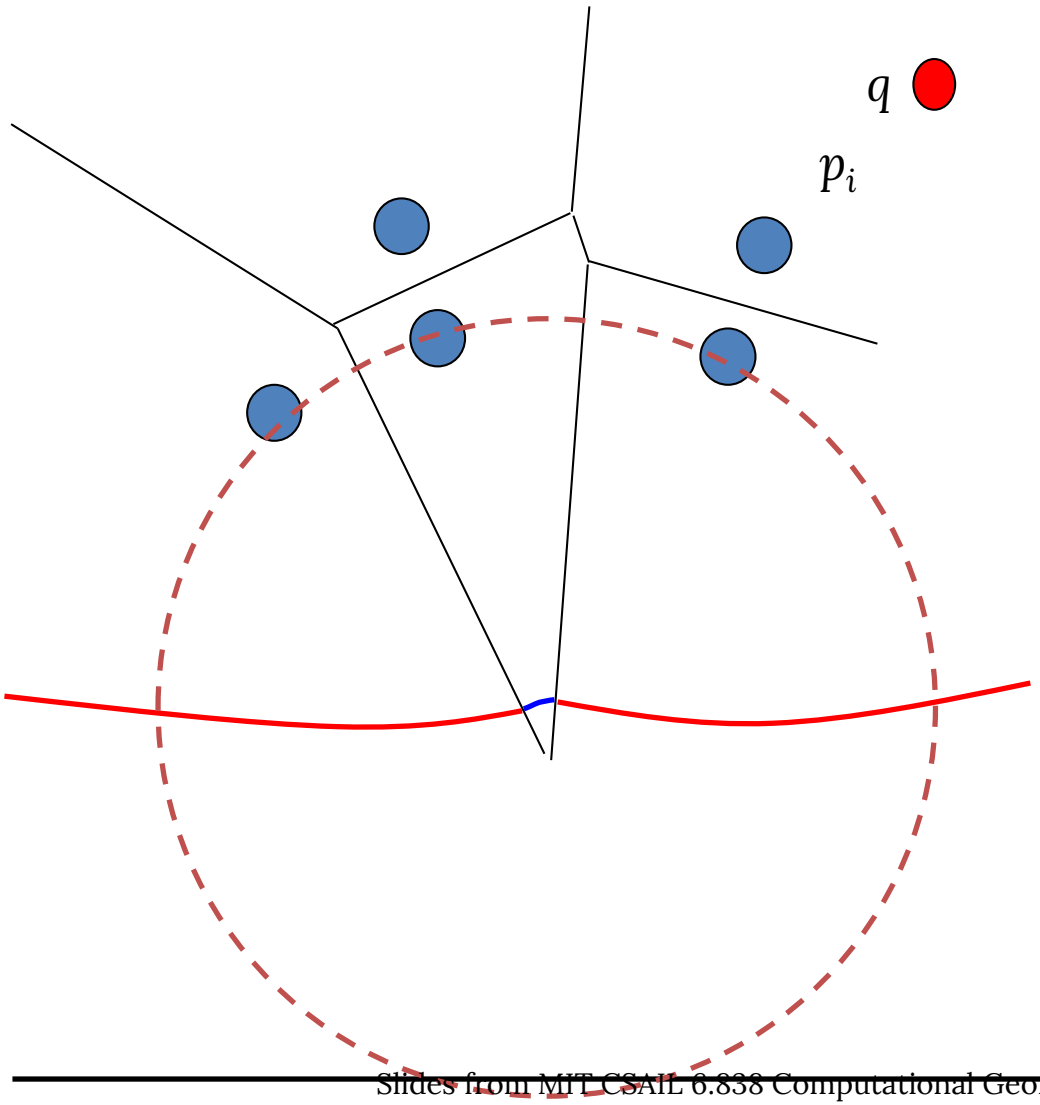


Sweep Line



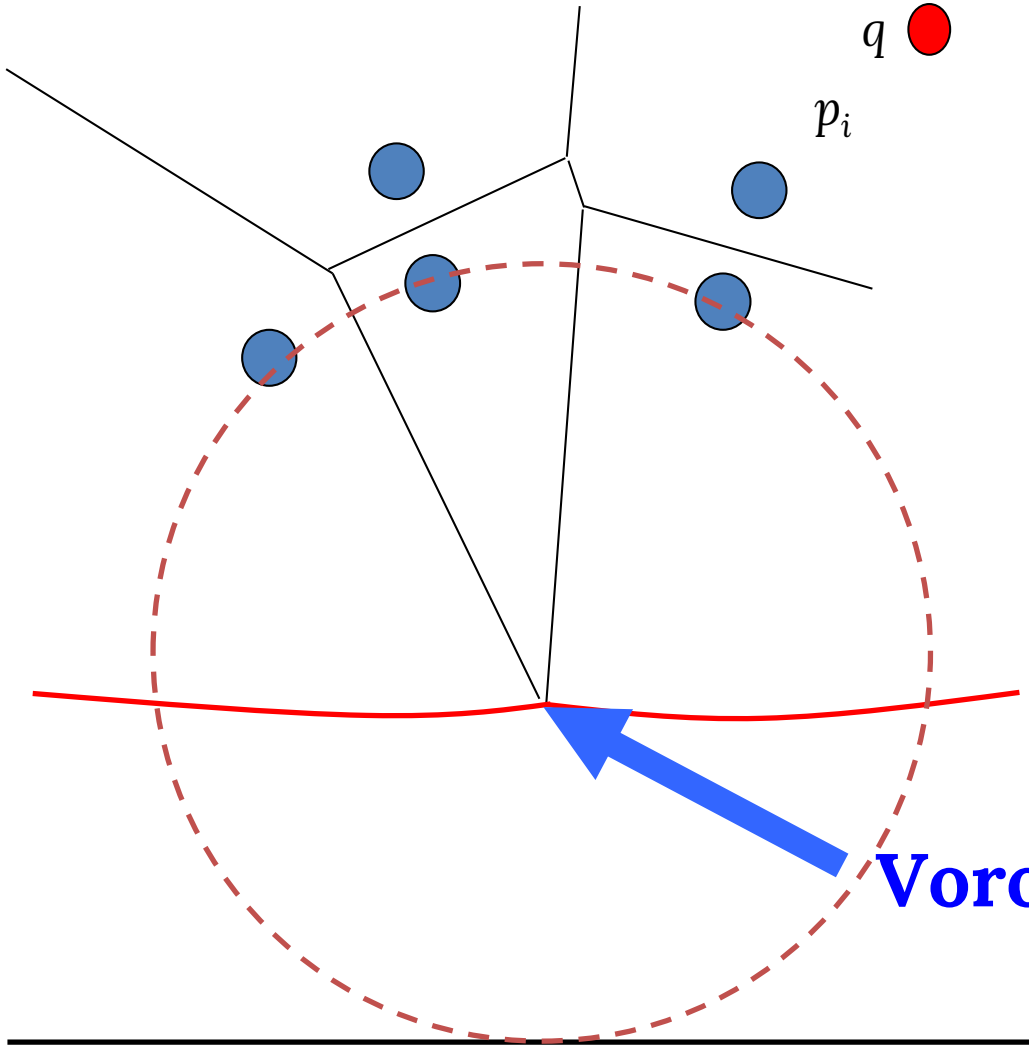
Constructing Voronoi Diagrams

Eventually, the middle arc disappears.



Constructing Voronoi Diagrams

We have detected a circle that is empty (contains no sites) and touches 3 or more sites.



Beach Line properties

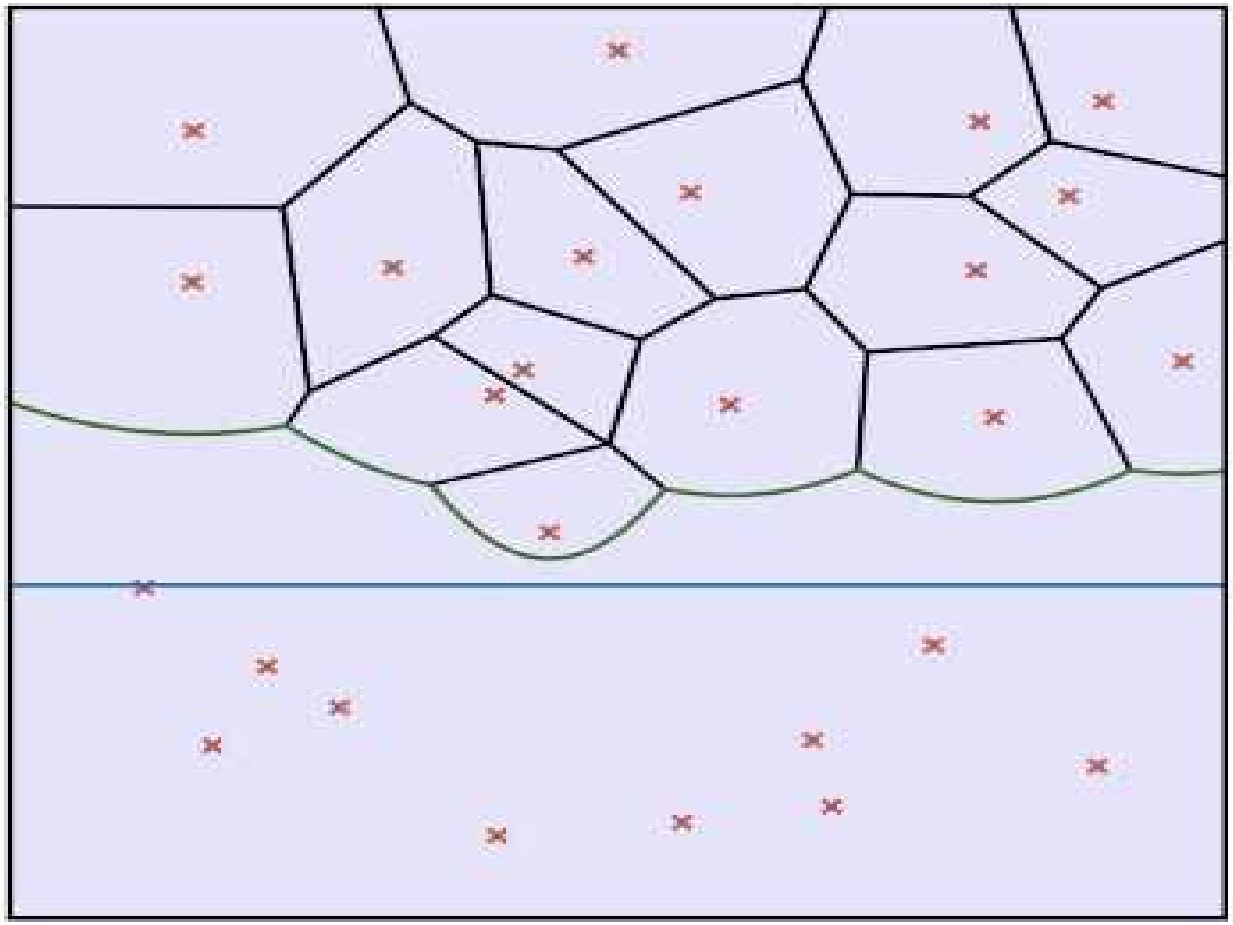
- Voronoi edges are traced by the break points
- Voronoi vertices are identified when two break points fuse
  - Decimation of an old arc identifies new vertex

Fortune's Algorithm

- Trace out the cells by line sweep
- Maintain and track the beach line
- No need to store parabolas, just store the participating vertex

- $O(n \log n)$ time

Fortune's Algorithm



Voronoi Diagram and Medial Axis

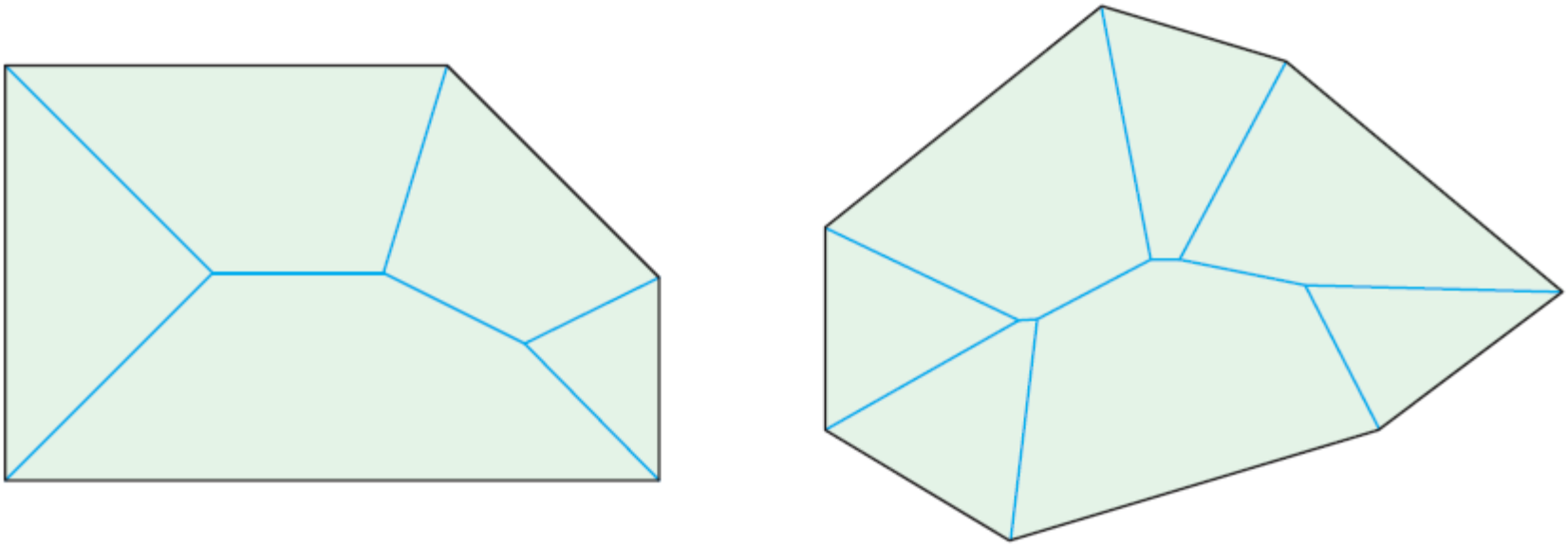


Figure 5.1: Convex polygons and their medial axes marked in blue.

Voronoi Diagram and Medial Axis

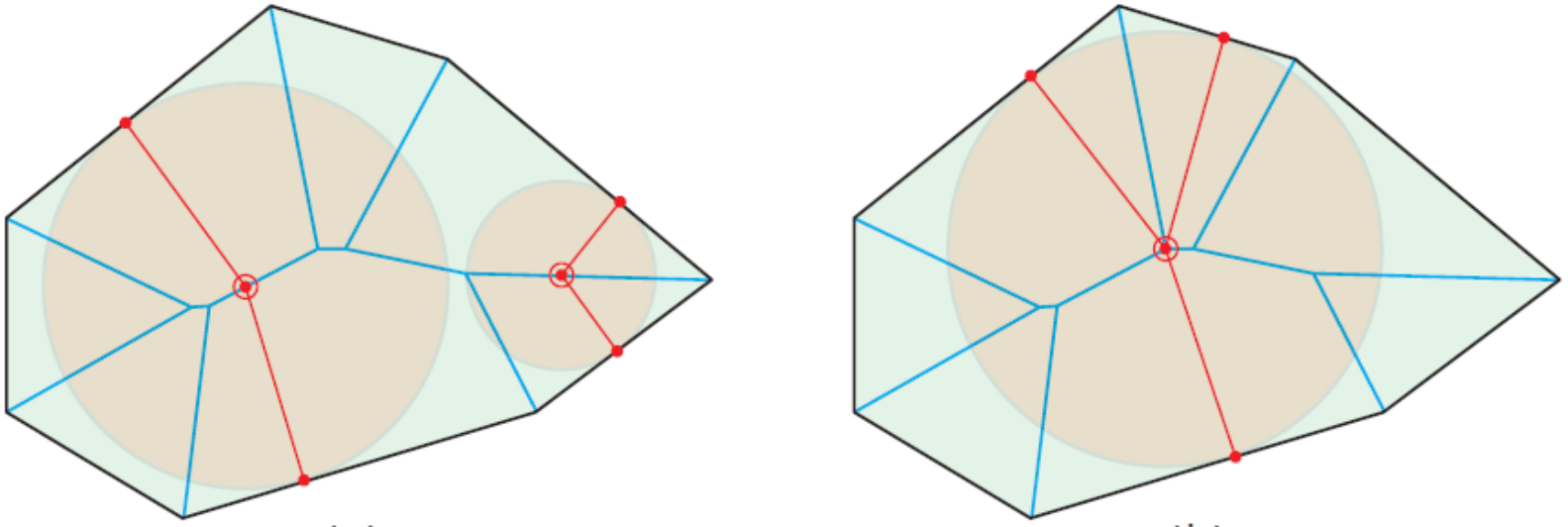
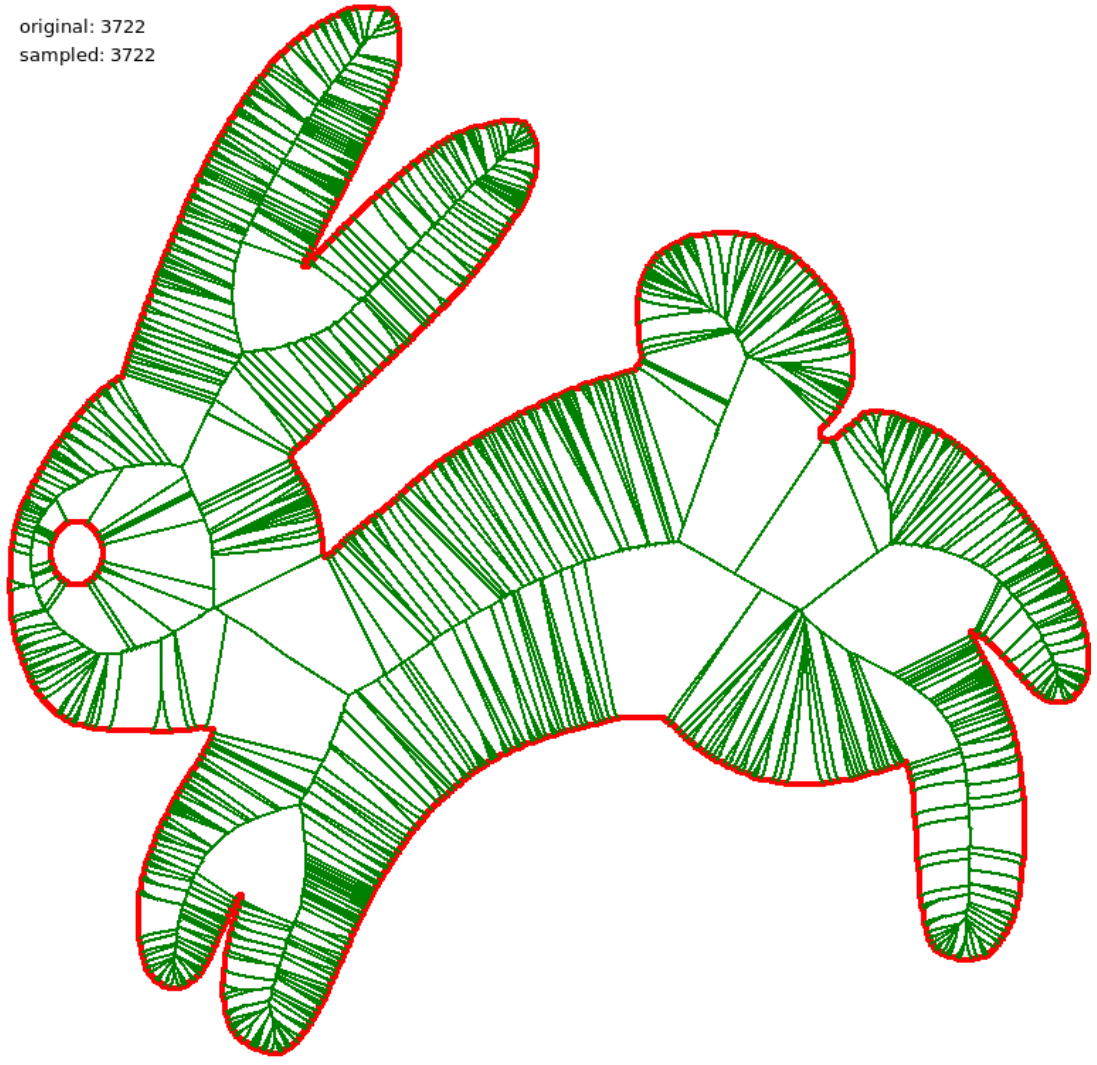


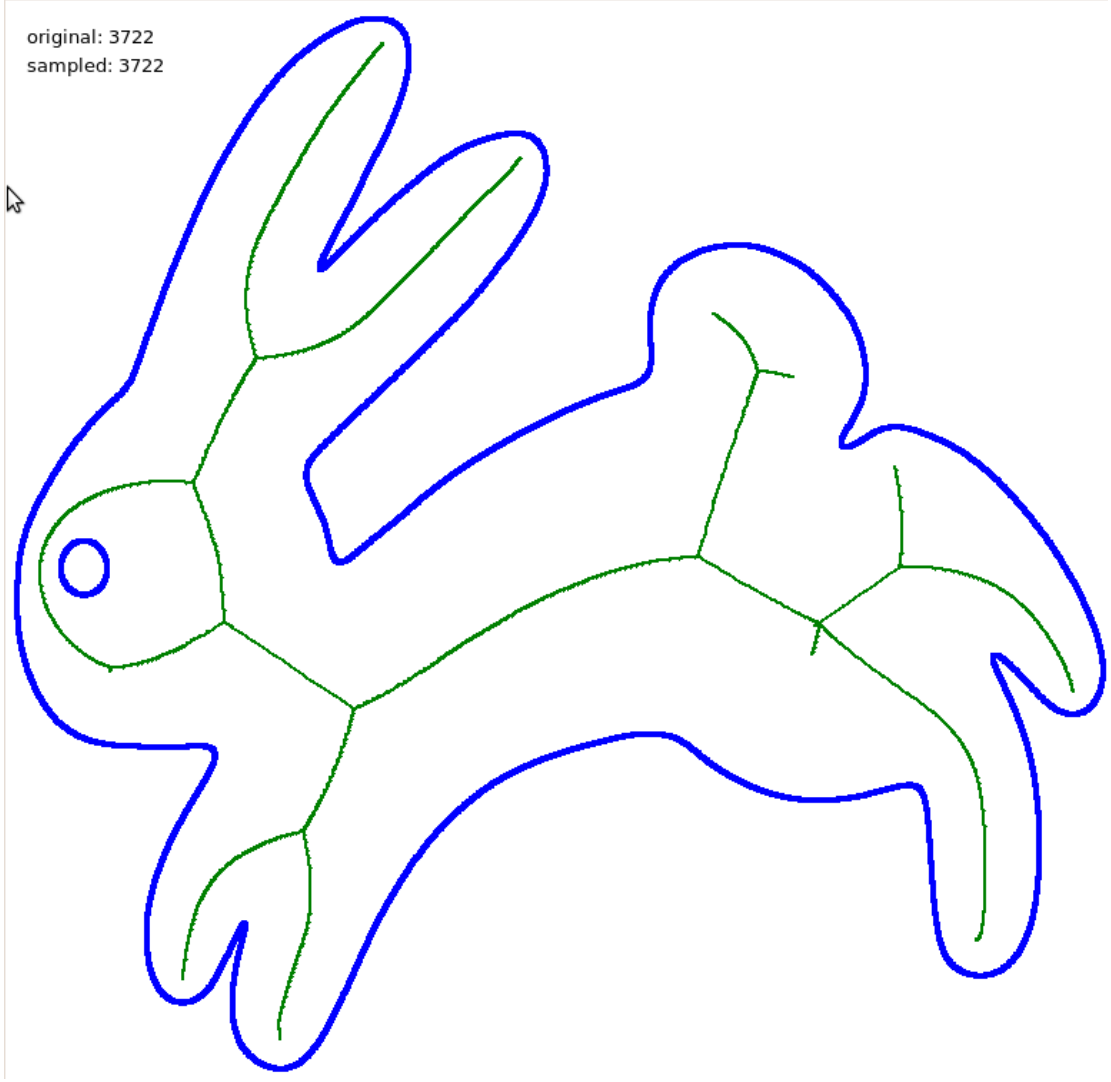
Figure 5.2: The maximal disks associated to (a) interior segments of $M(P)$ and (b) a degree-3 vertex of $M(P)$.

Medial axis vs Voronoi diagram



"On the Evaluation of the Voronoi-Based Medial Axis"
by Adriana Schulz, Francisco Ganacim and Leandro Cruz

Medial axis vs Voronoi diagram



"On the Evaluation of the Voronoi-Based Medial Axis"
by Adriana Schulz, Francisco Ganacim and Leandro Cruz

Looks familiar?

Skeleton Extraction by Mesh Contraction

Oscar Kin-Chung Au* Chiew-Lan Tai* Hung-Kuo Chu† Daniel Cohen-Or‡ Tong-Yee Lee†
*The Hong Kong Univ. of Science and Technology †National Cheng Kung University ‡Tel Aviv University

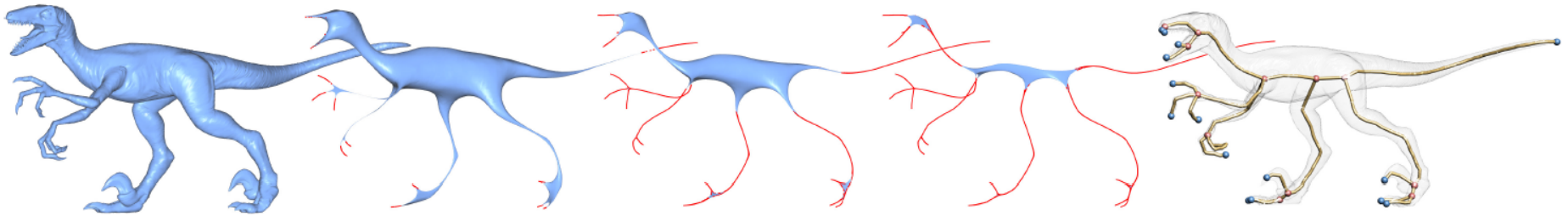


Figure 1: Our method extracts a 1D skeletal shape by performing geometry contraction using constrained Laplacian smoothing. Left to right are the original mesh and the results of the contraction after 1, 2 and 3 iterations. Faces with zero area are drawn in red. The rightmost ray-traced image shows the final skeleton after performing connectivity surgery and embedding refinement.

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The library offers data structures and algorithms like triangulations, Voronoi diagrams, Boolean operations on polygons and polyhedra, point set processing, arrangements of curves, surface and volume mesh generation, geometry processing, alpha shapes, convex hull algorithms, shape reconstruction, AABB and KD trees...
Learn more about CGAL by browsing through the Package Overview.
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