IFT 6113
MESH PARAMETERIZATION

Image from https://doc.cgal.org/latest/Surface_mesh_parameterization/index.html

Mikhail Bessmeltsev
What is Parameterization?

world

\( \mathbb{R}^3 \)

\( (x,y,z) \)

parameterization

surface

atlas

\( \mathbb{R}^2 \)

\( (u,v) \)

parameter domain
Problem Definition

**Given:** surface $S \subseteq \mathbb{R}^3$

domain $D \subset \mathbb{R}^2$

**Find** a bijective $f : S \rightarrow D$
Recall:

**Tangent Space**

\[ T_p S := \text{Image}(D\sigma_u) \]

\[ S \subset \mathbb{R}^3 \]

\[ V \cap S \]

\[ p \]

\[ \sigma \]

\[ U \subseteq \mathbb{R}^2 \]
Typical domains:

\[ \subseteq \mathbb{R}^2 \]

- Boundary
- Sphere (no boundary)
Typical domains:

Cross-Parameterization / Inter-surface Mapping
- all (closed) models
- usually utilize common base
Why Do We Need It?

Texture Mapping

Normal Mapping

Detail Transfer

Morphing

Mesh Completion

Editing

Databases

Remeshing

Surface Fitting
Why Do We Need It?

AtlasNet: A Papier-Mâché Approach to Learning 3D Surface Generation

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http://imagine.enpc.fr/~groueixt/atlasnet/

Figure 1. Given input as either a 2D image or a 3D point cloud (a), we automatically generate a corresponding 3D mesh (b) and its atlas parameterization (c). We can use the recovered mesh and atlas to apply texture to the output shape (d) as well as 3D print the results (e).
Texture Mapping

- Define color for each point on object surface

- Map 2D texture to model surface:
  - Texture pattern defined over 2D domain \((u, v)\)
  - Assign \((u, v)\) coordinates to each point on surface
Normal/Bump mapping

(a) 4M faces

(b) 8K faces

(c) 8K faces, normal-mapped

(d) normal-map
Morphing/Properties Transfer

Require cross-parameterization
Remeshing & Surface Fitting
Mesh Parameterization

\[ f : \Omega \to S, \text{ s.t.} \]

- \( f \) is piecewise linear
  - \( f|_t \) is linear
- \( f \) is \textbf{bijective}
  - at least locally

Parameter domain \( \Omega \subseteq \mathbb{R}^2 \)

Surface \( S \subseteq \mathbb{R}^3 \)
Earth maps!

- Distortion is inevitable
- But we can preserve some properties exactly

orthographic
\(\sim 500\) B.C.

stereographic
\(\sim 150\) B.C.

Mercator
1569

Lambert
1772

conformal
(angle-preserving)

equiareal
(area-preserving)
Intrinsic Descriptor

Theorema Egregium
(“Totally Awesome Theorem”):
Gaussian curvature is intrinsic.

\[ K := \kappa_1 \kappa_2 = \det \mathbb{II} \]
What is Distortion?

- Distortion (at $x$): How different is $f(D)$ from $D$
  - How to measure?

Small disk around $x$

Image of that disk
How to quantify distortion? Study **differential/Jacobian**.

\[
f(y) = f(x) + \left( \frac{\partial f}{\partial x_i} \right) (y - x) + O(\|y - x\|^2)
\]

Linearization \( \tilde{f}(y) \)

Semiaxes: \( r\sigma_1, r\sigma_2 \)
Linear Map Surgery

- **Singular Value Decomposition (SVD)** of $J_f$

$$J_f = U \Sigma V^T = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} V^T$$

with rotations $U \in \mathbb{R}^{3 \times 3}$, $V \in \mathbb{R}^{2 \times 2}$ and scale factors (singular values) $\sigma_1, \sigma_2 \geq 0$
How to quantify distortion?
Study **Metric Tensor** = 1\textsuperscript{st} fundamental form.

\[ I = \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_u \cdot f_v & f_v \cdot f_v \end{pmatrix} \]
How to quantify distortion?
Study **Metric Tensor** = 1st fundamental form.

\[
I = \begin{pmatrix}
    f_u \cdot f_u & f_u \cdot f_v \\
    f_u \cdot f_v & f_v \cdot f_v
\end{pmatrix} = J_f^T J_f
\]

“Gram Matrix”
How to quantify distortion?

Study **Metric Tensor = 1st fundamental form.**

\[
I = \begin{pmatrix}
    f_u \cdot f_u & f_u \cdot f_v \\
    f_u \cdot f_v & f_v \cdot f_v
\end{pmatrix} = J_f^T J_f
\]

\[
= (U \Sigma V^T)^T (U \Sigma V^T) = V \begin{pmatrix}
    \sigma_1^2 & 0 \\
    0 & \sigma_2^2
\end{pmatrix} V^T
\]

**Eigenvalues of** \( I \)
Pointwise distortion

Isometric

\[ \sigma_1 = \sigma_2 = 1 \]

Conformal or angle-preserving

\[ \sigma_1 = \sigma_2 \]

Equiareal or area-preserving

\[ \sigma_1 \cdot \sigma_2 = 1 \]

Defined pointwise on \( \Omega \)
Measuring Distortion

- **Local** distortion measure function of $\sigma_1$ and $\sigma_2$
  \[ E : (\mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}, \quad (\sigma_1, \sigma_2) \mapsto E(\sigma_1, \sigma_2) \]

- **Overall** distortion
  \[ E(f) = \int_{\Omega} E(\sigma_1(u,v), \sigma_2(u,v)) \, du \, dv \bigg/ \text{Area}(\Omega) \]

- On mesh constant per triangle
  \[ E(f) = \sum_{t \in \Omega} E(t) A(t) \bigg/ \sum_{t \in \Omega} A(t) \]
CONFORMAL MAPS
Stereographic Projection

Image from Wikipedia
Stereographic Projection

Parameterization: \( f(u, v) = (2ud, 2vd, (1 - u^2 - v^2)d) \) with \( d = \frac{1}{1+u^2+v^2} \)

Jacobian: \( J_f = \begin{pmatrix} 2d-4u^2d^2 & -4ud^2 \\ -4ud^2 & 2d-4v^2d^2 \end{pmatrix} \)

First fundamental form: \( I_f = \begin{pmatrix} 4d^2 & 0 \\ 0 & 4d^2 \end{pmatrix} \)

Eigenvalues: \( \lambda_1 = 4d^2, \quad \lambda_2 = 4d^2 \)

\( \sigma_1 = \sigma_2 = 2d \quad \Rightarrow \quad \text{conformal} \)
Stereographic Projection

Image from Wikipedia
Stereographic Projection
Does conformal map exist?

**Uniformization theorem**

"Every simply connected Riemann surface is conformally equivalent to one of three Riemann surfaces:

1. open unit disk
2. complex plane
3. Riemann sphere"
Parameterization: Practice
MESH PARAMETERIZATION METHODS

**Fixed Boundary**
- Bijectivity: easy
- Distortion: may be large

**Free boundary**
- Bijectivity: hard
- Distortion: minimum
MESH PARAMETERIZATION METHODS

**Fixed Boundary**
- Bijectivity: easy
- Distortion: may be large

**Free boundary**
- Bijectivity: hard
- Distortion: minimum
Spring Model

- Fix boundary vertices on a **convex** polygon
- Edges $\rightarrow$ springs
- Let go of the springs
  - “Relaxation”
Spring Model

Hooke’s Law:
Energy of spring between $p_i$ and $p_j$

$$E_{ij} = 0.5 \, D_{ij} \left\| u_i - u_j \right\|_2^2$$

Total energy

$$E = \sum_{(i,j) \in E} \frac{1}{2} D_{ij} \left\| u_i - u_j \right\|_2^2$$
Hooke’s Law:
Energy of spring between $p_i$ and $p_j$

$$E_{ij} = 0.5 D_{ij} |u_i - u_j|^2$$

Total energy

$$E = \sum_{(i,j) \in E} \frac{1}{2} D_{ij} |u_i - u_j|^2$$

Spring Model

Will this provide a bijective parameterization?
Spring Model

\[ E = \sum_{(i,j) \in E} \frac{1}{2} D_{ij} \| u_i - u_j \|^2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} \frac{1}{2} D_{ij} \| u_i - u_j \|^2 \]
Spring Model

$$E = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} \frac{1}{2} D_{ij} \| u_i - u_j \|^2$$

Stable state ⇔ minimum of total energy
Spring Model

\[ E = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} \frac{1}{2} D_{ij} \| u_i - u_j \|^2 \]

Stable state \iff minimum of total energy

\[ \frac{\partial E}{\partial u_i} = \sum_{j \in N_i} D_{ij} (u_i - u_j) = 0 \]
Spring Model

\[ E = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} \frac{1}{2} D_{ij} \| u_i - u_j \|^2 \]

\[ \sum_{j \in N_i} D_{ij} (u_i - u_j) = 0 \]

Rewrite:

\[ u_i = \sum_{j \in N_i} \lambda_{ij} u_j \]

\[ \lambda_{ij} = \frac{D_{ij}}{\sum_{k \in N_i} D_{ik}} \]
Spring Model

\[ E = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} \frac{1}{2} D_{ij} \| u_i - u_j \|^2 \]

\[ \sum_{j \in N_i} D_{ij} (u_i - u_j) = 0 \]

Rewrite:

\[ u_i = \sum_{j \in N_i} \lambda_{ij} u_j \]
\[ \lambda_{ij} = D_{ij} / \sum_{k \in N_i} D_{ik} \]

A generalization of barycentric coordinates
Linear System

- Let’s rearrange the variables:
  \[
  u_i = \sum_{j \in N_i, j \leq n} \lambda_{ij} u_j = \sum_{j \in N_i, j > n} \lambda_{ij} u_j = \bar{u}_i
  \]
  unknown parameter points          fixed

- Linear system
  \[
  A U = \bar{U},
  \]
  \[
  A = \begin{cases} 
  1, & i = j \\
  -\lambda_{ij}, & j \in N_i \\
  0, & \text{else}
  \end{cases}
  \]

  \[
  \lambda_{ij} = \frac{D_{ij}}{\sum_{k \in N_i} D_{ik}}
  \]

  – Solve separately for \( u \) and \( v \)
Linear System

• Let’s rearrange the variables:

\[ u_i - \sum_{j \in N_i, j \leq n} \lambda_{ij} u_j = \sum_{j \in N_i, j > n} \lambda_{ij} u_j = \bar{u}_i \]

unknown parameter points

fixed

• Linear system

\[ A U = \bar{U}, \]

\[ A = \begin{cases} 1, & i = j \\ -\lambda_{ij}, & j \in N_i \\ 0, & \text{else} \end{cases} \]

\[ \lambda_{ij} = D_{ij} / \sum_{k \in N_i} D_{ik} \]

What does the matrix remind you of?

– Solve separately for \( u \) and \( v \)
**Theorem** [Tutte’63,Floater’01,Maxwel’1864]:

If $G$ is a 3-connected planar graph (*triangular mesh*) then any convex combination embedding ($\lambda_{ij} > 0$) provides bijective parameterization
Choice of Weights: Uniform (Tutte)

\[ D_{ij} = 1 \quad \lambda_{ij} = \frac{1}{\#N_i} \]

No shape preservation – equilateral triangles

Graph Laplacian!
Choice of Weights: Uniform (Tutte)

\[ D_{ij} = 1 \quad \lambda_{ij} = \frac{1}{\#N_i} \]

No shape preservation – equilateral triangles

Graph Laplacian!

\[ E = \sum_{(i,j) \in E} \frac{1}{2} D_{ij} \| u_i - u_j \|^2 \]

is Dirichlet energy discretized on a graph!
Choice of Weights: Barycentric
Harmonic/Conformal/FEM Laplacian

\[ \omega_{ij} = \cot \gamma_{ij} + \cot \gamma_{ji} \]
\[ \lambda_{ij} = \omega_{ij} / \sum_k \omega_{ik} \]

\[ E = \sum_{(i,j) \in E} \frac{1}{2} D_{ij} \| u_i - u_j \|^2 \] is Dirichlet energy discretized on a mesh!
Issue

Point is inside, but the coordinate can be <0

Figure 1. Star-shaped polygon.
Issue

Point is inside, but the coordinate can be <0

⇒ ∃λ_{ij} < 0

Figure 1. Star-shaped polygon.
Issue

Point is inside, but the coordinate can be $< 0$
$\Rightarrow \exists \lambda_{ij} < 0$

Local non-bijectivity

Figure 1. Star-shaped polygon.
Recall:

Harmonic Functions

\[ \Delta f \equiv 0 \]

Mean value property:

\[ f(x) = \frac{1}{\pi r^2} \int_{B_r(x)} f(y) \, dA \]
Choice of Weights: Mean Value

\[ \omega_{ij} = \frac{\tan \frac{\alpha_{ij}}{2} + \tan \frac{\beta_{ji}}{2}}{r_{ij}} \]

\[ \lambda_{ij} = \frac{\omega_{ij}}{\sum_{k \in N_i} \omega_{ik}} \]
Choice of Weights: Mean Value

\[ \omega_{ij} = \frac{\tan \frac{\alpha_{ij}}{2} + \tan \frac{\beta_{ji}}{2}}{r_{ij}} \]

Always non-negative
Harmonic/Mean-Value Mappings

• Quasi-Conformal

• Linear precision
  – Reproduce planar inputs (same boundary)
Bijectivity (fold-overs)

- Can have fold-overs for negative coordinates
- Mean-value coordinates guaranteed to be positive

\[ \lambda_{ij} = \sum_{i} k_i N_i \cdot \omega_{ik} \]
Boundary Mapping

Chordal parameterization around convex shape
• circle
• rectangle
• triangle
• Choice often application specific
  – Reconstruction – rectangle
  – Mapping to base mesh – triangle
Examples

Parameterization with uniform weights [Tutte 1963] on a circular domain.

Parameterization with harmonic weights [Eck et al. 1995] on a circular domain.

Parameterization with mean value weights [Floater 2003] on a circular domain.
Parameterization: Free Boundary
Free Boundary Methods

• Direct energy minimization
  – Example: Least Squares Conformal Map (LSCM)....

• Indirect
  – Example: Angle Based Flattening (ABF)....

Free vs Fixed
LSCM – Geometric Interpretation

We’re minimizing conformal energy

\[ E_C = \frac{(\sigma_1 - \sigma_2)^2}{2} \]

Geometric Interpretation:

– Use triangle similarity
– Given angles \( \alpha_1, \alpha_2, \alpha_3 \) of a triangle \( P_1P_2P_3 \) in 2D we have

\[
P_3 - P_1 = \frac{\sin \alpha_2}{\sin \alpha_3} R_{\alpha_1} (P_2 - P_1),
\]

\[
R_{\alpha} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}
\]
LSCM

- In map from 3D to 2D might be impossible to keep angles exactly
  - Use least-squares

\[
\min \sum_i \left( P_3^i - P_1^i - \sin \alpha_2^i \frac{\sin \alpha_3^i}{\sin \alpha_3^i} R_{\alpha_1^i} (P_2^i - P_1^i) \right)^2
\]

- To solve need to fix two vertices
  - Obtain linear system
  - Choice of vertices affects solution
- Can have flips
Examples

Parameterization with mean value weights [Floater 2003] on a circular domain.

Parameterization with LSCM [Lévy et al. 2002].
ABF: Angle Based Flattening

• Triangular 2D mesh is defined by its angles
• Formulate parameterization as problem in angle space
• Angle based formulation:
  – Distortion as function of angles (conformality)
  – Validity: set of angle constraints
  – Convert solution to UV
ABF Formulation

- Distortion:
  - 2D/3D angle difference

\[
\sum_{t \in T, j=1...3} w_j^t (\alpha_j^t - \beta_j^t)^2, w_j^t = 1/\beta_j^t^2
\]
ABF Formulation

• **Constraints:**
  
  – Triangle validity:
  \[ \forall t \in T, \quad \alpha_1^t + \alpha_2^t + \alpha_3^t - \pi = 0; \]
  
  – Planarity:
  \[ \forall v \in V_{int}, \quad \sum_{(t,k) \in \nu^*} \alpha_k^t - 2\pi = 0 \]
  
  – Reconstruction
  \[ \forall v \in V_{int}, \quad \prod_{(t,k) \in \nu^*} \sin \alpha_{k+1}^t - \prod_{(t,k) \in \nu^*} \sin \alpha_k^t = 0 \]
  
  – Positivity
  \[ \alpha_j^t > 0 \]

• **Distortion:**
  \[ \sum_{t \in T, j=1 \ldots 3} w_j^t (\alpha_j^t - \beta_j^t)^2, \quad w_j^t = 1/\beta_j^{t^2} \]
Angle to UV Conversion

• Use computed angles as input to LSCM (it is a reproducing method..)
Examples

Parameterization with LSCM [Lévy et al. 2002].

Parameterization with ABF++ [Sheffer et al. 2005].
Examples

Parameterization with LSCM [Lévy et al. 2002].

Parameterization with ABF++ [Sheffer et al. 2005].
Cone Singularities [Kharevych:06]

- What separates boundary from interior in angle space?
- Answer: Sum of angles at vertex
- Formulation specific
  - ABF/ABF++
    - Planarity & Reconstruction
      \[ \forall v \in V_{int}, \sum_{(t,k)\in v^*} \alpha^t_k - 2\pi = 0 \]
- But... reconstruction can be enforced on boundaries

\[ \forall v \in V_{int}, \prod_{(t,k)\in v^*} \sin \alpha^t_{k\oplus 1} - \prod_{(t,k)\in v^*} \sin \alpha^t_{k\oplus 1} = 0 \]
Cone Singularities

• Idea: Reduce boundary to small set of vertices
• Implementation:
  – Enforce “interior” constraints at all other vertices
• To unfold choose any sequence of edges connecting “boundary” vertices
Circle Patterns + Cone Singularities
ABF + Cone Singularities
General Framework

- Choose an energy

<table>
<thead>
<tr>
<th>Name</th>
<th>$\mathcal{D}(\mathbf{J})$</th>
<th>$\mathcal{D}(\sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetric Dirichlet</td>
<td>$|\mathbf{J}|_F^2 + |\mathbf{J}^{-1}|_F^2$</td>
<td>$\sum_{i=1}^{n} (\sigma_i^2 + \sigma_i^{-2})$</td>
</tr>
<tr>
<td>Exponential Symmetric</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dirichlet</td>
<td>$\exp(s(|\mathbf{J}|_F^2 + |\mathbf{J}^{-1}|_F^2))$</td>
<td>$\exp(s \sum_{i=1}^{n} (\sigma_i^2 + \sigma_i^{-2}))$</td>
</tr>
<tr>
<td>Hencky strain</td>
<td>$|\log \mathbf{J}^\top \mathbf{J}|_F^2$</td>
<td>$\sum_{i=1}^{n} (\log^2 \sigma_i)$</td>
</tr>
<tr>
<td>AMIPS</td>
<td>$\exp(s \cdot \frac{1}{2} (\frac{\text{tr}(\mathbf{J}^\top \mathbf{J})}{\text{det}(\mathbf{J})}) + \frac{1}{2} (\text{det}(\mathbf{J}) + \text{det}(\mathbf{J}^{-1}))$)</td>
<td>$\exp(s \left( \frac{1}{2} \left( \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1} \right) + \frac{1}{4} \left( \sigma_1 \sigma_2 + \frac{1}{\sigma_1 \sigma_2} \right) \right)$</td>
</tr>
<tr>
<td>Conformal AMIPS 2D</td>
<td>$\frac{\text{tr}(\mathbf{J}^\top \mathbf{J})}{\text{det}(\mathbf{J})}$</td>
<td>$\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 \sigma_2}$</td>
</tr>
<tr>
<td>Conformal AMIPS 3D</td>
<td>$\frac{\text{tr}(\mathbf{J}^\top \mathbf{J})}{\text{det}(\mathbf{J})^{\frac{2}{3}}}$</td>
<td>$\frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{(\sigma_1 \sigma_2 \sigma_3)^{\frac{2}{3}}}$</td>
</tr>
</tbody>
</table>
Recall:

Gradient of a Hat Function

\[ \| \nabla f \| = \frac{1}{\ell_3 \sin \theta_3} = \frac{1}{h} \]

\[ \nabla f = \frac{e_{23}}{2A} \]

Length of \( e_{23} \) cancels “base” in \( A \)
Parameterization Jacobian

\[ \nabla u = \begin{bmatrix} \frac{\partial u}{\partial X} \\ \frac{\partial u}{\partial Y} \end{bmatrix} = \frac{1}{2A_T} \begin{bmatrix} Y_j - Y_k & Y_k - Y_i & Y_i - Y_j \\ X_k - X_j & X_i - X_k & X_j - X_i \end{bmatrix} \begin{pmatrix} u_i \\ u_j \\ u_k \end{pmatrix} \]

\[ X = \frac{x_j - x_i}{\|x_j - x_i\|} \]
\[ n = \frac{X \times (x_k - x_i)}{\|X \times (x_k - x_i)\|} \]
\[ Y = n \times X \]

**Figure 5.9.** Local \( X, Y \) basis in a triangle.
Parameterization Jacobian

\[ \nabla u = \begin{bmatrix} \frac{\partial u}{\partial X} \\ \frac{\partial u}{\partial Y} \end{bmatrix} = \frac{1}{2A_T} \begin{bmatrix} Y_j - Y_k & Y_k - Y_i & Y_i - Y_j \\ X_k - X_j & X_i - X_k & X_j - X_i \end{bmatrix} \begin{pmatrix} u_i \\ u_j \\ u_k \end{pmatrix} = M_T \]

\[ J_T = \begin{bmatrix} \frac{\partial u}{\partial X} & \frac{\partial v}{\partial X} \\ \frac{\partial u}{\partial Y} & \frac{\partial v}{\partial Y} \end{bmatrix} \]
Conformal Energies

- **Conformal** energy
  \[ E_C = \frac{(\sigma_1 - \sigma_2)^2}{2} \]

- **MIPS** energy
  \[ E_M = \kappa_F(J_f) = \|J_f\|_F \|J_f^{-1}\|_F = \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1} \]

- **Riemann theorem**: any \( C^1 \) continuous surface in \( \mathbb{R}^3 \) can be mapped conformally to fixed domain in \( \mathbb{R}^2 \)
  - Nearly true for meshes

[Pinkall & Polthier 1993]  
[Lévy et al. 2002]  
[Desbrun et al. 2002]  
[Hormann & Greiner 2000]
Detailed Example

Given a triangle $T$ with 2D texture coordinates $p_1, p_2, p_3$, $p_i = (s_i, t_i)$, and corresponding 3D coordinates $q_1, q_2, q_3$, the unique affine mapping $S(p) = S(s, t) = q$ is

$$S(p) = \frac{\langle p, p_2, p_3 \rangle q_1 + \langle p, p_3, p_1 \rangle q_2 + \langle p, p_1, p_2 \rangle q_3}{\langle p_1, p_2, p_3 \rangle}$$

$$S_s = \frac{\partial S}{\partial s} = \frac{q_1(t_2 - t_3) + q_2(t_3 - t_1) + q_3(t_1 - t_2)}{2A}$$

$$S_t = \frac{\partial S}{\partial t} = \frac{q_1(s_2 - s_1) + q_2(s_1 - s_3) + q_3(s_3 - s_2)}{2A}$$

$$A = \langle p_1, p_2, p_3 \rangle = \frac{(s_2 - s_1)(t_3 - t_1) - (s_3 - s_1)(t_2 - t_1)}{2}$$

Singular values: $\sqrt{0.5((a + c) \pm \sqrt{(a - c)^2 + 4b^2}}$

$$a = S_s \cdot S_s, \quad b = S_s \cdot S_t, \quad \text{and} \quad c = S_t \cdot S_t$$
General Framework

• Choose an energy
• Start with an initial bijective parameterization – E.g. Tutte
General Framework

• Choose an energy
• Start with an initial bijective parameterization
  – E.g. Tutte
• Use nonlinear optimization tools to minimize
  – Gradient descent
  – Quasi–Newton methods
  – ...

General Framework

• Choose an energy
• Start with an initial bijective parameterization
  – E.g. Tutte
• Use nonlinear optimization tools to minimize
  – Gradient descent
  – Quasi-Newton methods
  – …
• How to preserve bijectivity?
What is gradient of $E$ w/r to positions?
What is gradient of $E$ w/r to positions?

A vector field!
Scaling vector field

- Find a scale parameter, s.t. nothing flips if we add the vector field
- Limit line search step to this value
Issues

- Only local injectivity
Issues

- Only local injectivity
- Sometimes the step size is too small
  - One almost inverted triangle is enough

Image from “Blended Cured Quasi-Newton for Distortion Optimization” by Yufeng Zhu, Robert Bridson, and Danny M. Kaufman. SIGGRAPH 2018