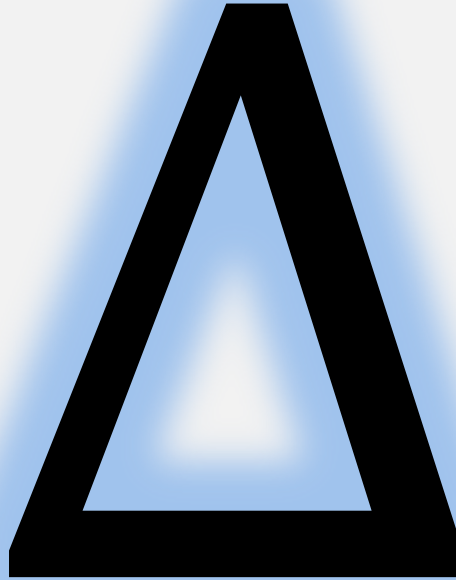


# IFT 6113

# DISCRETE LAPLACIAN

<http://tiny.cc/ift6113>

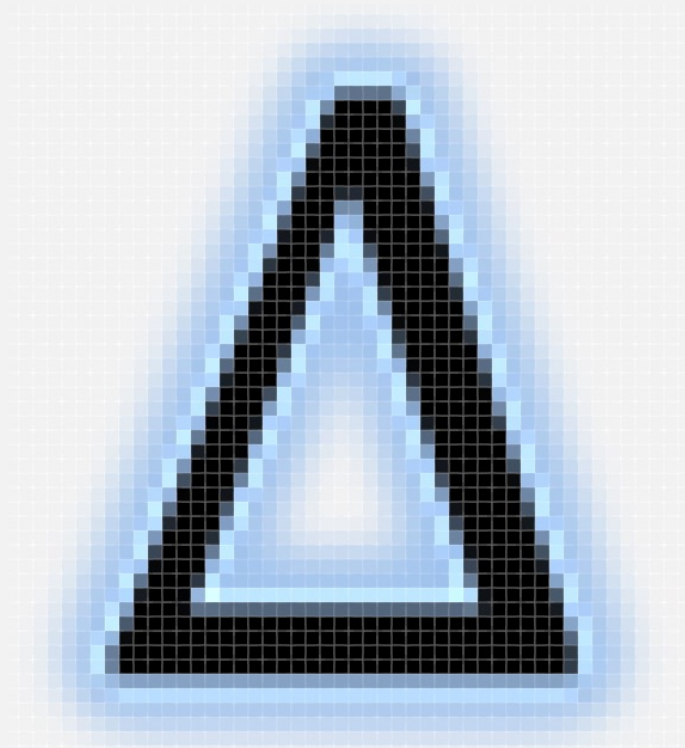


Mikhail Bessmeltsev

# IFT 6113

# DISCRETE LAPLACIAN

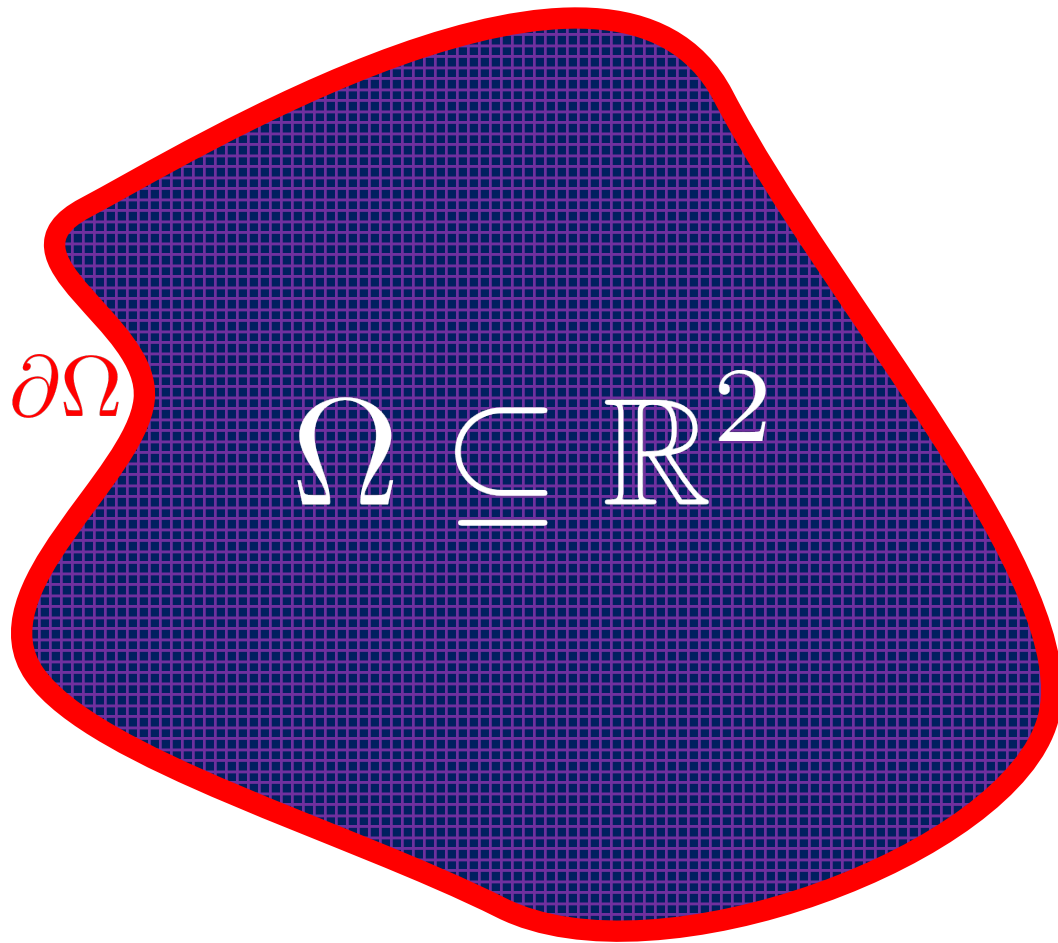
<http://tiny.cc/ift6113>



Mikhail Bessmeltsev

*Recall:*

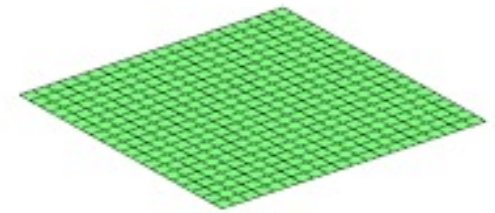
# Planar Region



**Wave equation:**

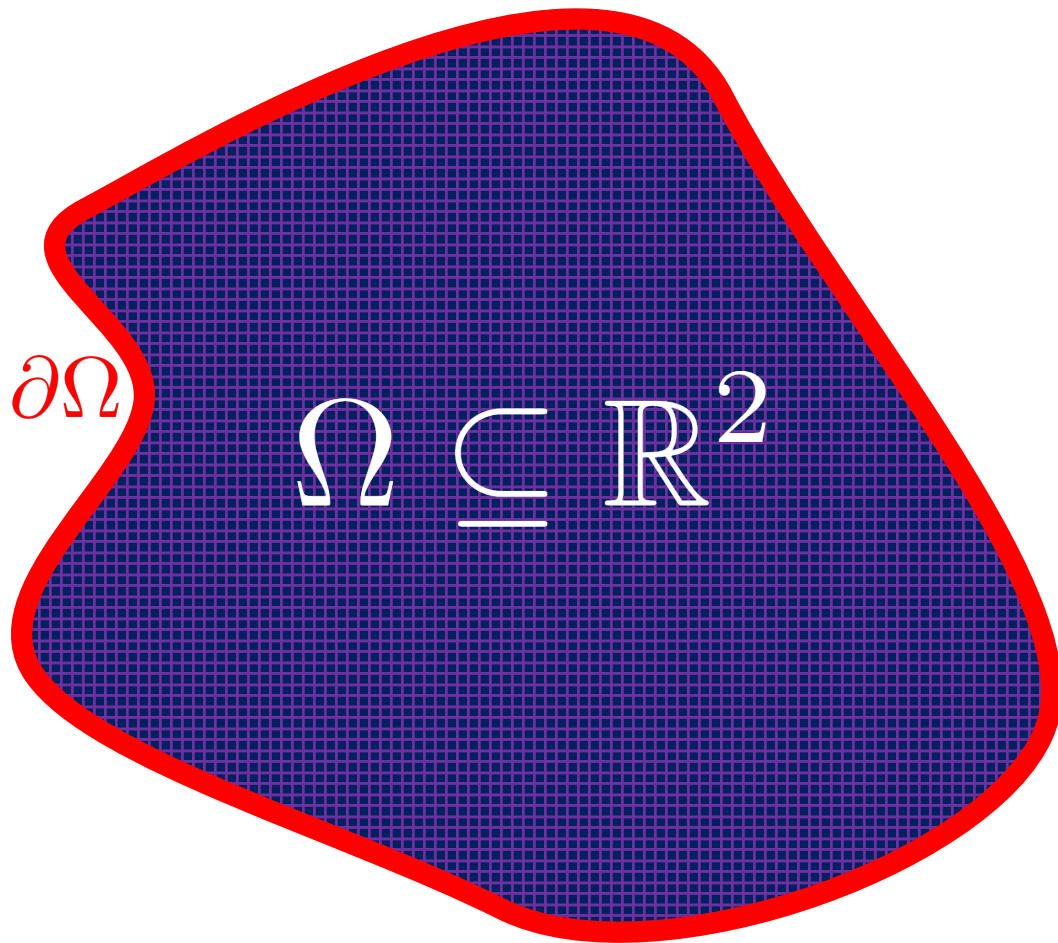
$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

$$\Delta := \sum_i \frac{\partial^2}{\partial x_i^2}$$



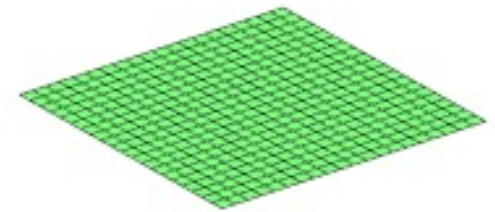
*Recall:*

# Planar Region



How to  
discretize on a  
mesh?

$$\Delta := \sum \frac{\partial^2}{\partial x_i^2}$$



# Discretizing the Laplacian

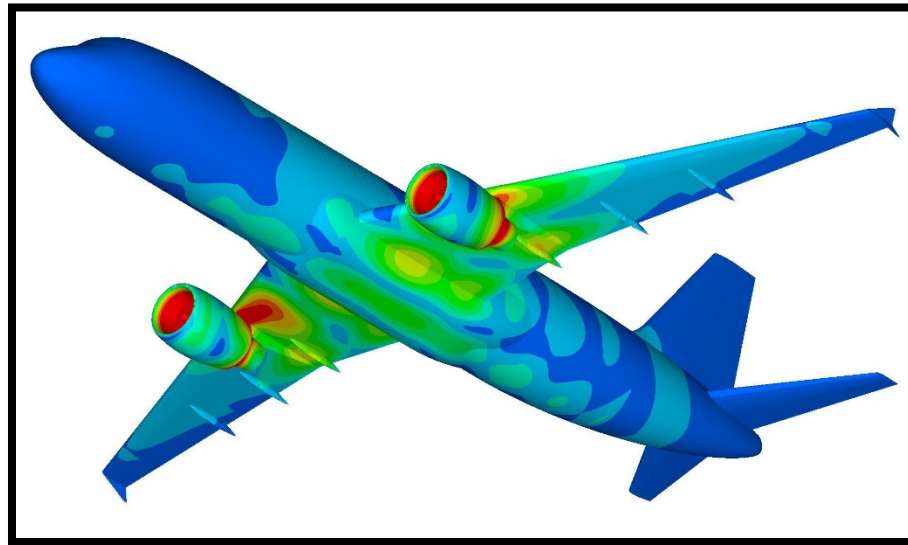
$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j f \right)$$

?!

# Today's Approach

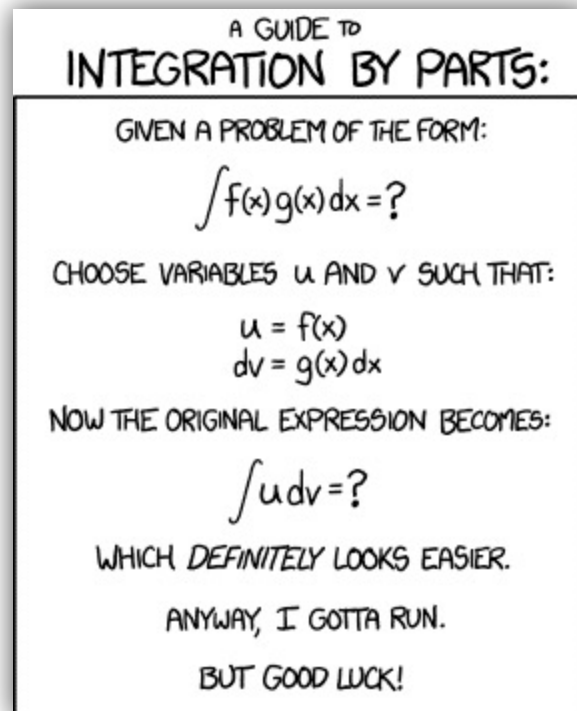
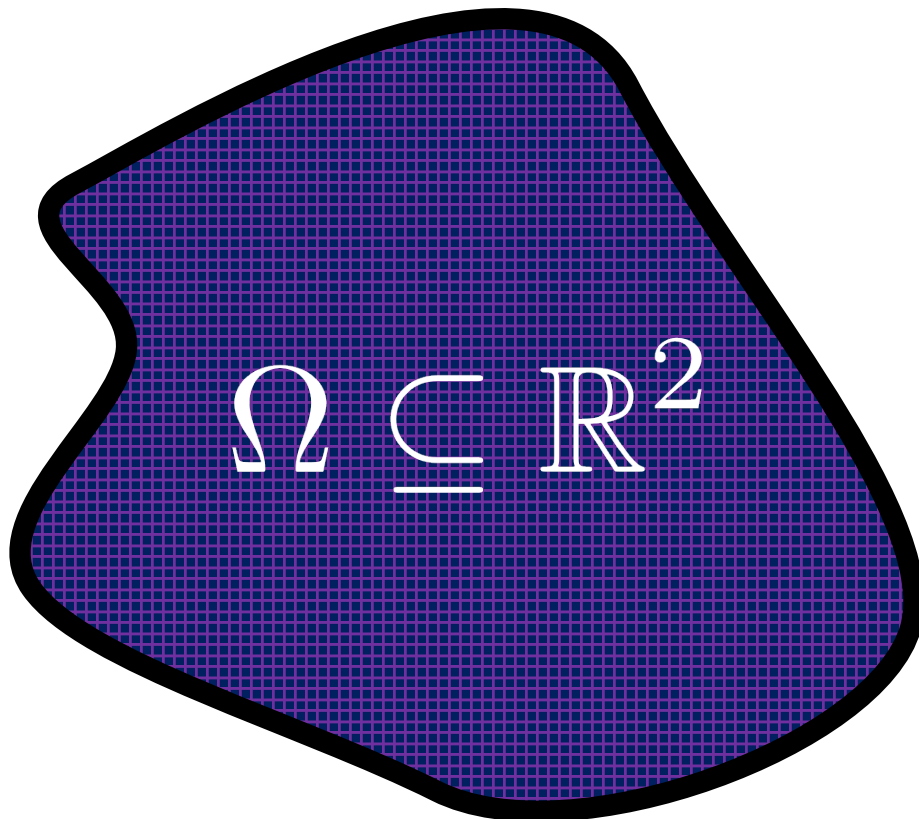
*First-order Galerkin*

## ***Finite element method (FEM)***



# Integration by Parts to the Rescue

$$\int_{\Omega} f \Delta g \, dA = \text{boundary terms} - \int_{\Omega} \nabla f \cdot \nabla g \, dA$$



# Slightly Easier?

$$\int_{\Omega} f \Delta g \, dA = \text{boundary terms} - \int_{\Omega} \nabla f \cdot \nabla g \, dA$$

Laplacian  
(second derivative)

Gradient  
(first  
derivative)



*Overview:*

# Galerkin FEM Approach

$$g = \Delta f$$

*Overview:*

# Galerkin FEM Approach

$$g = \Delta f \quad \Rightarrow \quad \langle g, \psi \rangle = \langle \Delta f, \psi \rangle$$

For any **test function**  $\psi$

*Overview:*

# Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA$$

For any **test function**  $\psi$

*Overview:*

# Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

For any **test function**  $\psi$

*Overview:*

# Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

If we have a function basis  $\{\psi_i\}$

$$f = \sum_{i=0}^{\infty} a_i \psi_i$$

$$g = \sum_{i=0}^{\infty} b_i \psi_i$$

*Overview:*

# Galerkin FEM Approach

We only need to test  $\psi = \psi_i$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

If we have a function basis  $\{\psi_i\}$

$$f = \sum_{i=0}^{\infty} a_i \psi_i$$

$$g = \sum_{i=0}^{\infty} b_i \psi_i$$

*Overview:*

# Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

Approximate  $f \approx \sum_i a_i \psi_i$  and  $g \approx \sum_i b_i \psi_i$

$$\implies \text{Linear system } \sum_i b_i \langle \psi_i, \psi_j \rangle = - \sum_i a_i \langle \nabla \psi_i, \nabla \psi_j \rangle$$

## Overview:

# Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

Approximate  $f \approx \sum_i a_i \psi_i$  and  $g \approx \sum_i b_i \psi_i$

$$\implies \text{Linear system } \sum_i b_i \langle \psi_i, \psi_j \rangle = - \sum_i a_i \langle \nabla \psi_i, \nabla \psi_j \rangle$$

Mass matrix:  $M_{ij} := \langle \psi_i, \psi_j \rangle$

Stiffness matrix:  $L_{ij} := \langle \nabla \psi_i, \nabla \psi_j \rangle$

$$\implies Mb = La$$

Which  
basis?



# BRACE YOURSELVES

- A2 out, due **Oct 18<sup>th</sup>**
- Project proposal due: **Oct 25<sup>th</sup>**
- Paper title due: **Oct 29<sup>th</sup>**



# Important to Note

Not the only way

*to approximate the Laplacian operator.*

- Divided differences
- Higher-order elements
- Boundary element methods
- Discrete exterior calculus
- ...

# $L^2$ Dual of a Function

**Function**  $f : M \rightarrow \mathbb{R}$



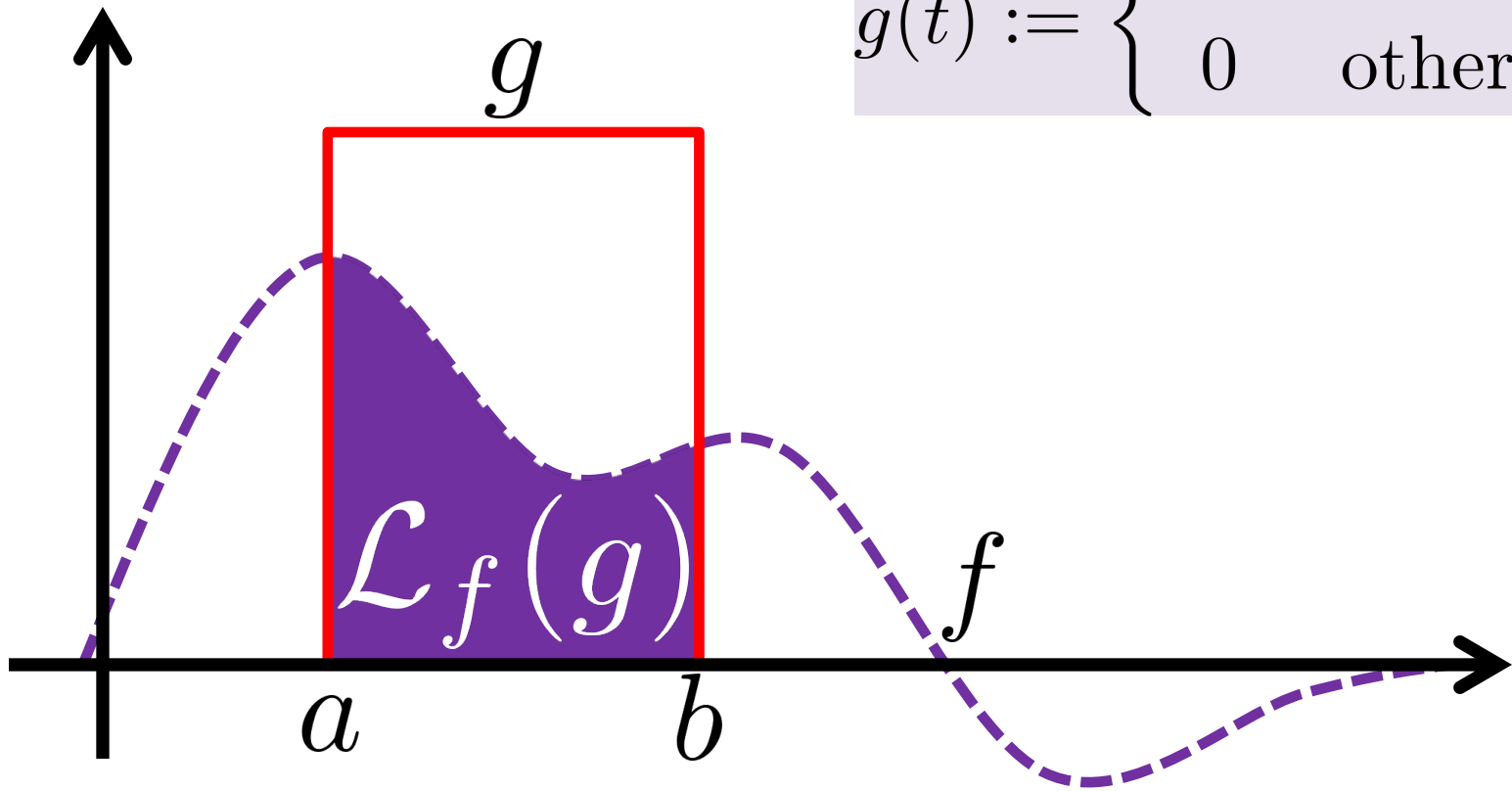
**Operator**  $\mathcal{L}_f : L^2(M) \rightarrow \mathbb{R}$

$$\mathcal{L}_f[g] := \int_M f(x)g(x) dA$$

  
**Test function**

# Observation

$$g(t) := \begin{cases} 1 & \text{if } t \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$



**Can recover function from dual**

# Dual of Laplacian

*Space of test functions (no boundary!):*

$$\{g \in L^\infty(M) : g|_{\partial M} \equiv 0\}$$

$$\begin{aligned}\mathcal{L}_{\Delta f}[g] &= \int_M g \Delta f \, dA \\ &= - \int_M \nabla g \cdot \nabla f \, dA\end{aligned}$$

**Use Laplacian without evaluating it!**

# Galerkin's Approach

*Choose one of each:*

- Function space

- Test functions

*Often the same!*

# One Derivative is Enough

$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

# First Order Finite Elements

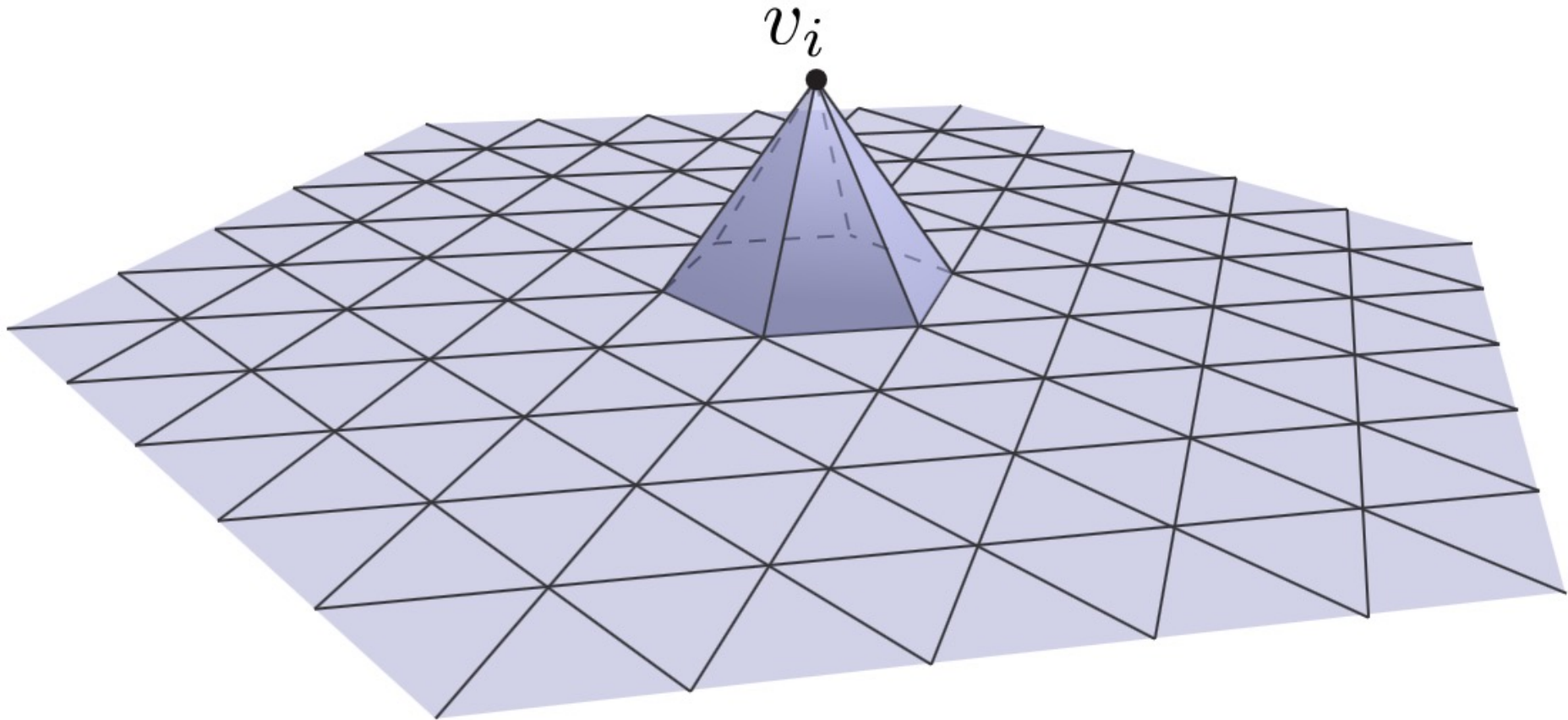
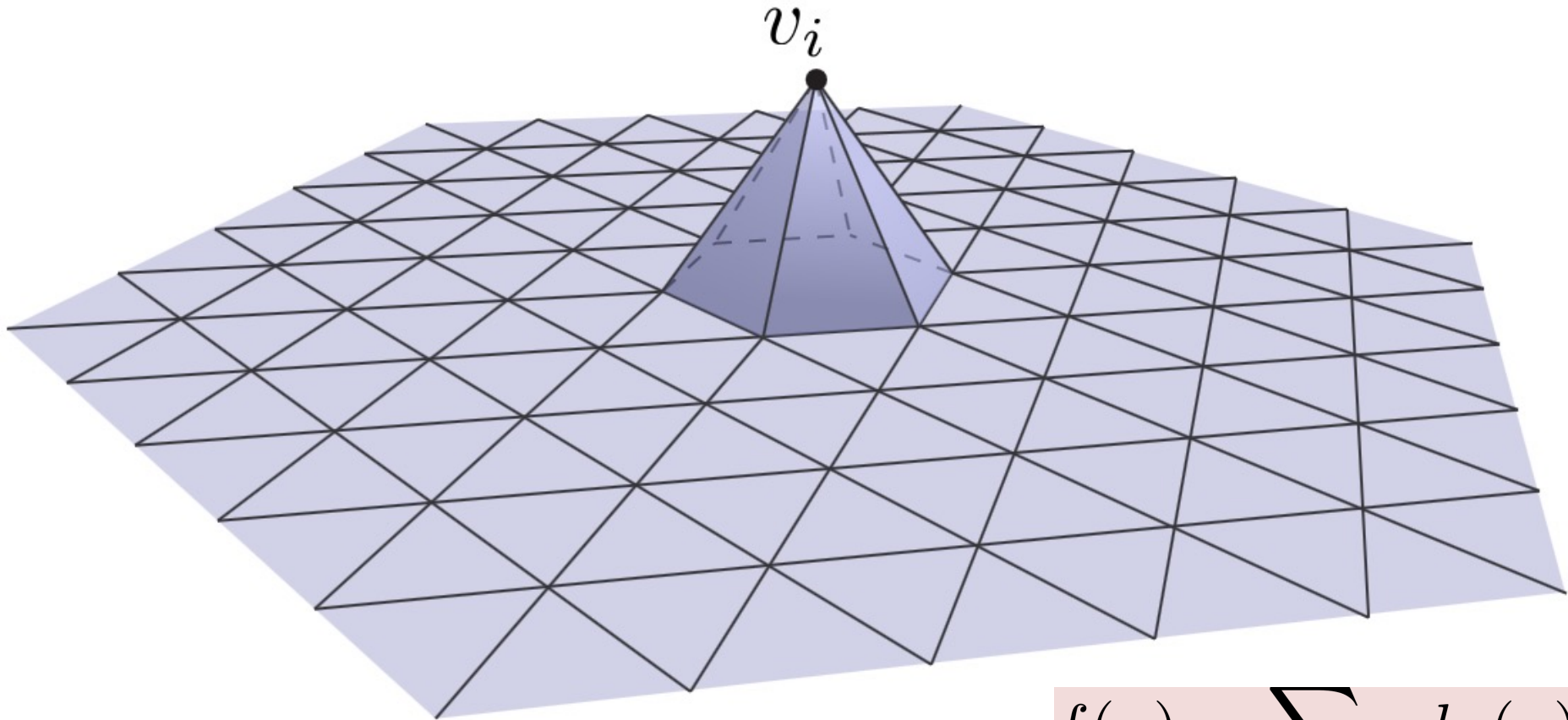


Image courtesy K. Crane, CMU

**One “hat function” per vertex**

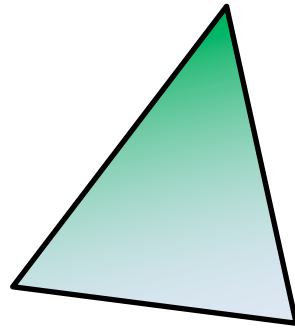


# Representing Functions



$$f(x) = \sum_i a_i h_i(x)$$
$$a \in \mathbb{R}^{|V|}$$

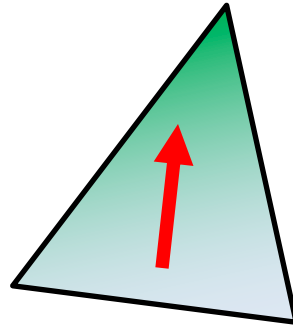
# What Do We Need



$$\mathcal{L}_{\Delta} f [g] = - \int_M \nabla g \cdot \nabla f \, dA$$

*Linear combination of hats  
(piecewise linear)*

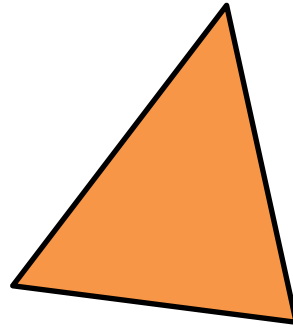
# What Do We Need



$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

*One vector per face*

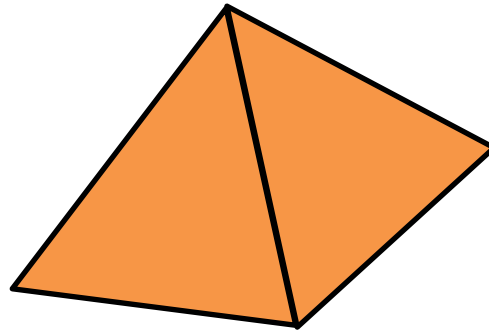
# What Do We Need



$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

*One scalar per face*

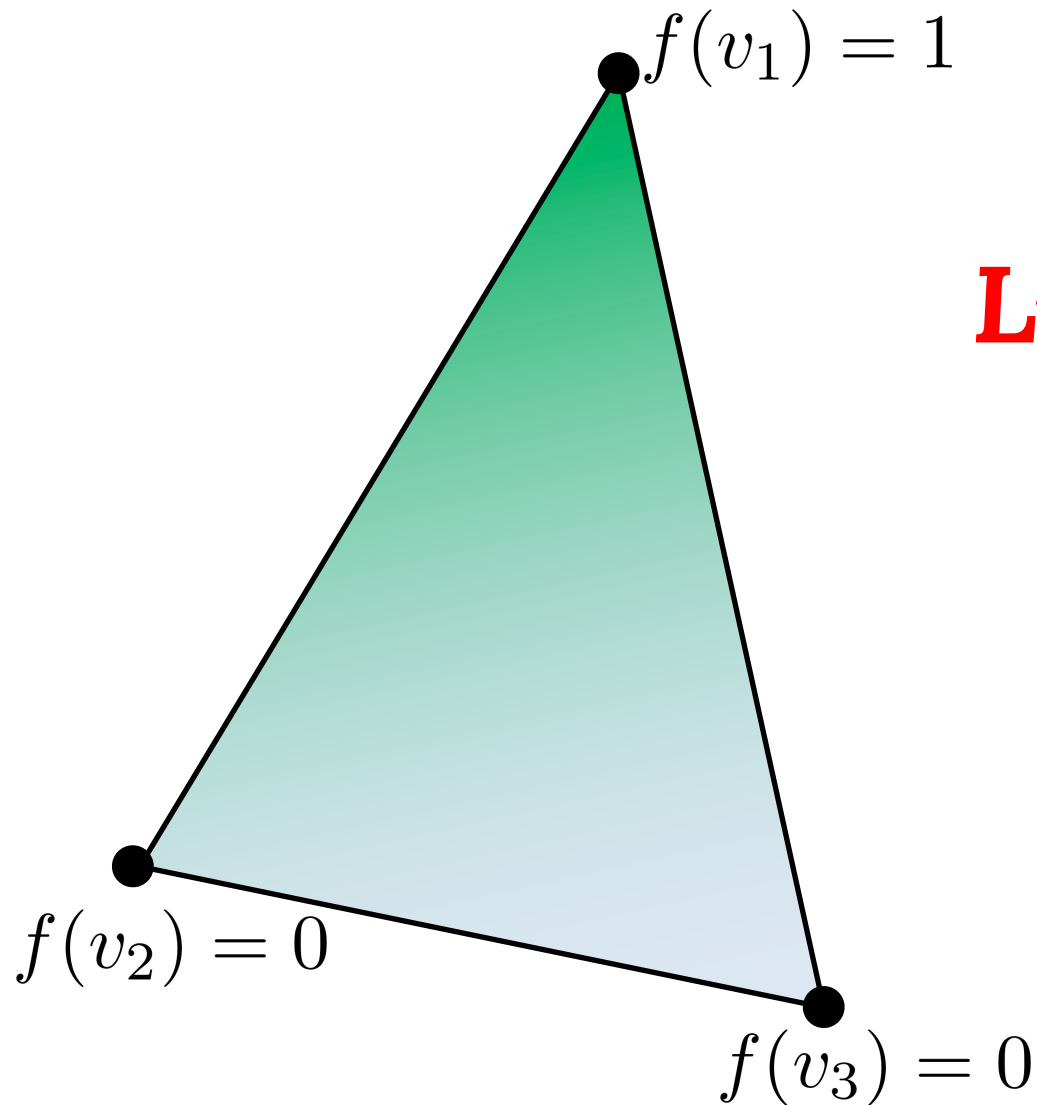
# What Do We Need



$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

**Sum scalars per face  
multiplied by face areas**

# Gradient of a Hat Function



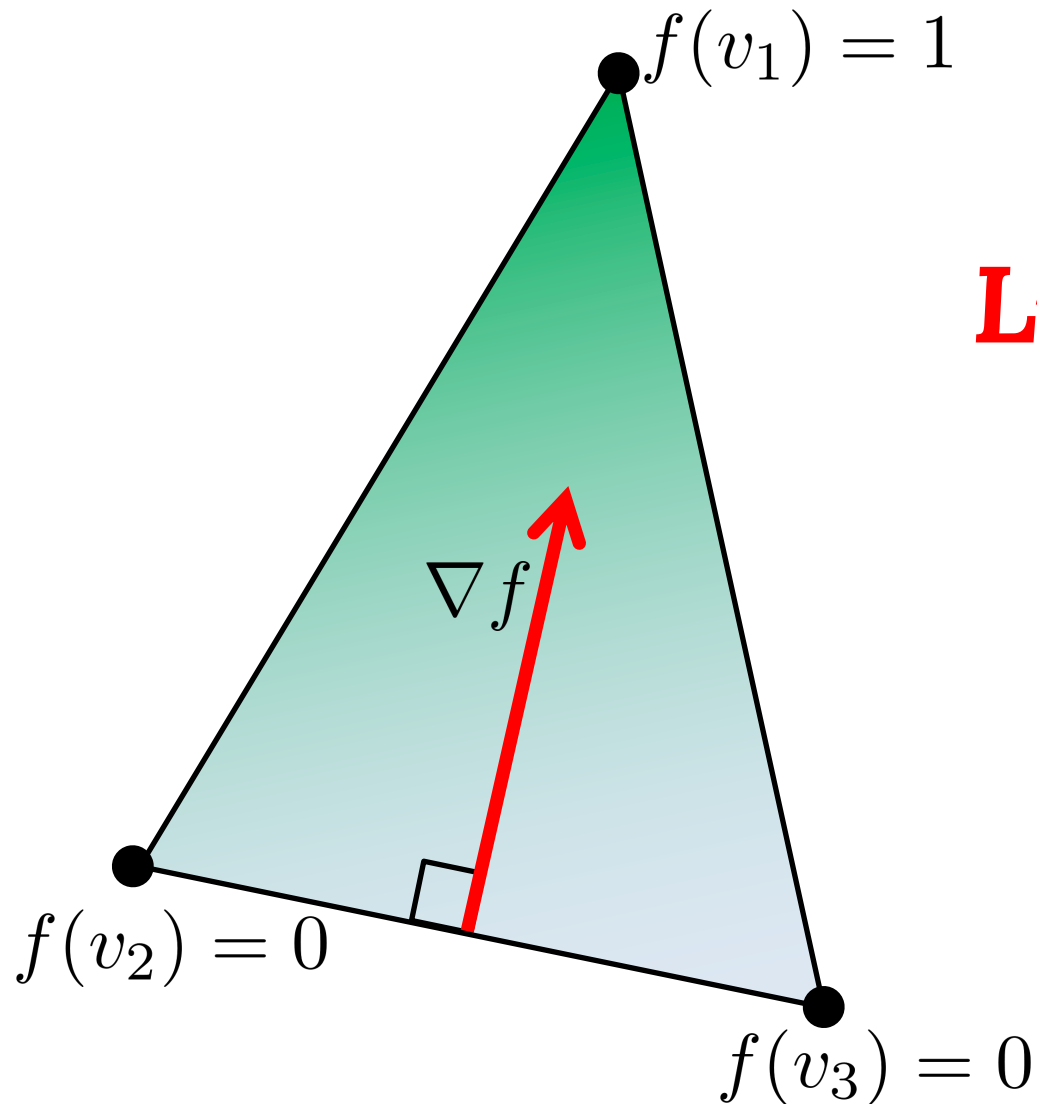
**Linear along edges**

$$\nabla f \cdot (v_1 - v_3) = 1$$

$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$

# Gradient of a Hat Function



**Linear along edges**

$$\nabla f \cdot (v_1 - v_3) = 1$$

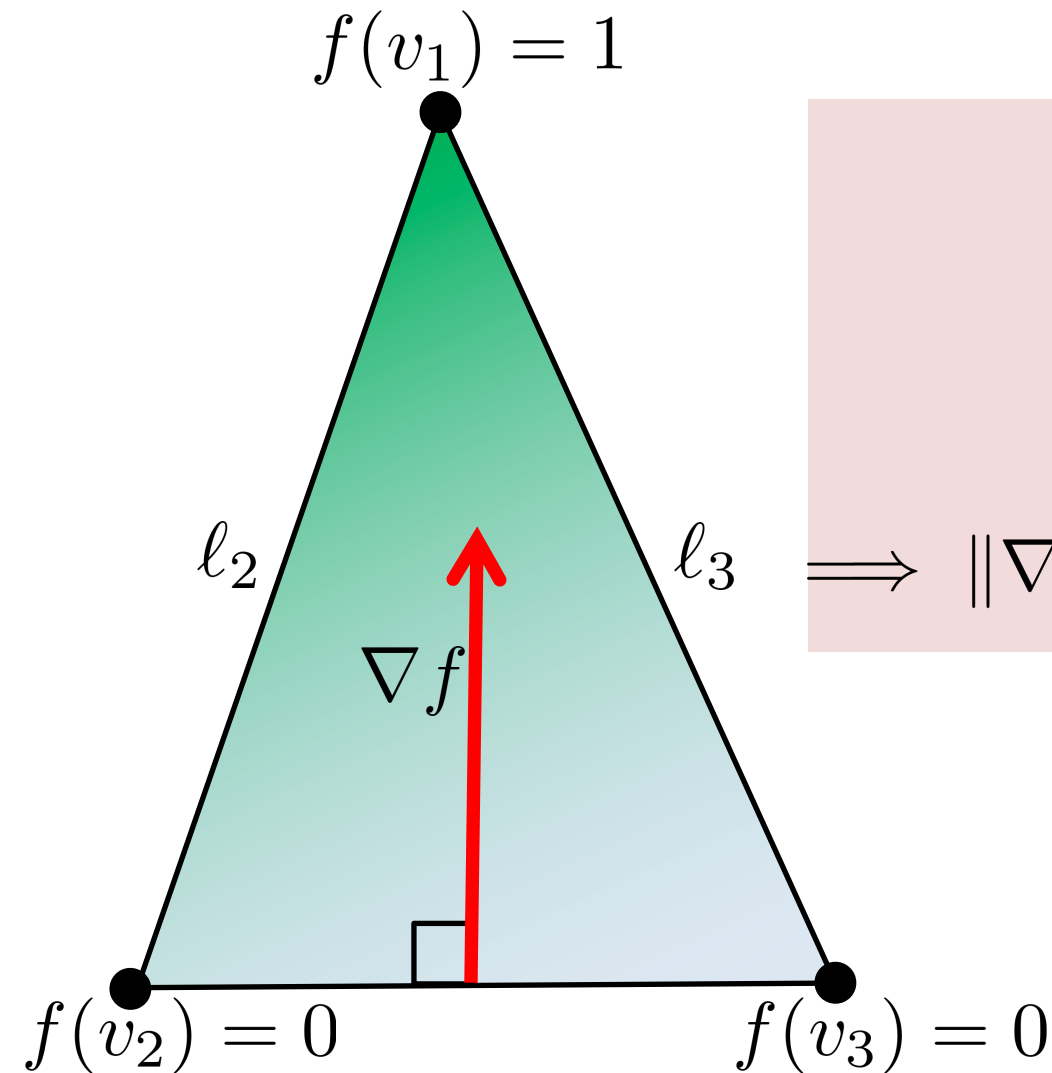
$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$



$$\nabla f \cdot (v_2 - v_3) = 0$$

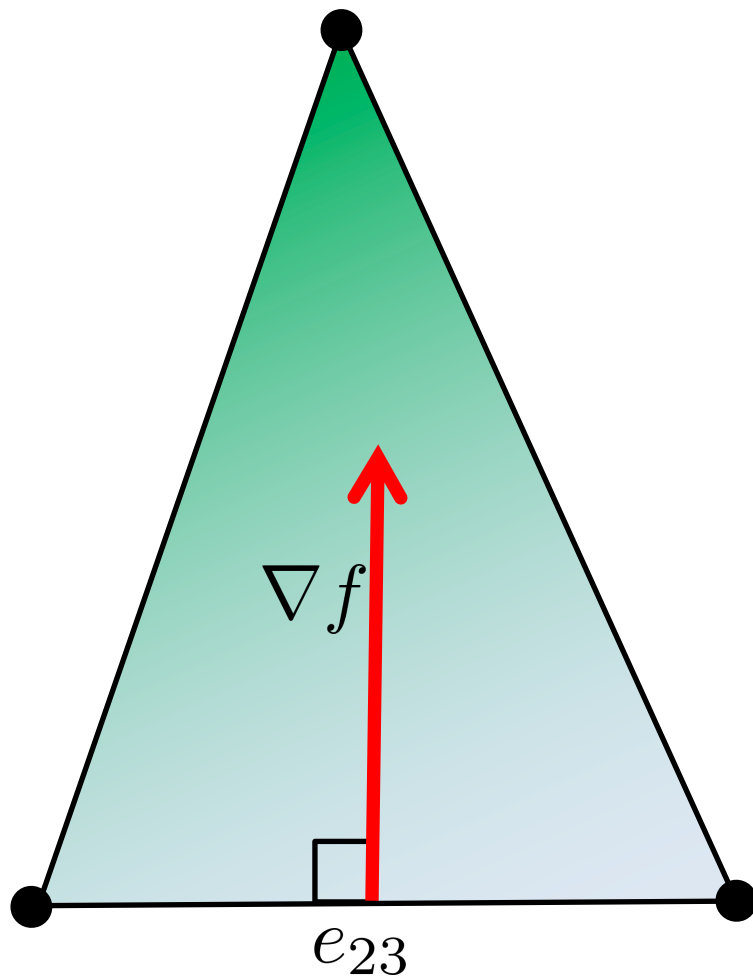
# Gradient of a Hat Function



$$\begin{aligned} 1 &= \nabla f \cdot (v_1 - v_3) \\ &= \|\nabla f\| l_3 \cos\left(\frac{\pi}{2} - \theta_3\right) \\ &= \|\nabla f\| l_3 \sin \theta_3 \\ \implies \|\nabla f\| &= \frac{1}{l_3 \sin \theta_3} = \frac{1}{h} \end{aligned}$$



# Gradient of a Hat Function



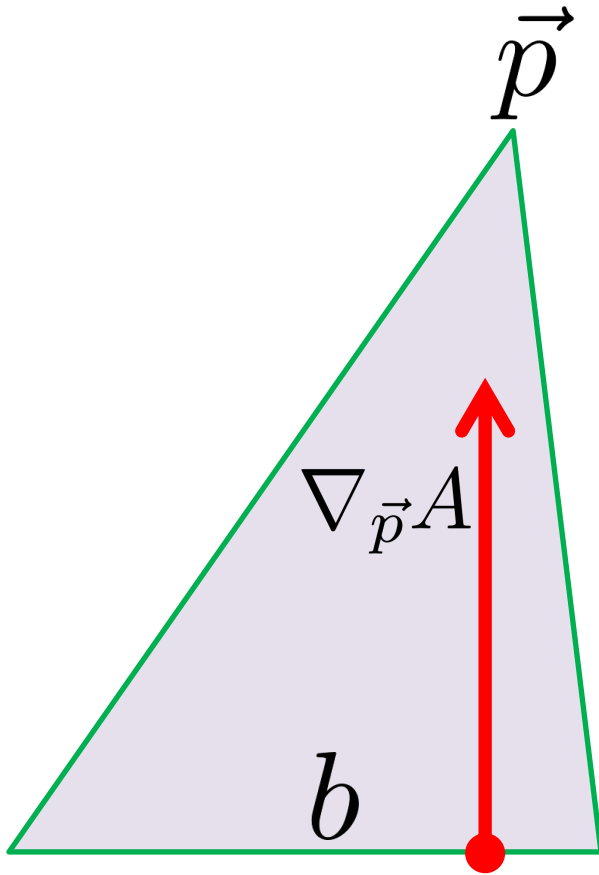
$$\|\nabla f\| = \frac{1}{l_3 \sin \theta_3} = \frac{1}{h}$$

$$\nabla f = \frac{e_{23}^\perp}{2A}$$

Length of  $e_{23}$  cancels  
"base" in  $A$

*Recall:*

# Single Triangle: Complete

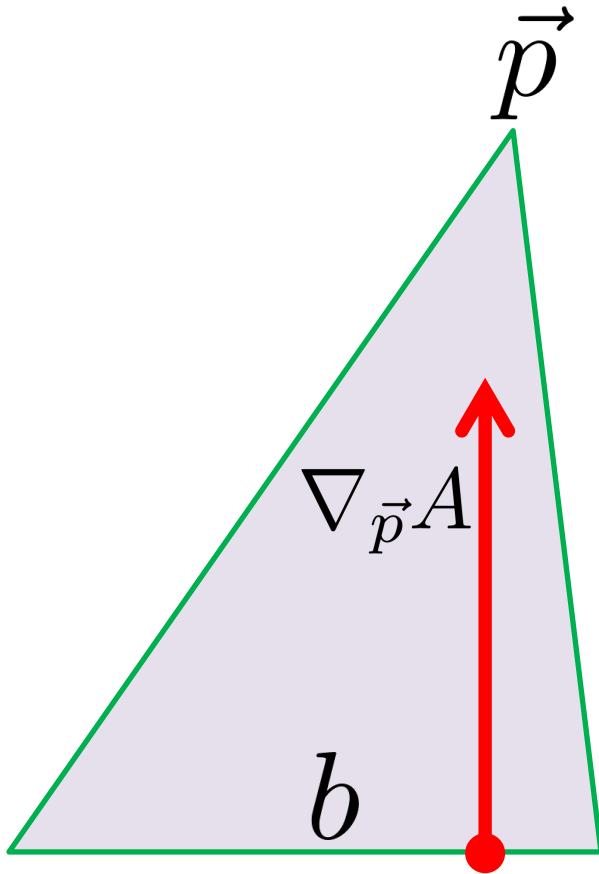


$$\vec{p} = p_n \vec{n} + p_e \vec{e} + p_{\perp} \vec{e}_{\perp}$$
$$A = \frac{1}{2} b \sqrt{p_n^2 + p_{\perp}^2}$$
$$\nabla_{\vec{p}} A = \frac{1}{2} b \vec{e}_{\perp}$$

***Similar expression***

*Recall:*

# Single Triangle: Complete



$$\vec{p} = p_n \vec{n} + p_e \vec{e} + p_{\perp} \vec{e}_{\perp}$$
$$A = \frac{1}{2} b \sqrt{p_n^2 + p_{\perp}^2}$$
$$\nabla_{\vec{p}} A = \frac{1}{2} b \vec{e}_{\perp}$$

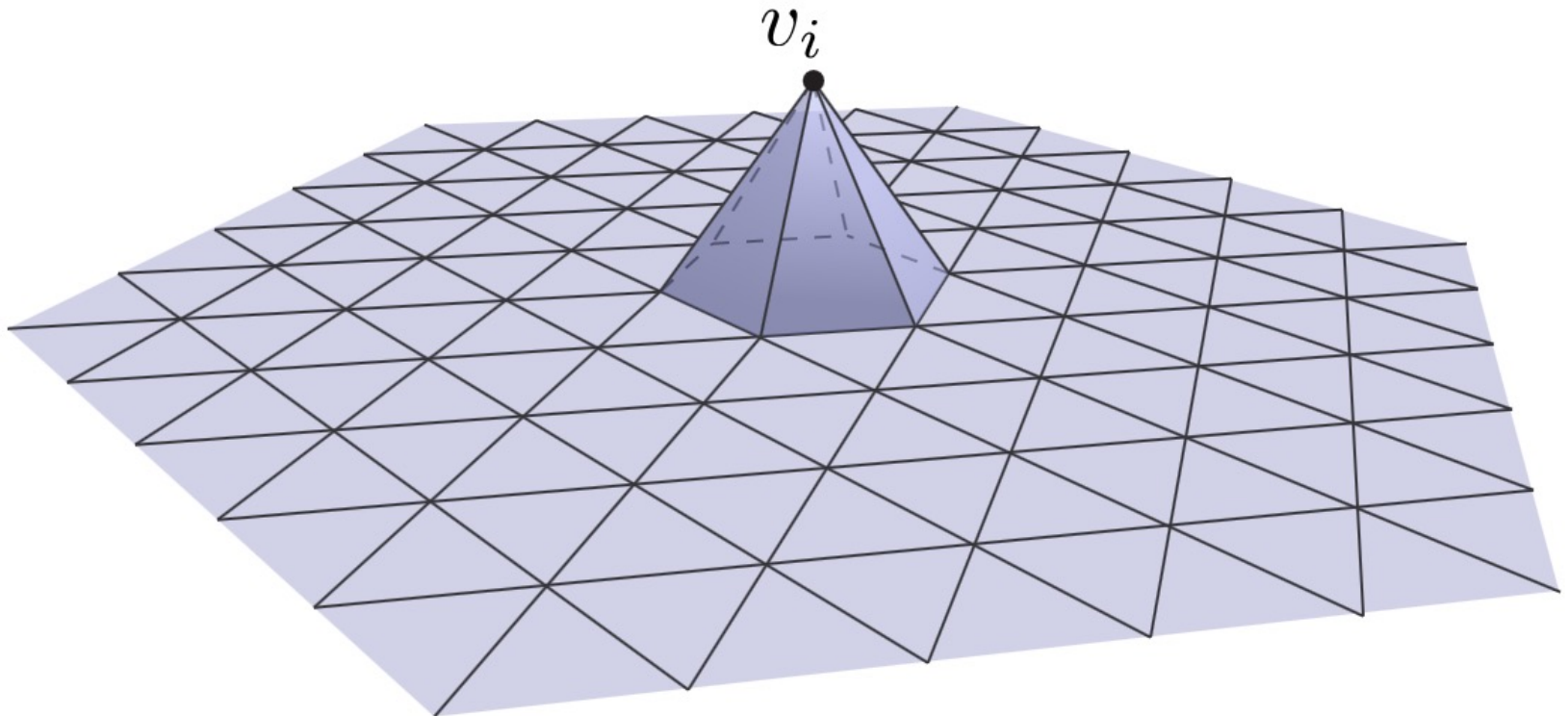
$$\nabla f = \frac{e_{23}^{\perp}}{2A} = \frac{\vec{e}_{\perp}}{h} = \frac{\nabla_{\vec{p}} A}{A}$$

**Similar expression**

# What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = - \int_M \boxed{\nabla g \cdot \nabla f} dA$$

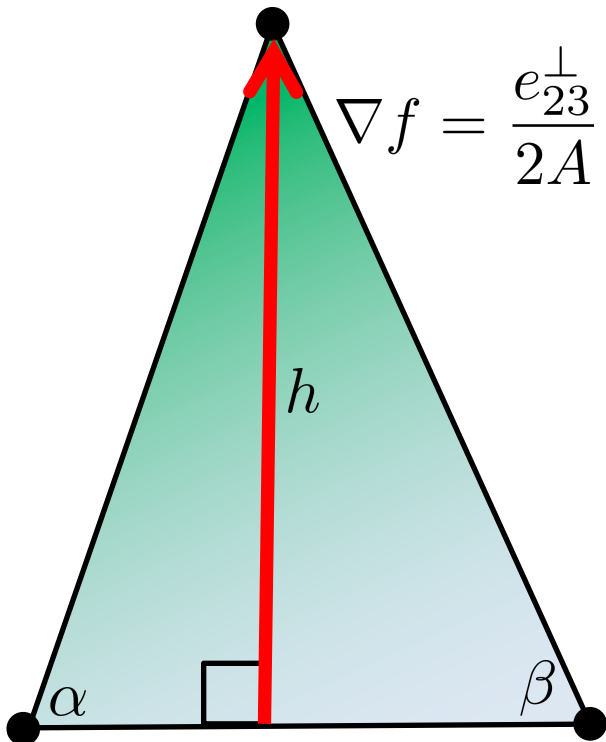
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# What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = - \int_M \boxed{\nabla g \cdot \nabla f} dA$$

---



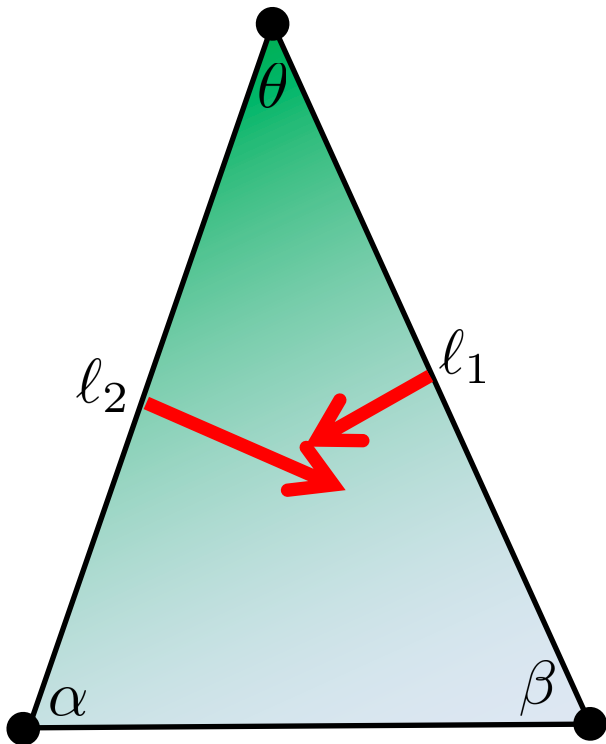
## Case 1: Same vertex

$$\begin{aligned} \int_T \langle \nabla f, \nabla f \rangle dA &= A \|\nabla f\|^2 \\ &= \frac{A}{h^2} = \frac{b}{2h} \\ &= \frac{1}{2} (\cot \alpha + \cot \beta) \end{aligned}$$

# What We Actually Need

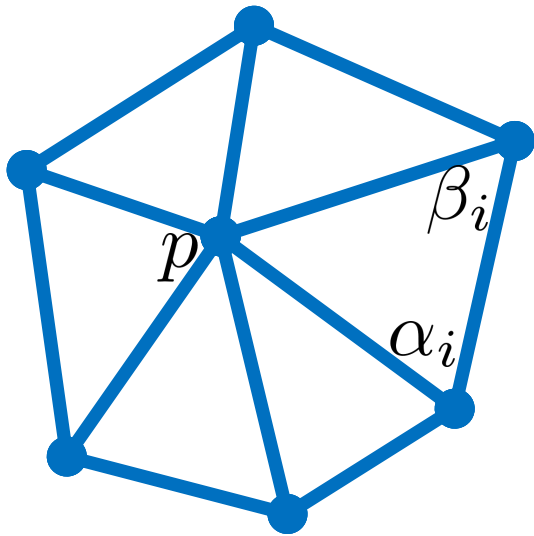
$$\mathcal{L}_{\Delta f}[g] = - \int_M \boxed{\nabla g \cdot \nabla f} dA$$

## Case 2: Different vertices

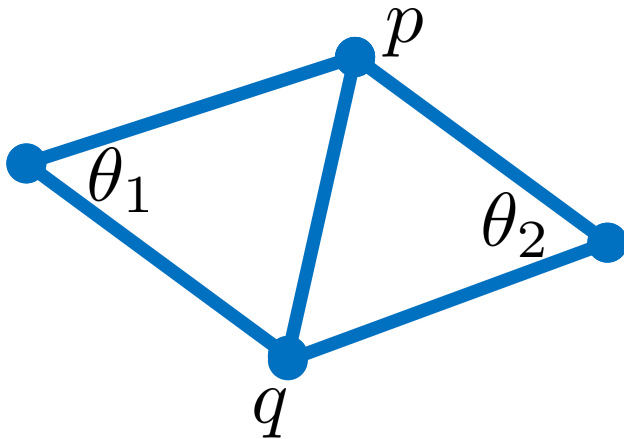


$$\begin{aligned} \int_T \langle \nabla f_\alpha, \nabla f_\beta \rangle dA &= A \langle \nabla f_\alpha, \nabla f_\beta \rangle \\ &= \frac{1}{4A} \langle e_{31}^\perp, e_{12}^\perp \rangle = -\frac{l_1 l_2 \cos \theta}{4A} \\ &= \frac{-h^2 \cos \theta}{4A \sin \alpha \sin \beta} = \frac{-h \cos \theta}{2b \sin \alpha \sin \beta} \\ &= -\frac{\cos \theta}{2 \sin(\alpha + \beta)} = -\frac{1}{2} \cot \theta \end{aligned}$$

# Summing Around a Vertex



$$\langle \nabla h_p, \nabla h_p \rangle = \frac{1}{2} \sum_i (\cot \alpha_i + \cot \beta_i)$$

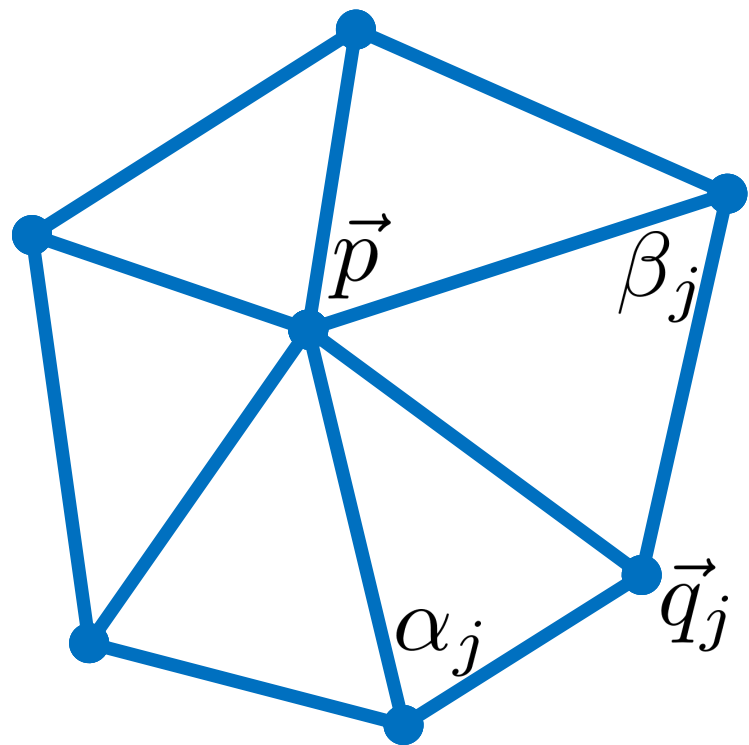


$$\langle \nabla h_p, \nabla h_q \rangle = -\frac{1}{2} (\cot \theta_1 + \cot \theta_2)$$

*Recall:*

# Summing Around a Vertex

$$\nabla_{\vec{p}} A = \frac{1}{2} \sum_j (\cot \alpha_j + \cot \beta_j) (\vec{p} - \vec{q}_j)$$



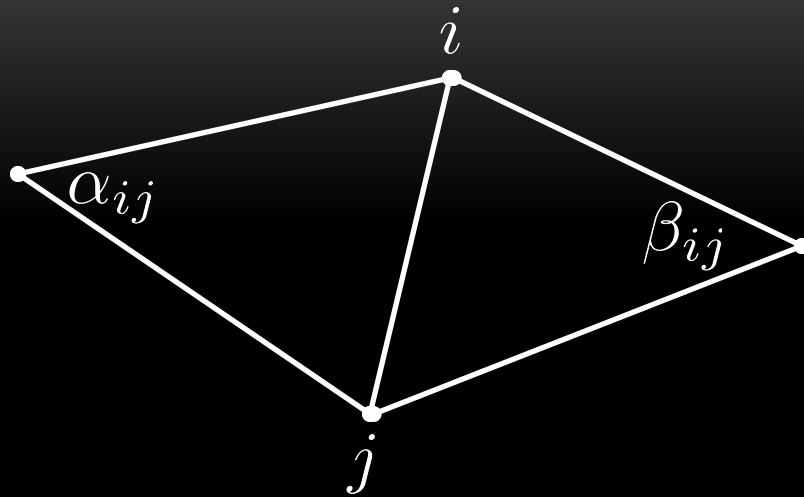
$$\nabla_{\vec{p}} A = \frac{1}{2} ((\vec{p} - \vec{r}) \cot \alpha + (\vec{p} - \vec{q}) \cot \beta)$$

Same weights up  
to sign!



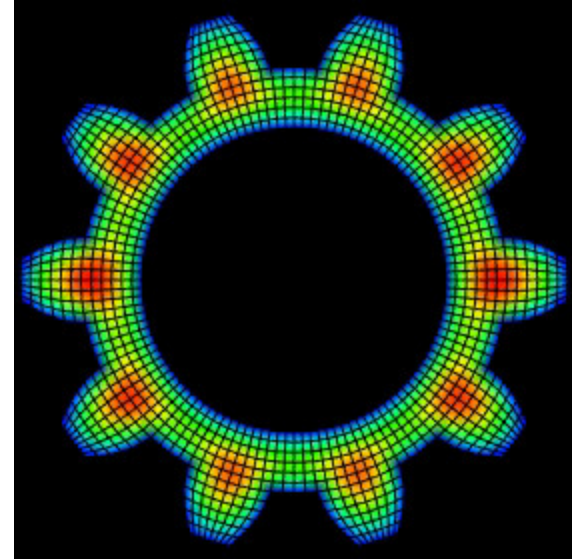
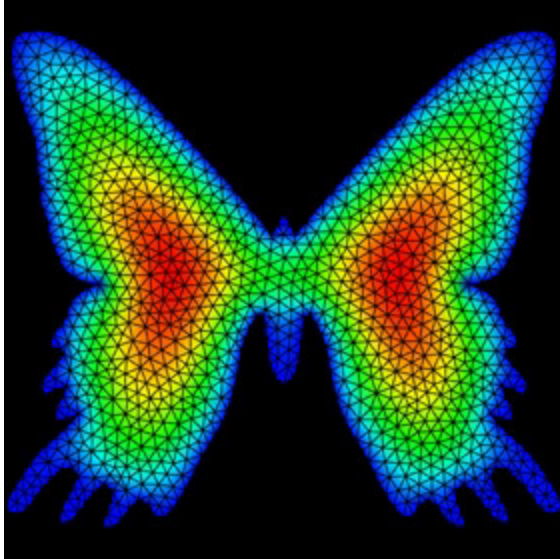
# THE COTANGENT LAPLACIAN

$$L_{ij} = \begin{cases} \frac{1}{2} \sum_{i \sim k} (\cot \alpha_{ik} + \cot \beta_{ik}) & \text{if } i = j \\ -\frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}) & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$



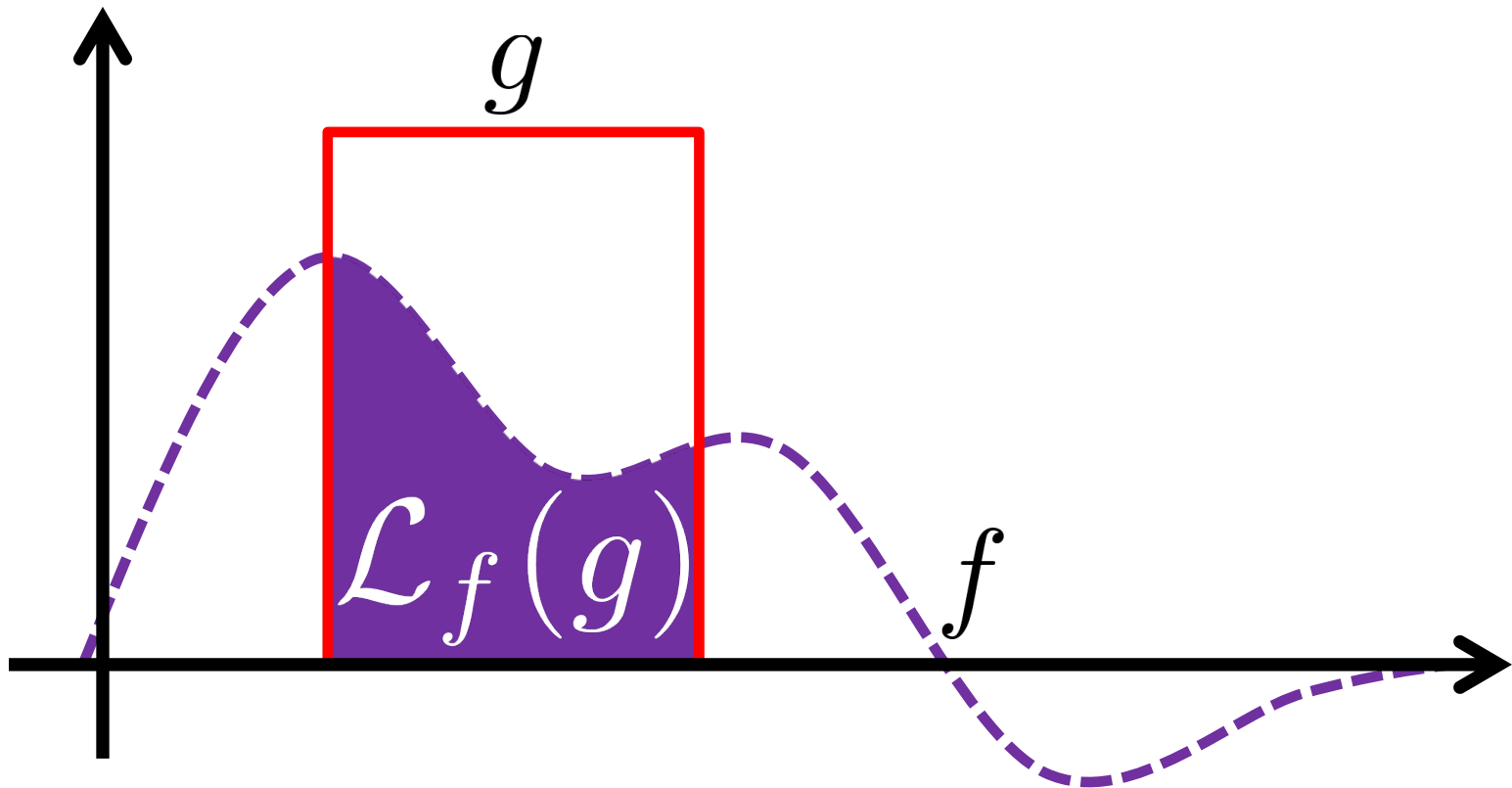
# Poisson Equation

$$\Delta f = g$$



# Weak Solutions

$$\int_M \phi \Delta f \, dA = \int_M \phi g \, dA \quad \forall \text{ test functions } \phi$$



# FEM Hat Weak Solutions

$$\int_M h_i \Delta f \, dA = \int_M h_i g \, dA \quad \forall \text{ hat functions } h_i$$

$$\begin{aligned} \int_M h_i \Delta f \, dA &= - \int_M \nabla h_i \cdot \nabla f \, dA \\ &= - \int_M \nabla h_i \cdot \nabla \sum_j a_j h_j \, dA \\ &= - \sum_j a_j \int_M \nabla h_i \cdot \nabla h_j \, dA \\ &= \sum_j L_{ij} a_j \end{aligned}$$

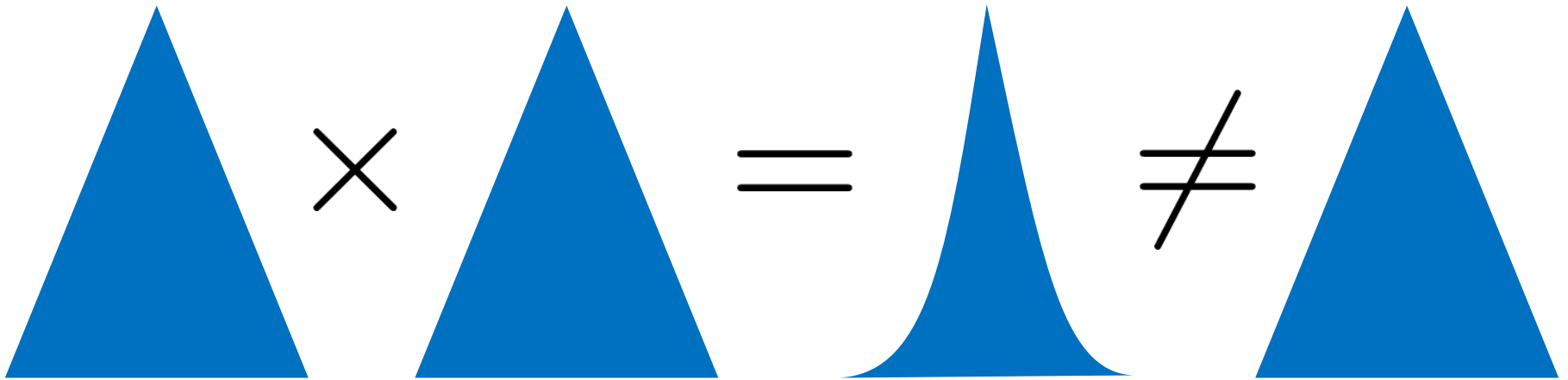
# Stacking Integrated Products

$$\begin{pmatrix} \int_M h_1 \Delta f dA \\ \int_M h_2 \Delta f dA \\ \vdots \\ \int_M h_{|V|} \Delta f dA \end{pmatrix} = \begin{pmatrix} \sum_j L_{1j} a_j \\ \sum_j L_{2j} a_j \\ \vdots \\ \sum_j L_{|V|j} a_j \end{pmatrix} = L \vec{a}$$

**Multiply by Laplacian matrix!**

# Problematic Right-Hand Side

$$\int_M h_i \Delta f \, dA = \int_M h_i g \, dA \quad \forall \text{ hat functions } h_i$$



***Product of hats is quadratic***

# A Few Ways Out

- ***Just do the integral***  
*“Consistent” approach*
- ***Approximate some more***
- ***Redefine  $g$***

# A Few Ways Out

- ***Just do the integral***

*“Consistent” approach*

- ***Approximate some more***

- ***Redefine  $g$***

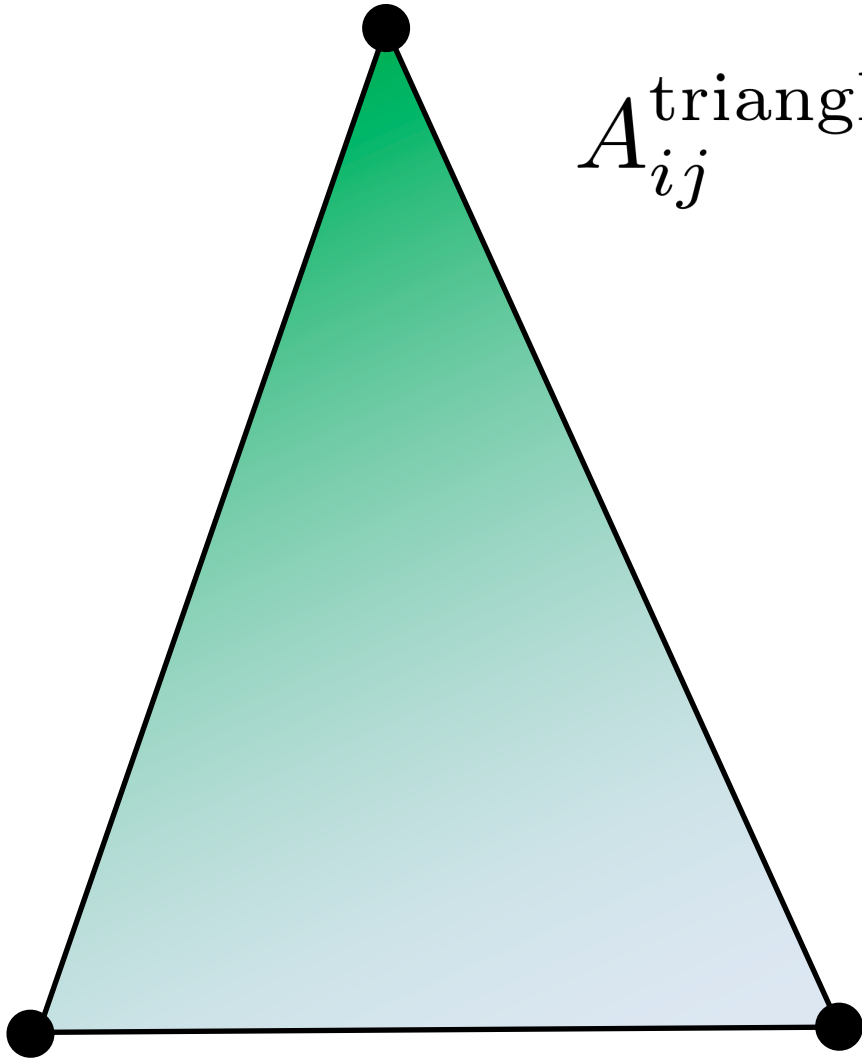


# The Mass Matrix

$$A_{ij} := \int_M h_i h_j dA$$

- **Diagonal elements:**  
Norm of  $h_i$
- **Off-diagonal elements:**  
Overlap between  $h_i$  and  $h_j$

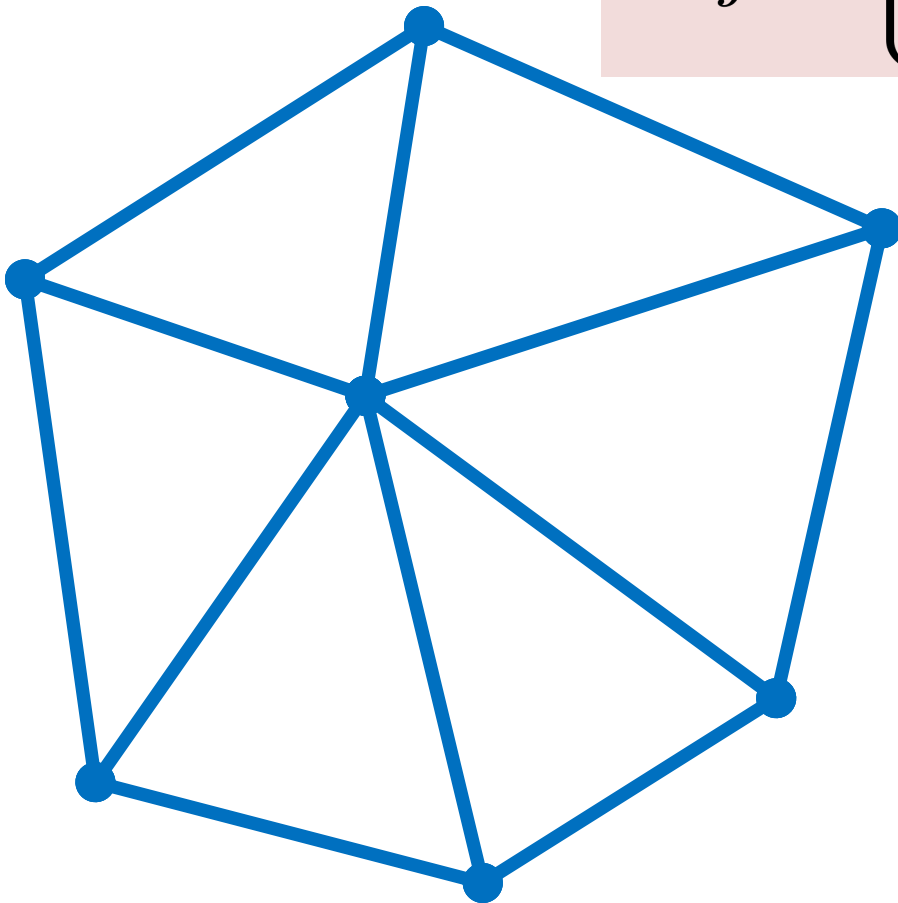
# Consistent Mass Matrix



$$A_{ij}^{\text{triangle}} = \begin{cases} \frac{\text{area}}{6} & \text{if } i = j \\ \frac{\text{area}}{12} & \text{if } i \neq j \end{cases}$$

# Non-Diagonal Mass Matrix

$$M_{ij} = \begin{cases} \frac{\text{one-ring area}}{6} & \text{if } i = j \\ \frac{\text{adjacent area}}{12} & \text{if } i \neq j \end{cases}$$



# Properties of Mass Matrix

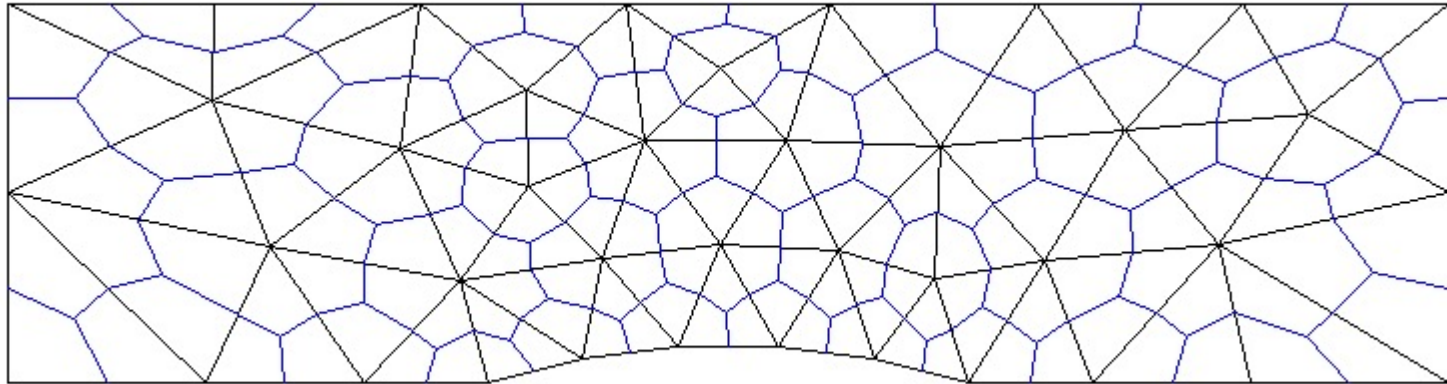
- Rows sum to one ring area / 3
- Involves only vertex and its neighbors
- Partitions surface area

***Issue: Not trivial to invert!***

# Use for Integration

$$\begin{aligned}\int_M f &= \int_M \sum_j a_j h_j(\cdot) \mathbf{1} \\ &= \int_M \sum_j a_j h_j \sum_i h_i \\ &= \sum_{ij} A_{ij} a_j \\ &= \mathbf{1}^\top A \vec{a}\end{aligned}$$

# Lumped Mass Matrix



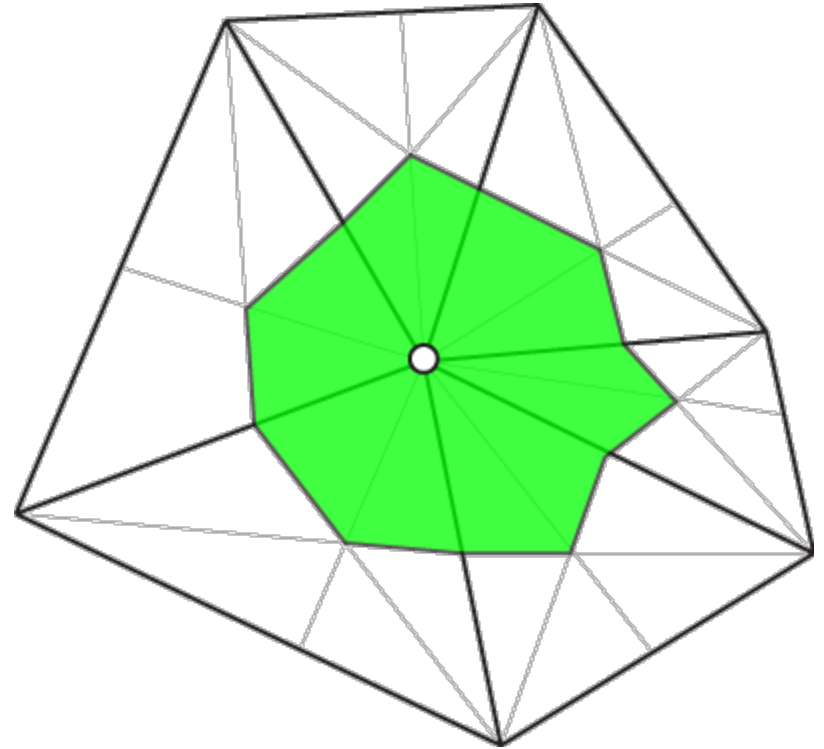
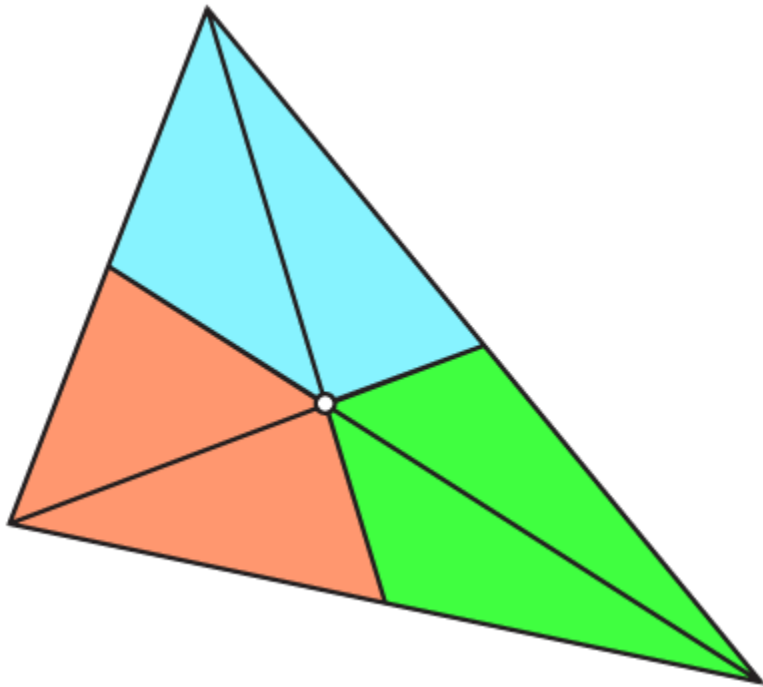
$$\tilde{a}_{ii} := \text{Area}(\text{cell } i)$$

Won't make big difference for smooth functions

<http://users.led-inc.eu/~phk/mesh-dualmesh.html>

Approximate with diagonal matrix

# Simplest: Barycentric Lumped Mass



<http://www.alecjacobson.com/weblog/?p=1146>

***Area/3 to each vertex***

# Ingredients

- ***Cotangent Laplacian  $L$***

*Per-vertex function to integral of its Laplacian against each hat*

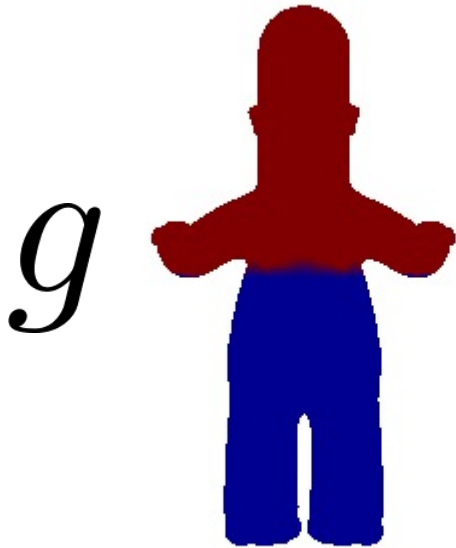
- ***Area weights  $A$***

*Integrals of pairwise products of hats (or approximation thereof)*



# Solving the Poisson Equation

$$\Delta f = g \rightarrow L \vec{f} = A \vec{g}$$



**Must integrate  
to zero**



**Determined up  
to constant**

# Helmholtz Equation

$$\Delta f = \lambda f$$

$$\int \Delta f \cdot \psi = -\int \nabla f \cdot \nabla \psi = \int \lambda f \cdot \psi$$

$$\Rightarrow La = \lambda Ma$$

Generalized  
Eigenvalue Problem

# Important Detail: Boundary Conditions

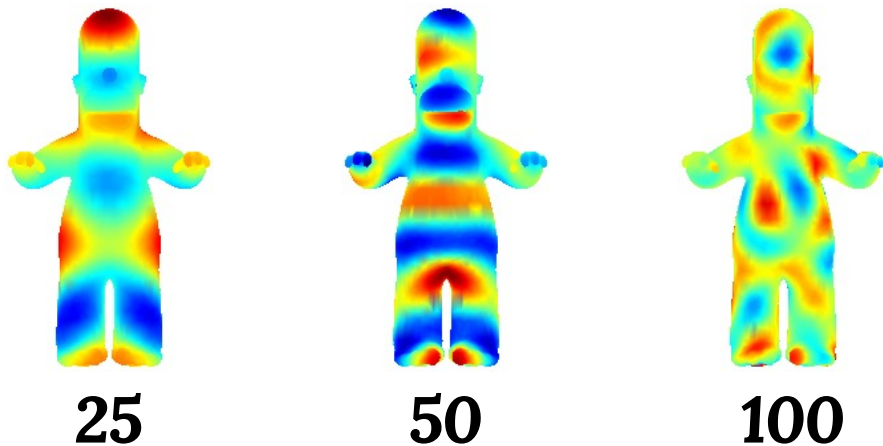
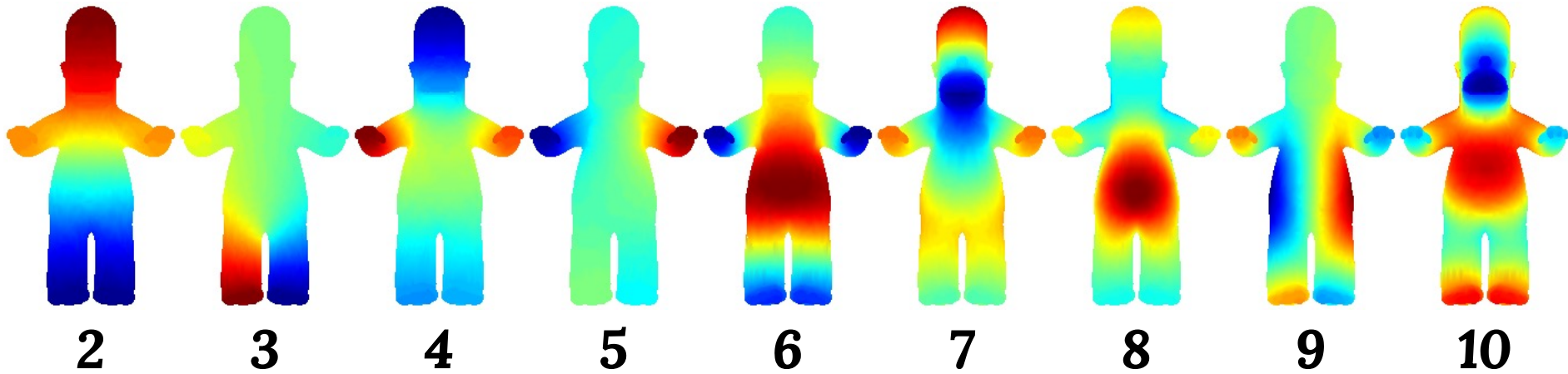
$$\begin{aligned}\Delta f(x) &= g(x) \quad \forall x \in \Omega \\ f(x) &= u(x) \quad \forall x \in \Gamma_D \\ \nabla f \cdot n &= v(x) \quad \forall x \in \Gamma_N\end{aligned}$$

Strong  
form

$$\begin{aligned}\int_{\Omega} \nabla f \cdot \nabla \phi &= \int_{\Gamma_N} v(x) \phi(x) d\Gamma - \int_{\Omega} f(x) \phi(x) d\Omega \\ f(x) &= u(x) \quad \forall x \in \Gamma_D\end{aligned}$$

Weak form

# Eigenhomers



What is  
smallest  
eigenvalue?

**On board:  
FEM approach?**

# Higher-Order Elements

https://www.femtable.org

finite element method

## Periodic Table of the Finite Elements

**The  $\mathcal{P}_r^- \Lambda^k$  family**

	k=0	k=1	k=2	k=3
n=1				
n=2				
n=3				

**The  $\mathcal{P}_r \Lambda^k$  family**

	k=0	k=1	k=2	k=3
n=1				
n=2				
n=3				

**The  $\mathcal{Q}_r^- \Lambda^k$  family**

	k=0	k=1	k=2	k=3
n=1				
n=2				
n=3				

**The  $\mathcal{S}_r \Lambda^k$  family**

	k=0	k=1	k=2	k=3
n=1				
n=2				
n=3				

<https://www.femtable.org/>

# Point Cloud Laplace: Easiest Option

$$W_{ij} = \exp\left(-\frac{\|x_i - x_j\|^2}{t}\right)$$

$$D_{ii} = \sum_j W_{ji}$$

$$L = D - W$$

$$Lf = \lambda Df$$

