IFT 6113
CONTINUOUS LAPLACIAN
http://tiny.cc/ift6113

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Famous Motivation

CAN ONE HEAR THE SHAPE OF A DRUM?

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To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

“La Physique ne nous donne pas seulement l’occasion de résoudre des problèmes..., elle nous fait présentir la solution.” H. Poincaré.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

Fig. 1

1. And now to the theme and the title.
It has been known for well over a century that if a membrane $\Omega$, held fixed along its boundary $\Gamma$ (see Fig. 1), is set in motion its displacement (in the direction perpendicular to its original plane)

$$F(x, y; t) = F(\hat{r}; t)$$

obeys the wave equation

$$\frac{\partial^2 F}{\partial t^2}$$
An Experiment

Is this possible?
Unreasonable to Ask?

Length of string
1D: length of a string

\[ u(x, t) \] - string height

Boundary conditions: \( u(0, t) = 0 = u(l, t) \)

PDE: \[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]
Standing waves?

(on the board)
Can you hear the length of an interval?

Yes!

\[ \lambda_k = \left( \frac{\pi k}{\ell} \right)^2 \]
Spoiler Alert

“No, but…”

• Has to be a weird drum
• Spectrum tells you a lot!

https://en.wikipedia.org/wiki/Hearing_the_shape_of_a_drum
You can learn a lot about a shape by hitting it (lightly) with a hammer!
Spectral Geometry

What can you learn about its shape from vibration frequencies and oscillation patterns?

\[ \Delta f = \lambda f \]
Objectives

- Make “vibration modes” more **precise**

- **Progressively more complicated** domains
  - Line segments
  - Regions in $\mathbb{R}^2$
  - Graphs
  - Surfaces/manifolds

- Next time: **Discretization, applications**
Vector Spaces and Linear Operators

\[ \mathcal{L}[\vec{x} + \vec{y}] = \mathcal{L}[\vec{x}] + \mathcal{L}[\vec{y}] \]

\[ \mathcal{L}[c\vec{x}] = c\mathcal{L}[\vec{x}] \]
In Finite Dimensions

\[ A \xrightarrow{\vec{x}} \]

\text{matrix vector}

\[ \vec{x} \mapsto A\vec{x} \]

\text{linear operator}
Recall: Spectral Theorems in $\mathbb{C}^n$

**Theorem.** Suppose $A \in \mathbb{C}^{n \times n}$ is Hermitian. Then, $A$ has an orthogonal basis of $n$ eigenvectors. If $A$ is positive semidefinite, the corresponding eigenvalues are nonnegative.
Our Progression

- **Line segments**
- **Regions in** \( \mathbb{R}^2 \)
- **Graphs**
- **Surfaces/manifolds**
Minus Second Derivative Operator

\[ \{ f(\cdot) \in C^\infty([a, b]) : f(0) = f(\ell) = 0 \} \]

\[ \mathcal{L}[f(\cdot)] := -f''(\cdot) \]

“Dirichlet boundary conditions”

Eigenfunctions:

\[ f_k(x) = \sin \left( \frac{\pi k x}{\ell} \right), \quad \lambda_k = \left( \frac{\pi k}{\ell} \right)^2 \]
Physical Intuition: Wave Equation

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0
\]

Minus second derivative operator!
Observation

\[ \{ f(\cdot) \in C^\infty([a, b]) : f(0) = f(\ell) = 0 \} \]

\[ \langle f, \mathcal{L}[f] \rangle = -\int_0^\ell f(x) f''(x) \, dx \]

\[ = -[f(x)f'(x)]_0^\ell + \int_0^\ell f'(x)^2 \, dx \]

\[ \geq 0 \]
Hilbert-Schmidt Theorem

Theorem. Let $H \neq 0$ be an infinite-dimensional, separable Hilbert space and let $K \in L(H)$ be compact and self-adjoint. Then, there exists a countable orthonormal basis of $H$ consisting of eigenvectors of $K$.

**Hilbert space:** Space with inner product  
**Separable:** Admits countable, dense subset  
**Compact operator:** Bounded sets to relatively compact sets  
**Self-adjoint:** $\langle Kv, w \rangle = \langle v, Kw \rangle$
Our Progression

• **Line segments**

• **Regions in** $\mathbb{R}^2$

• **Graphs**

• **Surfaces/manifolds**
Planar Region

\[ \Omega \subseteq \mathbb{R}^2 \]

Wave equation:

\[
\frac{\partial^2 u}{\partial t^2} = \Delta u
\]

\[
\Delta := \sum_i \frac{\partial^2}{\partial x_i^2}
\]
Typical Notation

\[ \Delta = \nabla \cdot \nabla \]

More later...

Gradient operator:

\[ \nabla := \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right) \]
Positivity, Self-Adjointness

\[ \{ f(\cdot) \in C^\infty(\Omega) : f|_{\partial\Omega} \equiv 0 \} \]

"Dirichlet boundary conditions"

\[ \mathcal{L}[f] := -\Delta f \]

\[ \langle f, g \rangle := \int_{\Omega} f(x)g(x) \, dx \]

On board:

1. Positive: \( \langle f, \mathcal{L}[f] \rangle \geq 0 \)

2. Self-adjoint: \( \langle f, \mathcal{L}[g] \rangle = \langle \mathcal{L}[f], g \rangle \)
Dirichlet Energy

\[ E[f] := \int_{\Omega} \langle \nabla f, \nabla f \rangle \, dA \]

On board:

\[
\begin{align*}
\min_{f} E[f] \\
\text{s.t. } f|_{\partial\Omega} = g
\end{align*}
\]

\[ \Delta f \equiv 0 \]

"Laplace equation"  
"Harmonic function"

Images made by E. Vouga
Harmonic Functions

\[ \Delta f \equiv 0 \]

Mean value property:

\[ f(x) = \frac{1}{\pi r^2} \int_{B_r(x)} f(y) \, dA \]
Intrinsic Operator

Images made by E. Vouga

Coordinate-independent
Another Interpretation of Eigenfunctions

Find critical points of $E[f]$

$s.t. \int_{\Omega} f^2 = 1$

Small eigenvalue: smooth function
Our Progression

• **Line segments**

• **Regions in** $\mathbb{R}^2$

  • **Graphs**

• **Surfaces/manifolds**
Basic Setup

• **Function:**
  One value per vertex
Dirichlet energy of a function on a graph?
Differencing Operator

Orient edges arbitrarily

\[ D_{ev} := \begin{cases} 
-1 & \text{if } E_{e1} = v \\
1 & \text{if } E_{e2} = v \\
0 & \text{otherwise} 
\end{cases} \]

\[ D \in \{-1, 0, 1\}^{|E| \times |V|} \]
Dirichlet Energy on a Graph

\[ E[f] := \| Df \|_2^2 = \sum_{(v,w) \in E} (f_v - f_w)^2 \]

\[ D_{ev} := \begin{cases} 
-1 & \text{if } E_{e1} = v \\
1 & \text{if } E_{e2} = v \\
0 & \text{otherwise}
\end{cases} \]
(Unweighted) Graph Laplacian

\[ E[f] = \| D f \|_2^2 = f^\top (D^\top D) f := f^\top L f \]

\[ L_{\nu w} = A - D = \begin{cases} 
1 & \text{if } \nu \sim w \\
-\text{degree}(\nu) & \text{if } \nu = w \\
0 & \text{otherwise}
\end{cases} \]

<table>
<thead>
<tr>
<th>Labeled graph</th>
<th>Degree matrix</th>
<th>Adjacency matrix</th>
<th>Laplacian matrix</th>
</tr>
</thead>
</table>
| ![Graph](https://en.wikipedia.org/wiki/Laplacian_matrix) | \[
\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
2 & -1 & 0 & 0 & -1 & 0 \\
-1 & 3 & -1 & 0 & -1 & 0 \\
-1 & 3 & -1 & 0 & -1 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 3 & -1 & 1 \\
-1 & -1 & 0 & -1 & 3 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
\end{pmatrix}
\] |

- Symmetric
- Positive definite
Second-Smallest Eigenvector

\[ Lx = \lambda x \]

Fiedler vector ("algebraic connectivity")

Used for graph partitioning
Mean Value Property

\[
L_{vw} = A - D = \begin{cases} 
1 & \text{if } v \sim w \\
-\text{degree}(v) & \text{if } v = w \\
0 & \text{otherwise}
\end{cases}
\]

\[
(Lx)_v = 0
\]

Value at \( v \) is average of neighboring values
For More Information...

Graph Laplacian encodes lots of information!

Example: Kirchoff’s Theorem
Number of spanning trees equals
\[
\frac{1}{n! \lambda_2 \lambda_3 \cdots \lambda_n}
\]
Hear the Shape of a Graph?

No!

“Enneahedra”
Our Progression

- Line segments
- Regions in $\mathbb{R}^2$
- Graphs
- Surfaces/manifolds
Recall:
Scalar Functions

\[ f : \rightarrow \mathbb{R} \]

Map points to real numbers
Differential of a Map

Suppose \( f : S \to \mathbb{R} \) and take \( p \in S \). For \( v \in T_pS \), choose a curve \( \alpha : (-\varepsilon, \varepsilon) \to S \) with \( \alpha(0) = p \) and \( \alpha'(0) = v \). Then the differential of \( f \) is \( df : T_pS \to \mathbb{R} \) with

\[
(df)_p(v) := \frac{d}{dt} \big|_{t=0} (f \circ \alpha)(t) = (f \circ \alpha)'(0).
\]

On the board (time-permitting):
- Does not depend on choice of \( \alpha \)
- Linear map

Following *Curves and Surfaces*, Montiel & Ros
Gradient Vector Field

\[ \nabla f : S \rightarrow \mathbb{R}^3 \text{ with } \]

\[
\begin{align*}
\langle (\nabla f)(p), v \rangle &= (df)_p(v), v \in T_pS \\
\langle (\nabla f)(p), N(p) \rangle &= 0
\end{align*}
\]

Following *Curves and Surfaces*, Montiel & Ros
Dirichlet Energy

\[ E[f] := \int_{S} \| \nabla f \|_{2}^{2} \, dA \]

Images made by E. Vouga
From Inner Product to Operator

On the board:

“Motivation” from finite-dimensional linear algebra.

\[ \langle f, g \rangle_\Delta := \int_S \nabla f(x) \cdot \nabla g(x) \, dA \]

\[ \equiv \langle f, \Delta g \rangle \]

Implies

\[ \langle f, f \rangle \geq 0 \]

Laplace–Beltrami operator
What is Divergence?

$V : S \rightarrow \mathbb{R}^3$ where $V(p) \in T_p S$

$dV_p : T_p S \rightarrow \mathbb{R}^3$

$\{e_1, e_2\} \subset T_p S$ orthonormal basis

$$(\nabla \cdot V)_p := \sum_{i=1}^{2} \langle e_i, dV(e_i) \rangle_p$$

Things we should check (but probably won’t):

• Independent of choice of basis
  • $\Delta = \nabla \cdot \nabla$
Eigenfunctions

\[ \Delta \psi_i = \lambda_i \psi_i \]

Vibration modes of surface (not volume!)

http://alice.loria.fr/publications/papers/2008/ManifoldHarmonics//photo/dragon_mhb
Chladni Plates

https://www.youtube.com/watch?v=CGiiSlMFFlI
Performance Art?

https://www.youtube.com/watch?v=Fyzqd2_T09Q
Practical Application

https://www.youtube.com/watch?v=3uMZZzVvnSiU
Additional Connection to Physics

\[ \frac{\partial u}{\partial t} = -\Delta u \]

Heat equation

http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf
Spherical Harmonics

Weyl’s Law

\[ N(\lambda) := \# \text{ eigenfunctions } \leq \lambda \]
\[ \omega_d := \text{ volume of unit ball in } \mathbb{R}^d \]
\[ \lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = (2\pi)^{-d} \omega_d \text{vol}(\Omega) \]

Corollary: \( \text{vol}(\Omega) = (2\pi)^d \lim_{R \to \infty} \frac{N(R)}{R^{d/2}} \)

For surfaces: \( \lambda_n \sim \frac{4\pi}{\text{vol}(\Omega)} n \)
Laplacian of \( xyz \) function

\[
\Delta \vec{x} = \frac{1}{2} (\kappa_1 + \kappa_2) \mathcal{N}
\]

Intuition:
Laplacian measures difference with neighbors.