

IFT 6113

CURVES



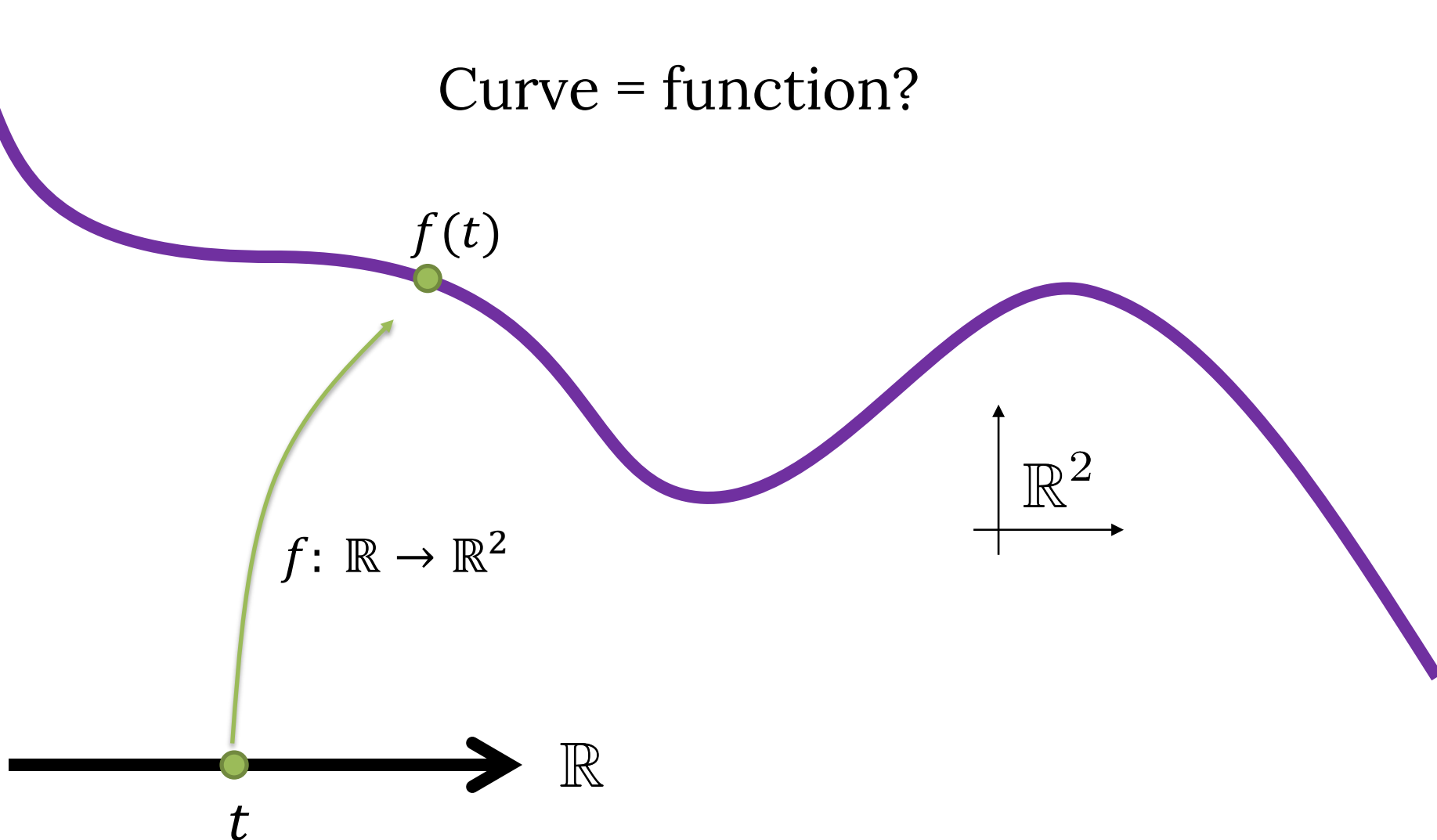
<http://tiny.cc/ift6113>

Mikhail Bessmeltsev

Some slides from Justin Solomon and Alla Sheffer

DEFINING “CURVE”

Curve = function?

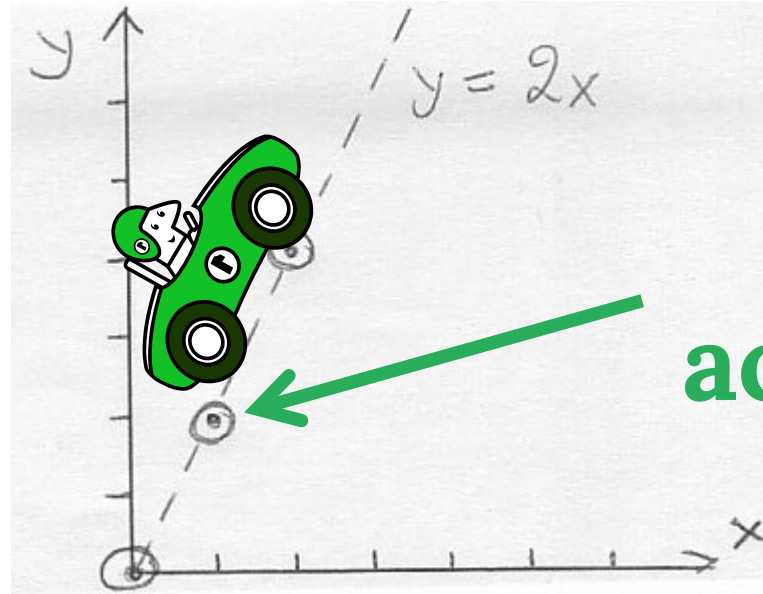


SUBTLETY

$$\gamma(t) = (0,0)$$

Not all functions are curves!

DIFFERENT FROM CALCULUS



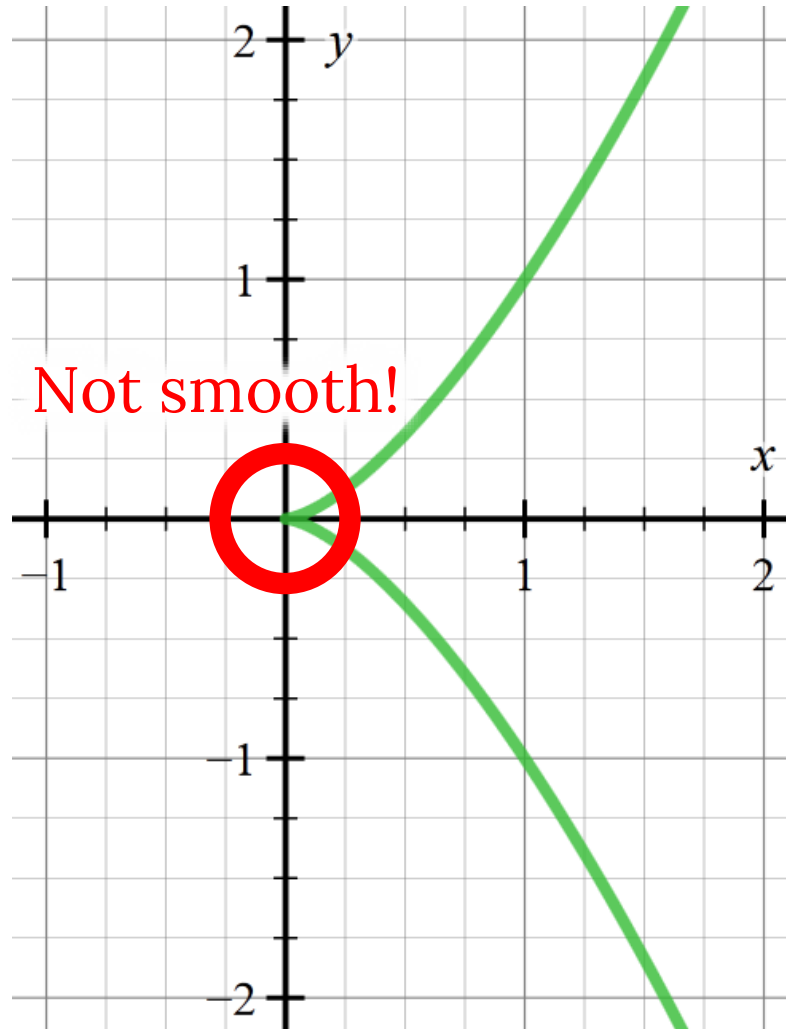
**Sudden
acceleration**

Same curves?

$$f_1(t) = (t, 2t)$$

$$f_2(t) = \begin{cases} (t, 2t) & t \leq 1 \\ (2(t - \frac{1}{2}), 4(t - \frac{1}{2})) & t > 1 \end{cases}$$

GRAPHS OF SMOOTH FUNCTIONS



$$f(t) = (t^2, t^3)$$

Smooth functions

GEOMETRY OF A CURVE

A curve is a
set of points
with certain properties.

It is not a function.

GEOMETRIC DEFINITION



Set of points that locally **looks like a line.**

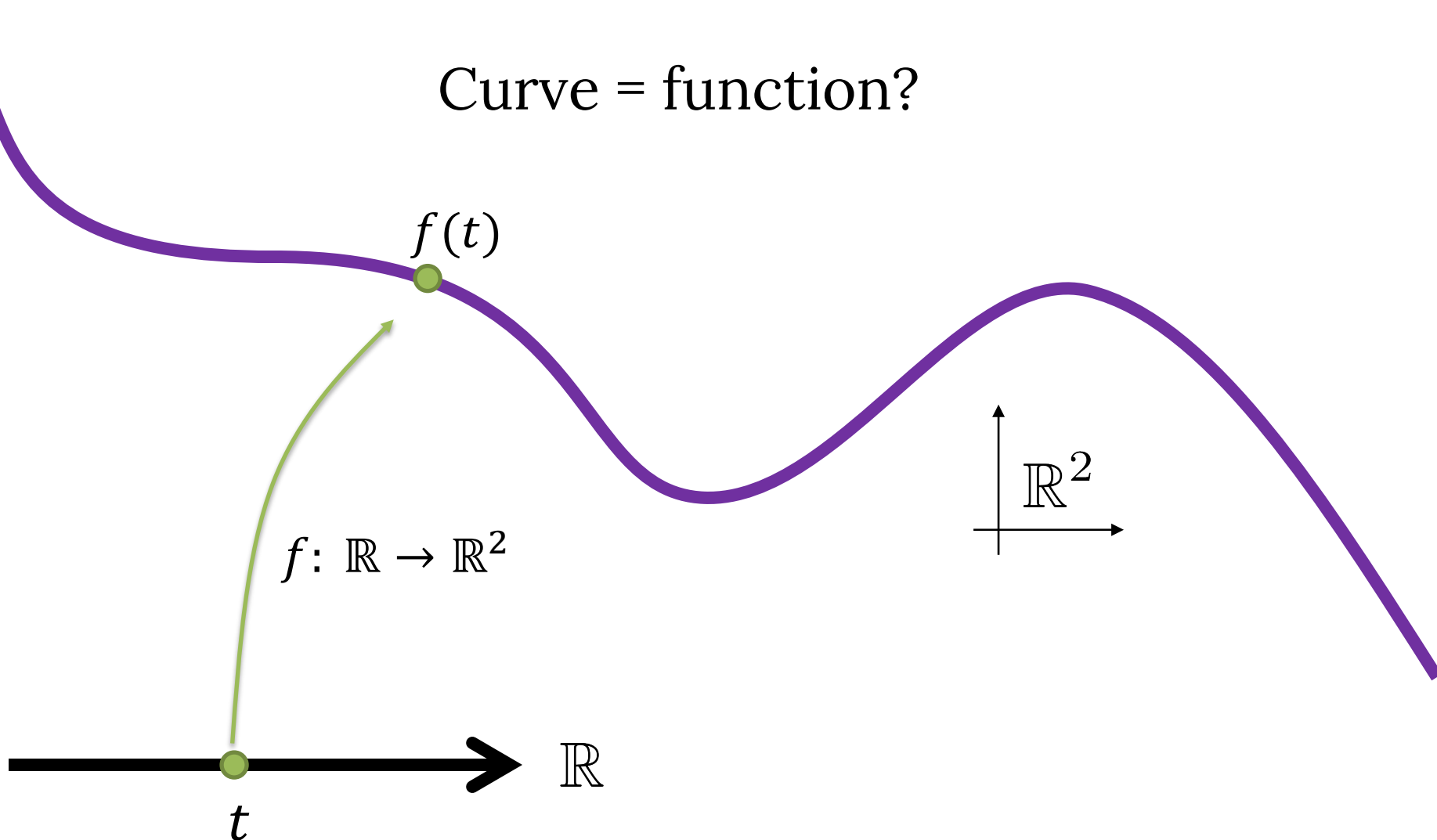
GEOMETRIC DEFINITION



Too restrictive?

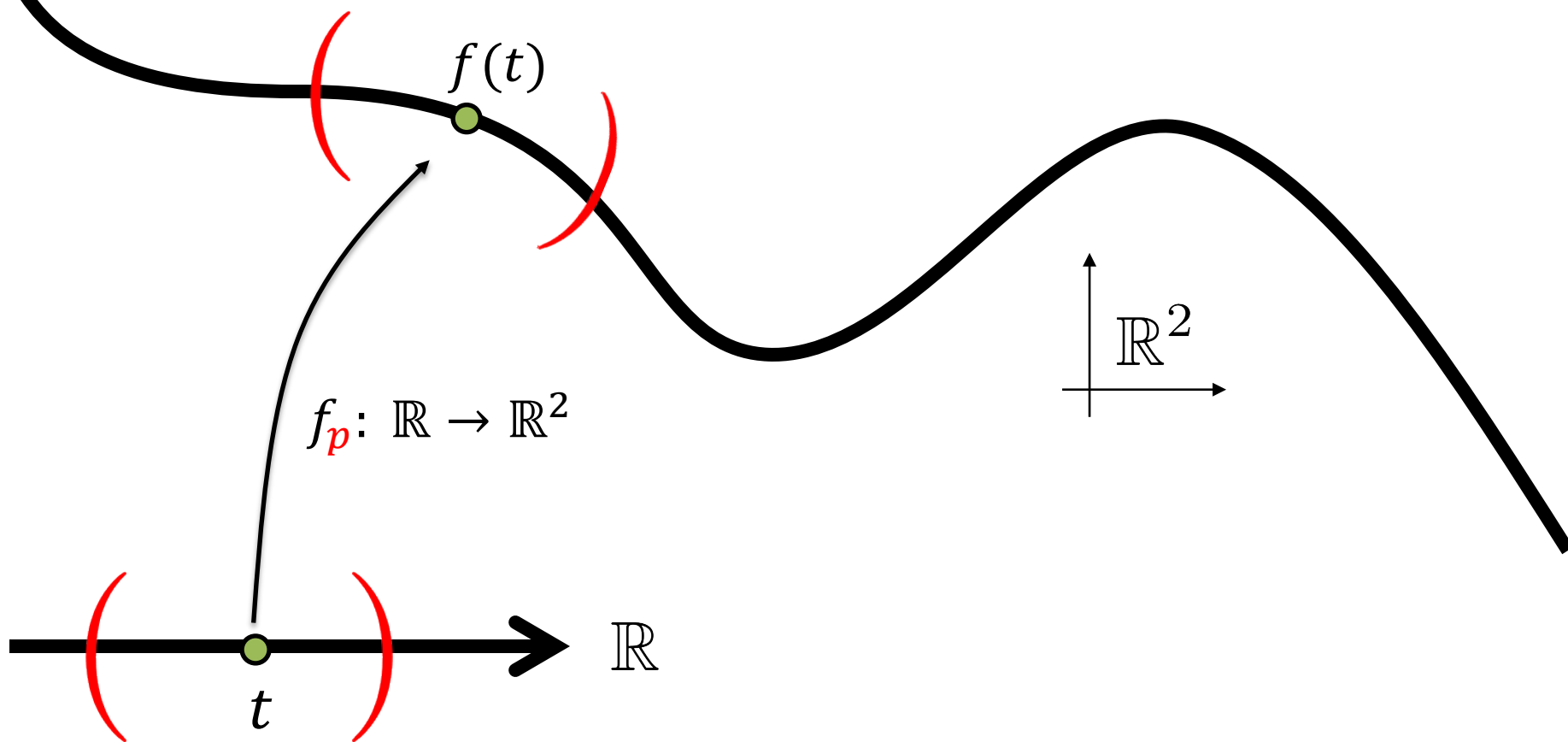
DEFINING “CURVE”

Curve = function?

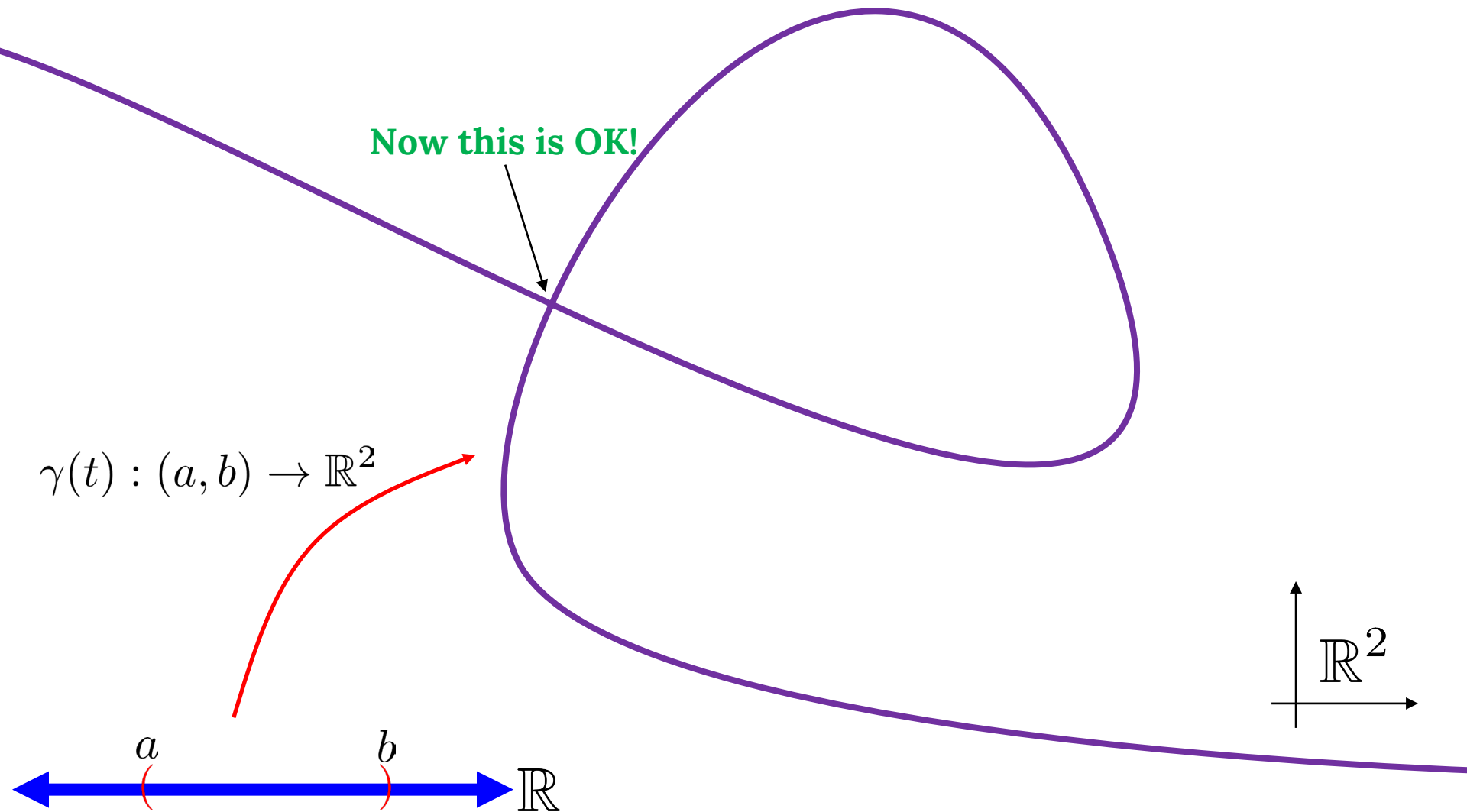


DEFINING “CURVE”

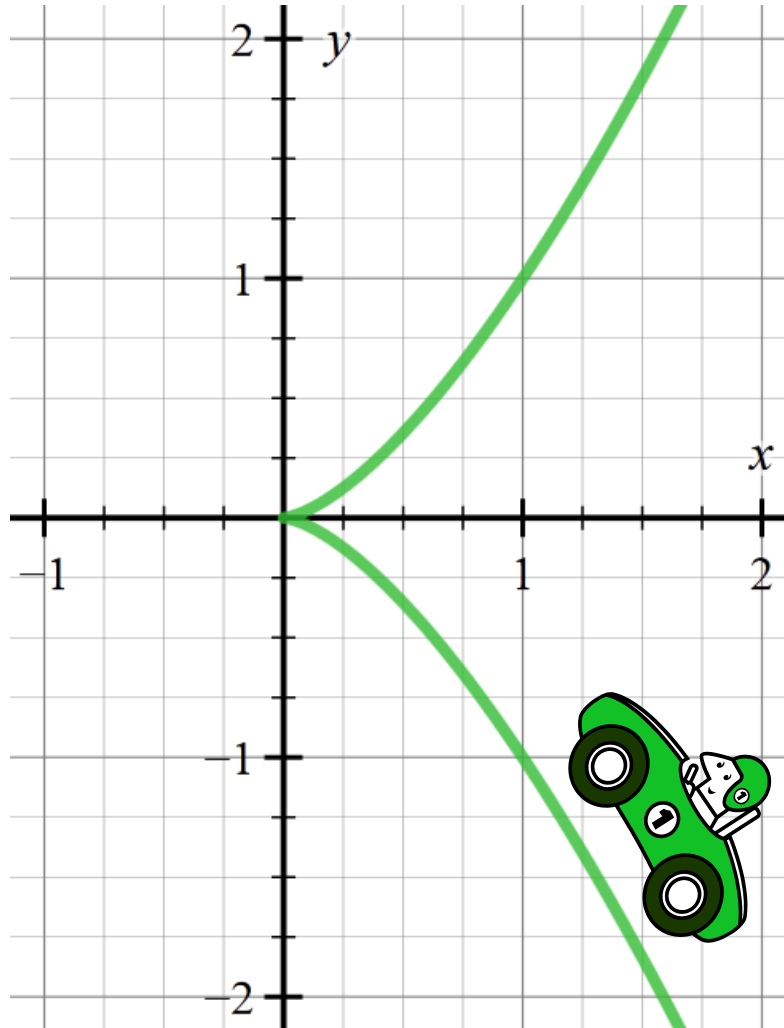
Curve = function?



PARAMETERIZED CURVE



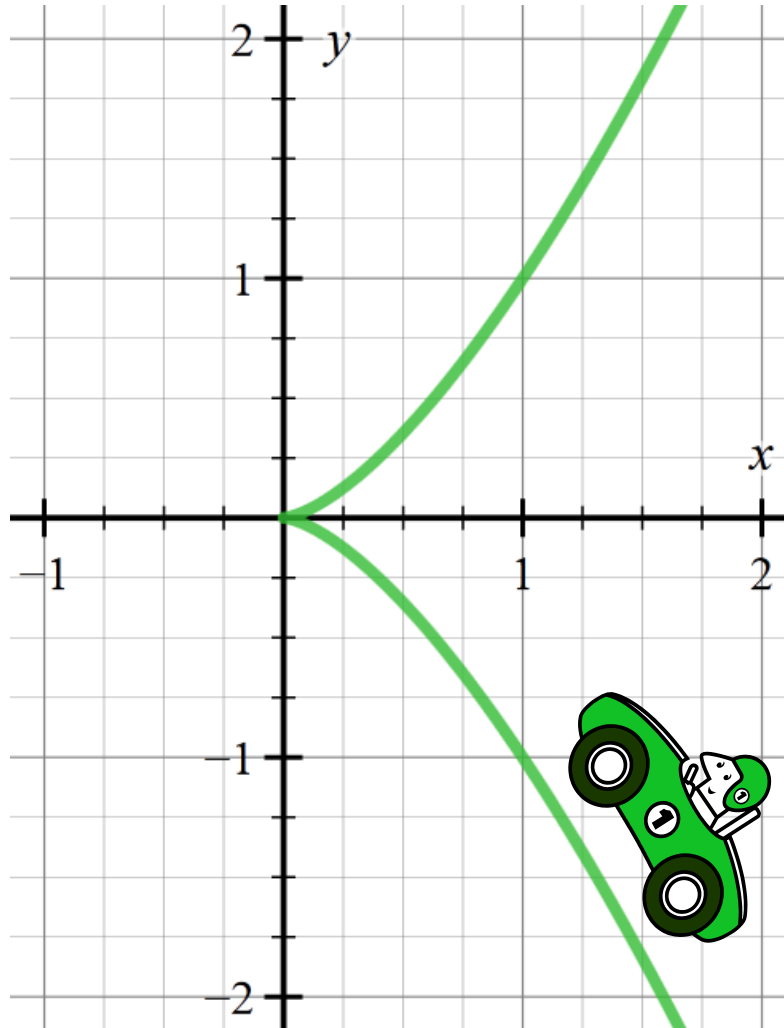
WHAT WAS THE PROBLEM HERE?



$$f(t) = (t^2, t^3)$$

Smooth functions

FIXING



$$f'(t) \neq 0$$

Non-zero velocity!

DEPENDENCE OF VELOCITY

$$\tilde{\gamma}(s) := \gamma(\phi(s))$$

On the board:

Effect on velocity and acceleration.

CHANGE OF PARAMETER

Geometric measurements should be
invariant
to changes of parameter.



ARC LENGTH

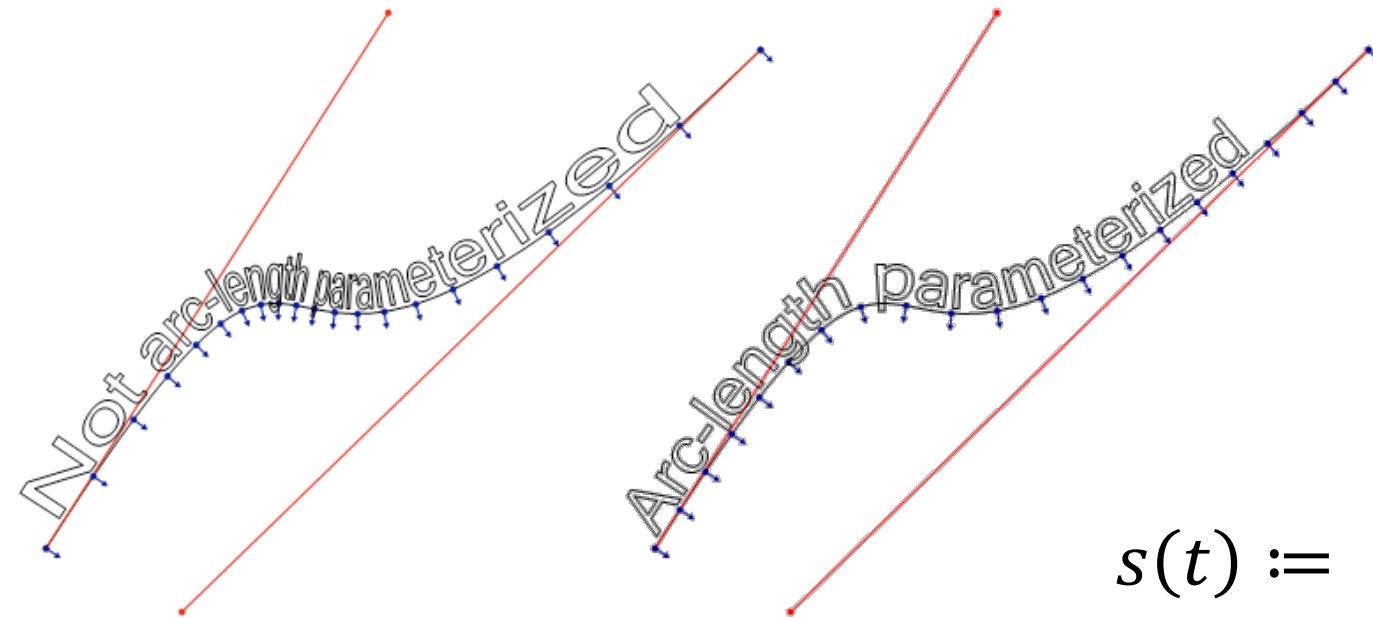
$$\int_a^b \|\gamma'(t)\| dt$$

On the board:

Independence of parameter

PARAMETERIZATION BY ARC LENGTH

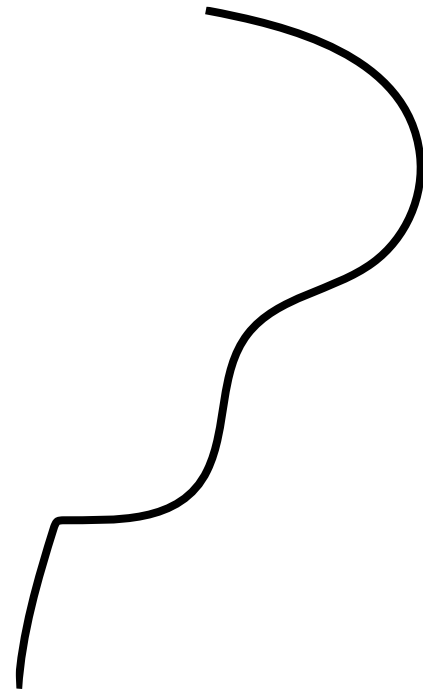
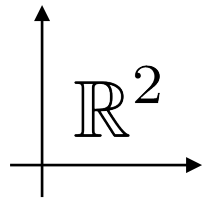
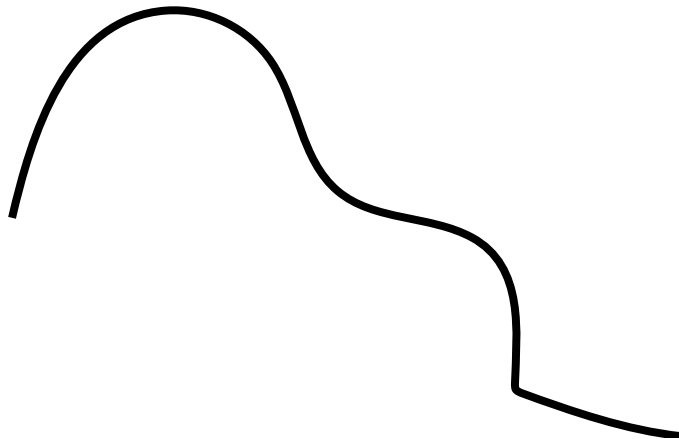
<http://www.planetclegg.com/projects/WarpingTextToSplines.html>



$$s(t) := \int_{t_0}^t \|\gamma'(t)\| dt$$
$$t(s) := \text{inverse of } s(t)$$
$$\gamma(s) := \gamma(t(s))$$

Constant-speed parameterization

SAME CURVE?



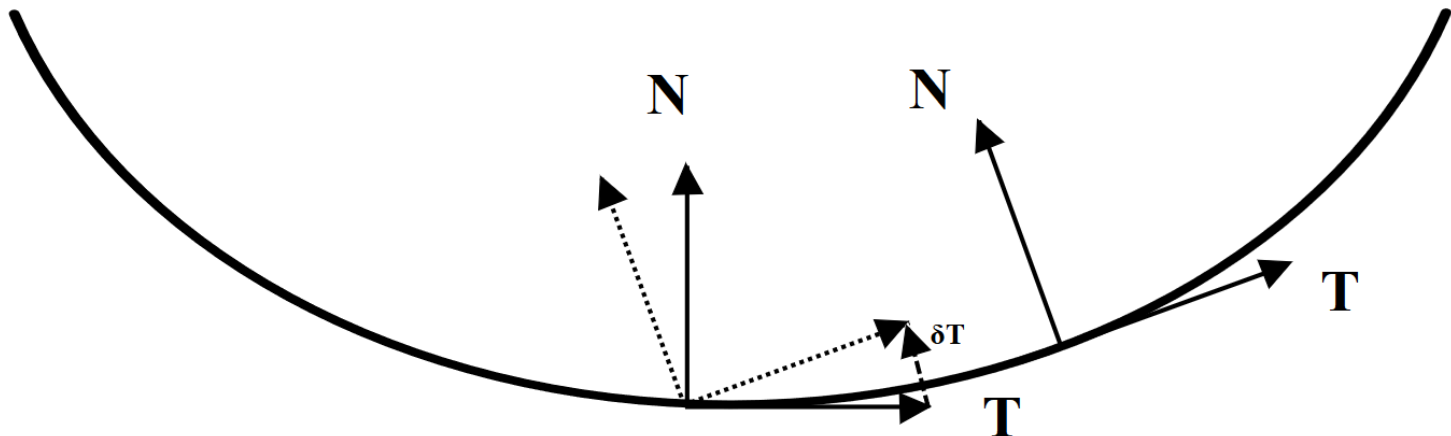
MOVING FRAME IN 2D

Tangent $T(s) := \gamma'(s)$

$$\|\gamma'(s)\| = 1$$

Normal

$$N(s) := \text{Rot}\left(\frac{\pi}{2}\right) T(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T(s)$$



PHILOSOPHICAL POINT

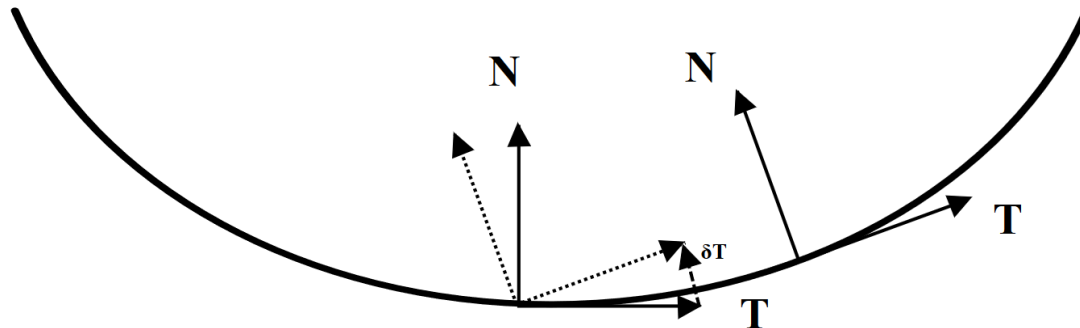
Differential geometry “should” be
coordinate-invariant.

Referring to x and y is a hack!
(but sometimes convenient...)

FRENET-SERRET FORMULAS

On the board:

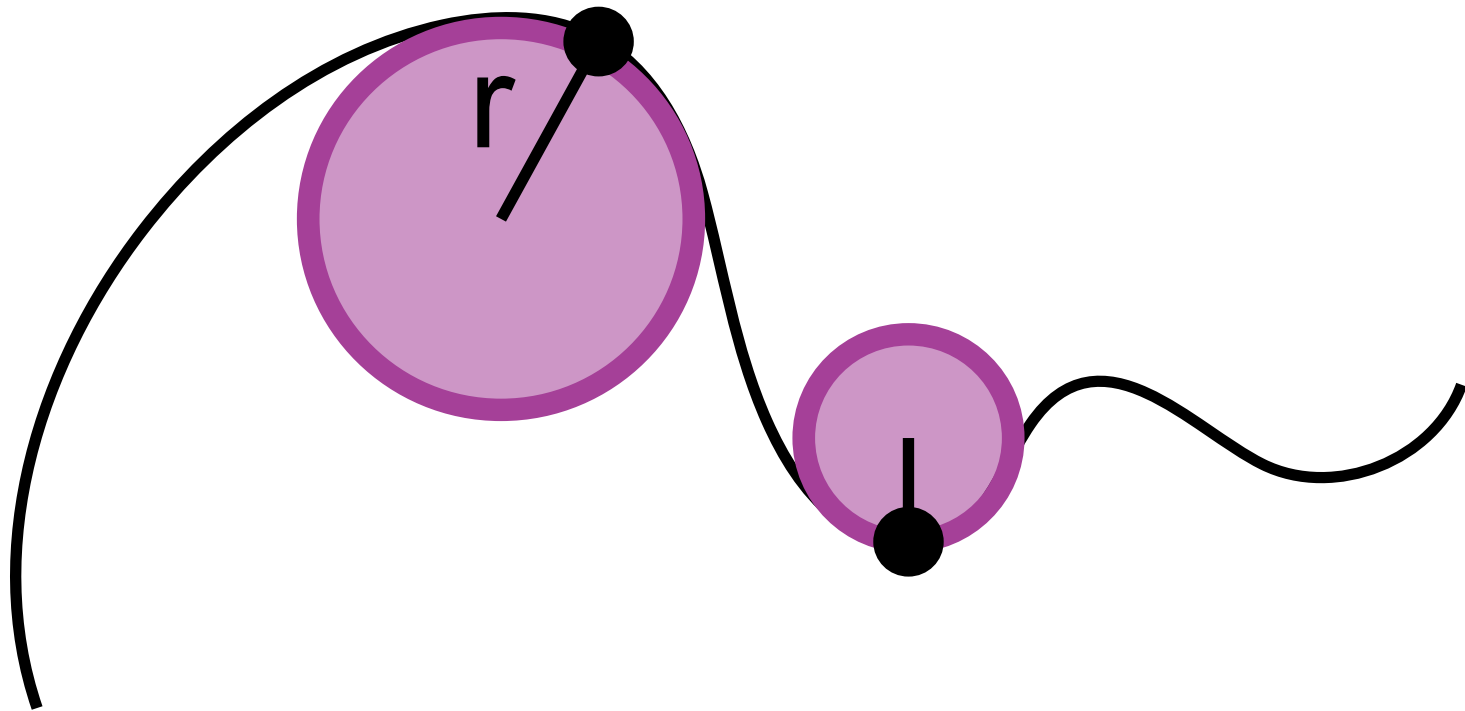
$$\frac{d}{ds} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix} := \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix}$$



https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

Use coordinates *from* the curve
to express its shape!

RADIUS OF CURVATURE

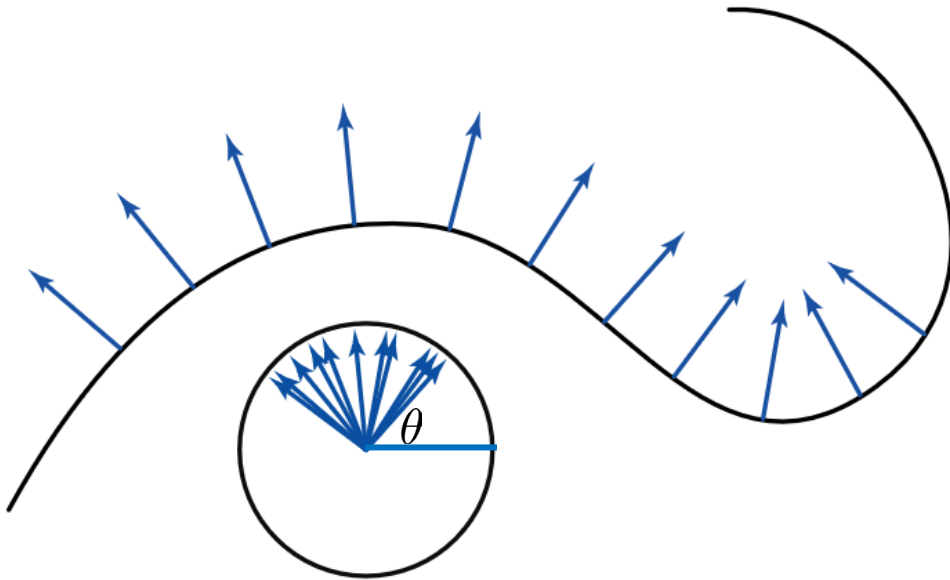


$$r(s) = \frac{1}{k(s)}$$

Fundamental theorem of the
local theory of plane curves:

**$k(s)$ characterizes a planar
curve up to rigid motion.**

IDEA OF PROOF



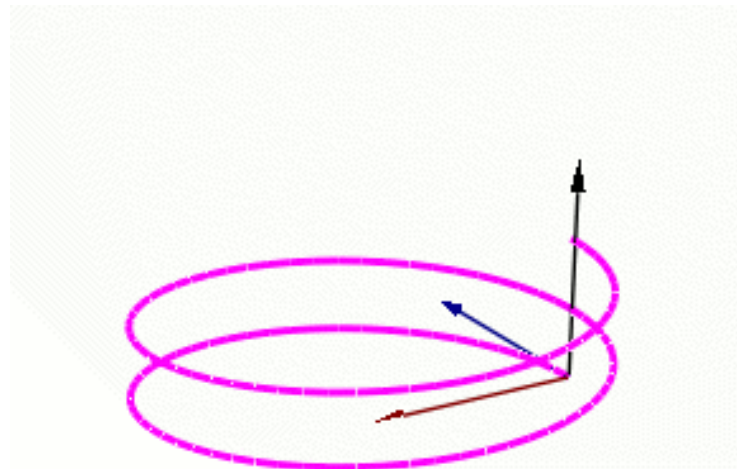
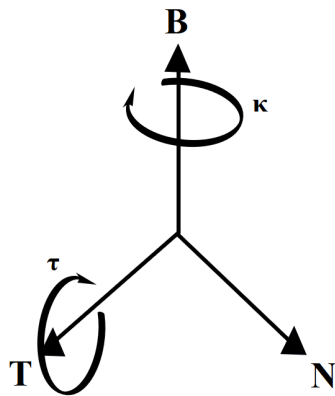
$$T(s) := (\cos \theta(s), \sin \theta(s))$$
$$\implies k(s) := \theta'(s)$$

Image from DDG course notes by E. Grinspun

Provides intuition for curvature

FRENET FRAME: CURVES IN \mathbb{R}^3

- **Curvature:** In-plane motion
 - **Binormal:** $T \times N$
- **Torsion:** Out-of-plane motion

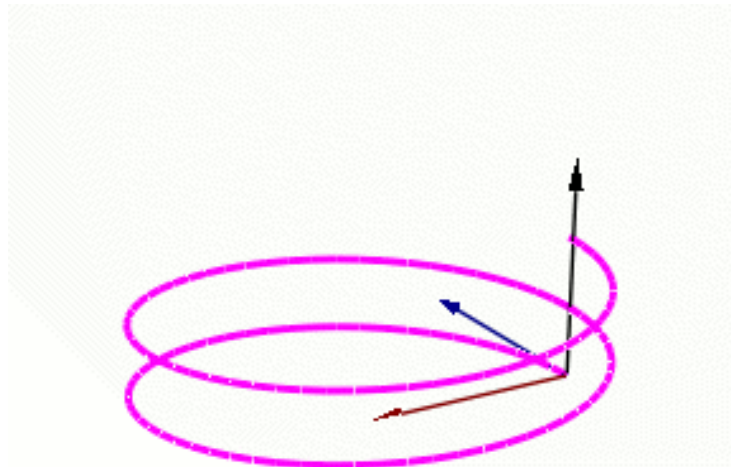


FRENET FRAME: CURVES IN \mathbb{R}^3

$$\frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

- **Curvature:** In-plane motion
- **Torsion:** Out-of-plane motion

Binormal: $T \times N$

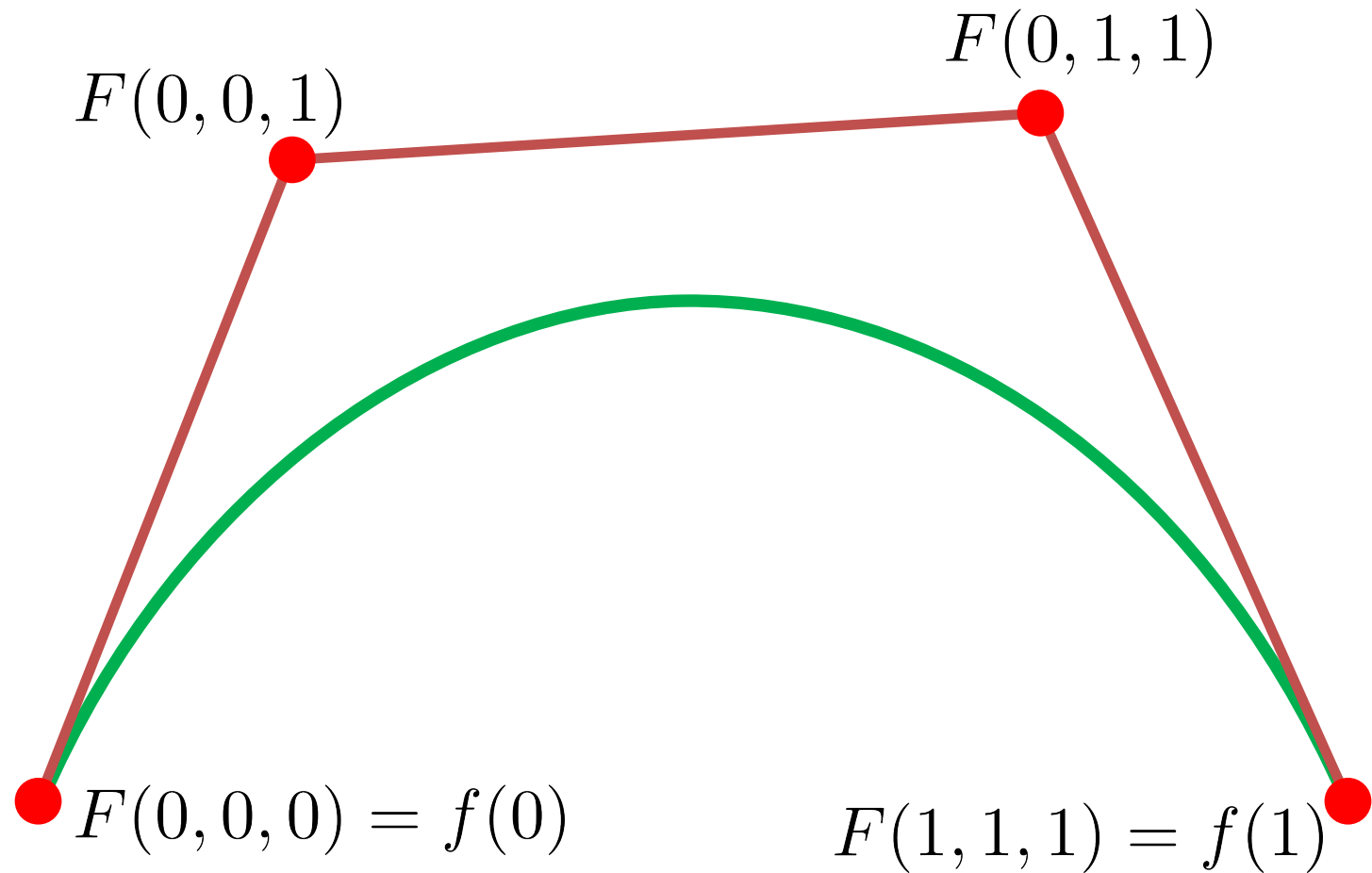


Fundamental theorem of the
local theory of space curves:

**Curvature and torsion
characterize a 3D curve up
to rigid motion.**

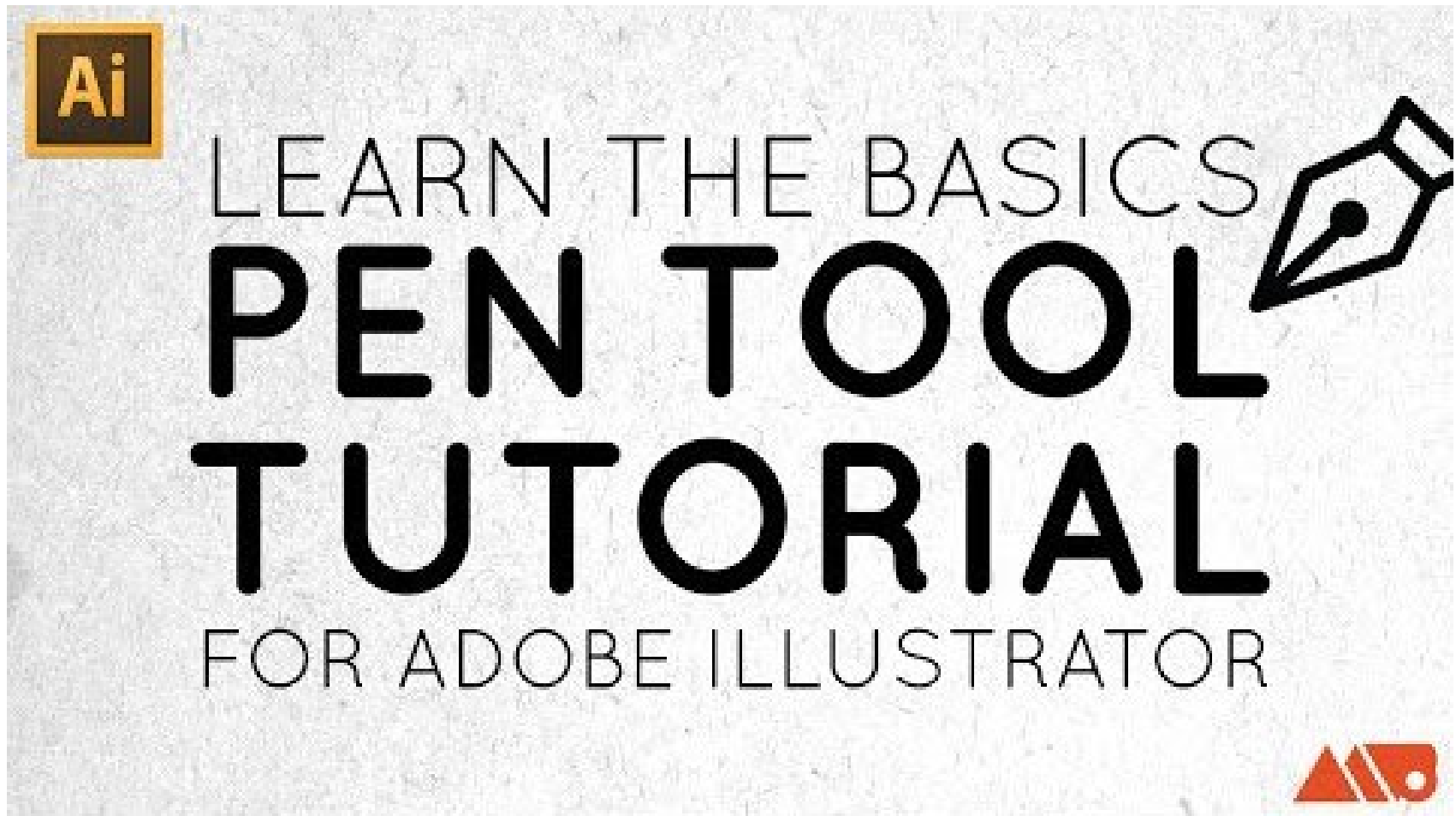
Representing curves digitally

TRADITIONAL APPROACH



Piecewise smooth approximations

MODELING CURVES



NOT THE ONLY MODELING TOOL

κ -Curves: Interpolation at Local Maximum Curvature

ZHIPEI YAN, Texas A&M University

STEPHEN SCHILLER, Adobe Research

GREGG WILENSKY, Adobe

NATHAN CARR, Adobe Research

SCOTT SCHAEFER, Texas A&M University

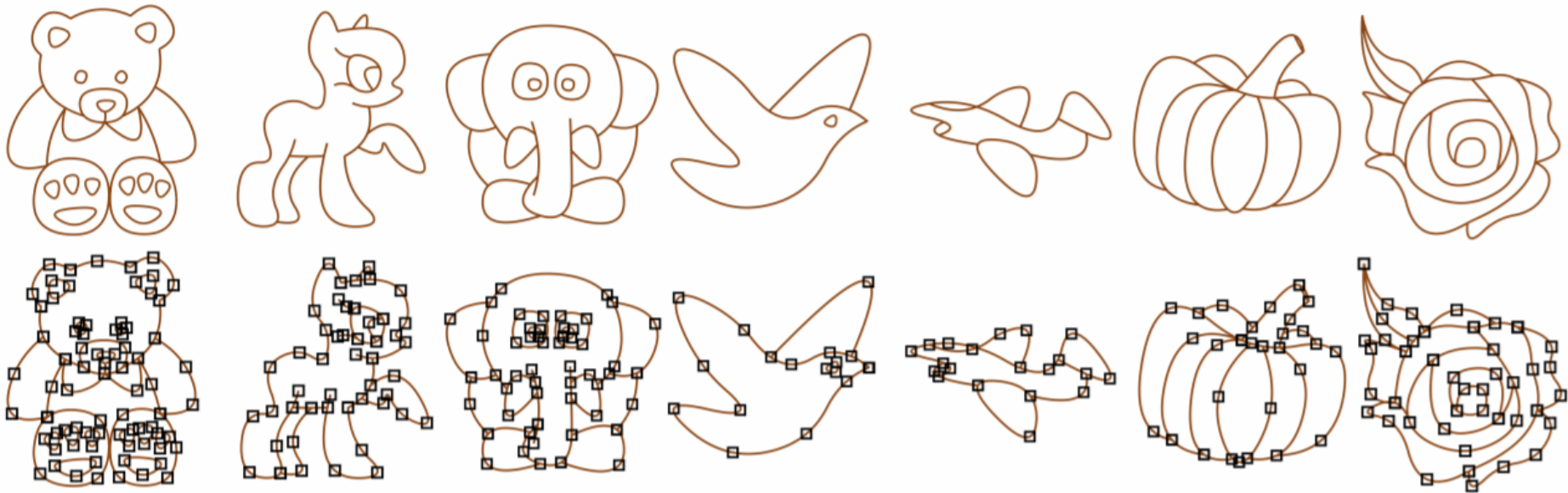


Fig. 1. Top row shows example shapes made from the control points below. In all cases, local maxima of curvature only appear at the control points, and the curves are G^2 almost everywhere.

QUESTION

What is the arc length of a cubic Bézier curve?

$$\int_a^b \|\gamma'(t)\| dt$$

QUESTION

What is the arc length of a cubic Bézier curve?

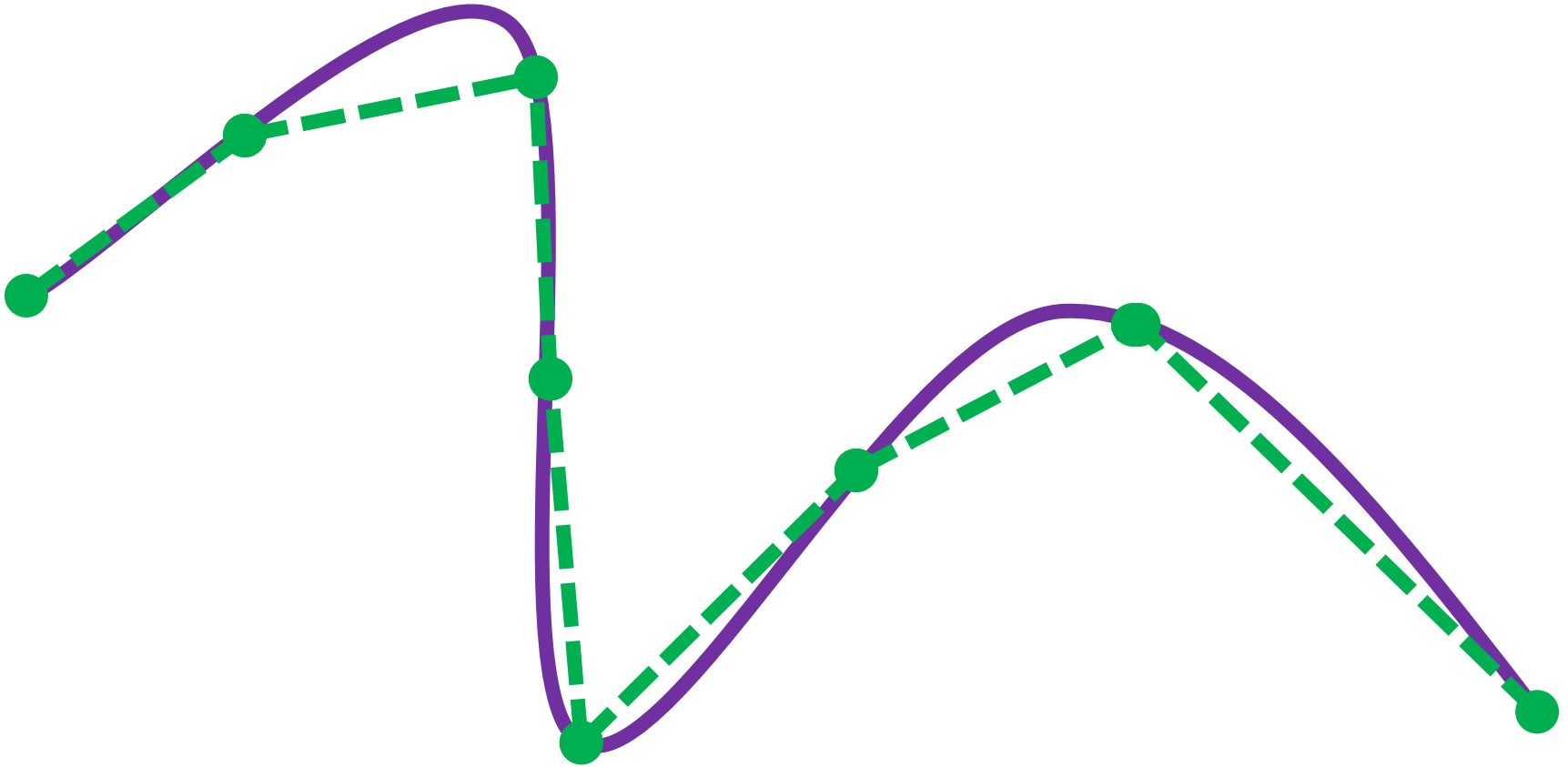
$$\int_a^b \|\gamma'(t)\| dt$$

Not known in closed form.

ONLY APPROXIMATIONS ANYWAY

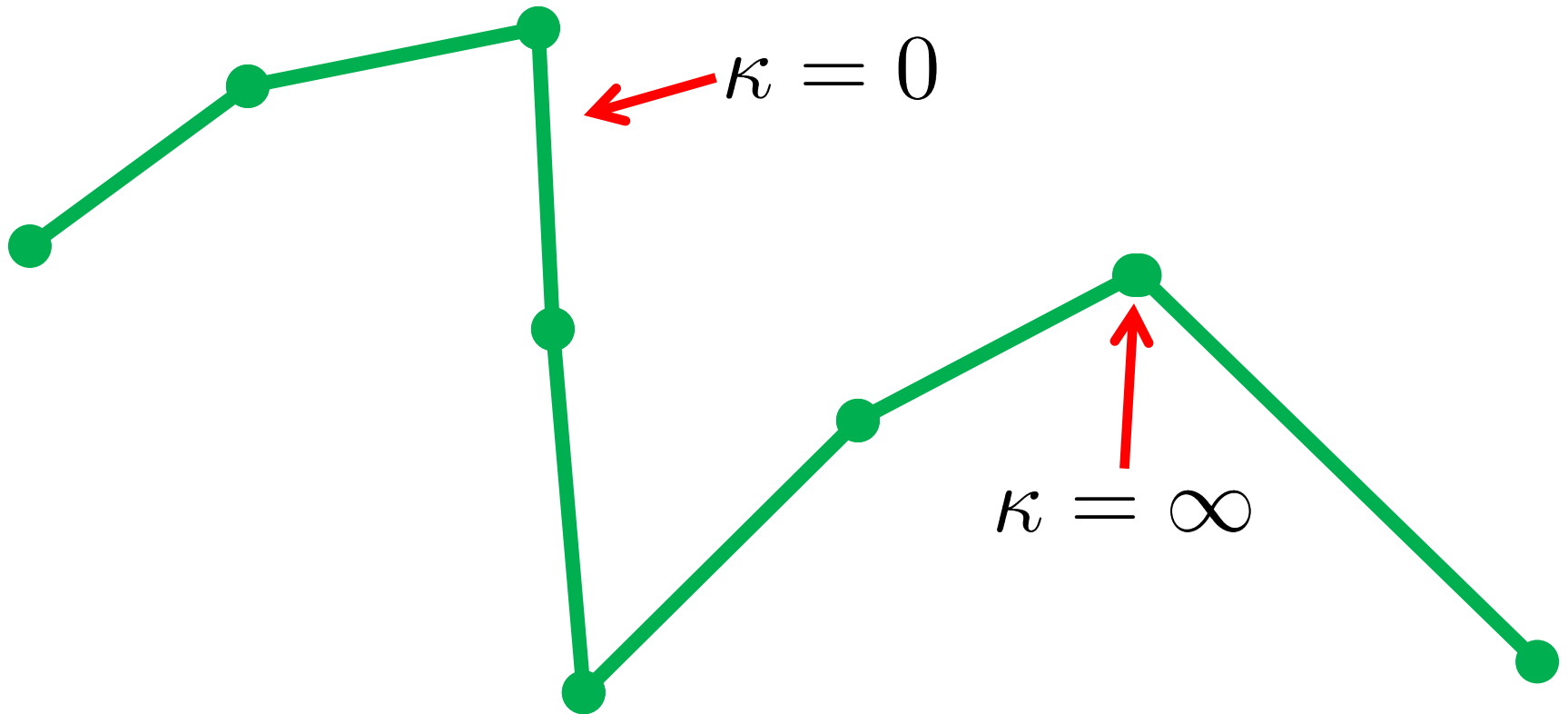
$$\{\text{Bézier curves}\} \subsetneq \{\gamma : \mathbb{R} \rightarrow \mathbb{R}^3\}$$

ANOTHER REASONABLE APPROXIMATION



Piecewise linear

BIG PROBLEM



Boring differential structure

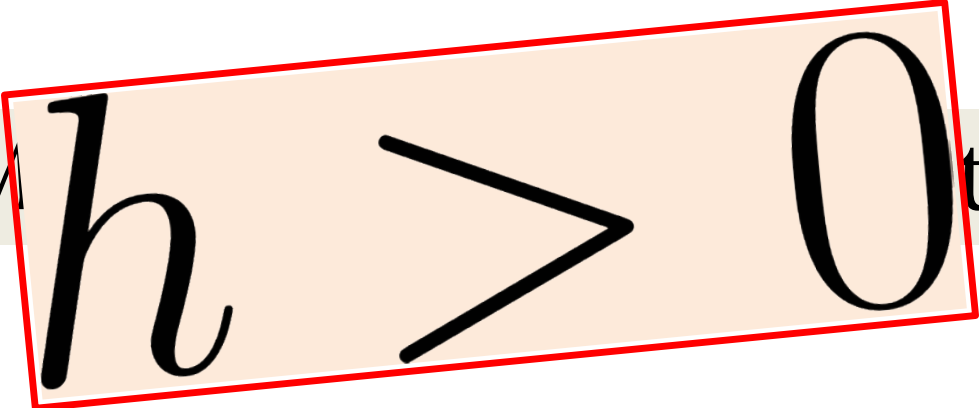
FINITE DIFFERENCE APPROACH

$$f'(x) \approx \frac{1}{h} [f(x + h) - f(x)]$$

THEOREM: As $\Delta h \rightarrow 0$, [insert statement].

REALITY CHECK

$$f'(x) \approx \frac{1}{h} [f(x+h) - f(x)]$$

THEOREM  statement].

TWO KEY CONSIDERATIONS

- Convergence to continuous theory
- Discrete behavior

GOAL

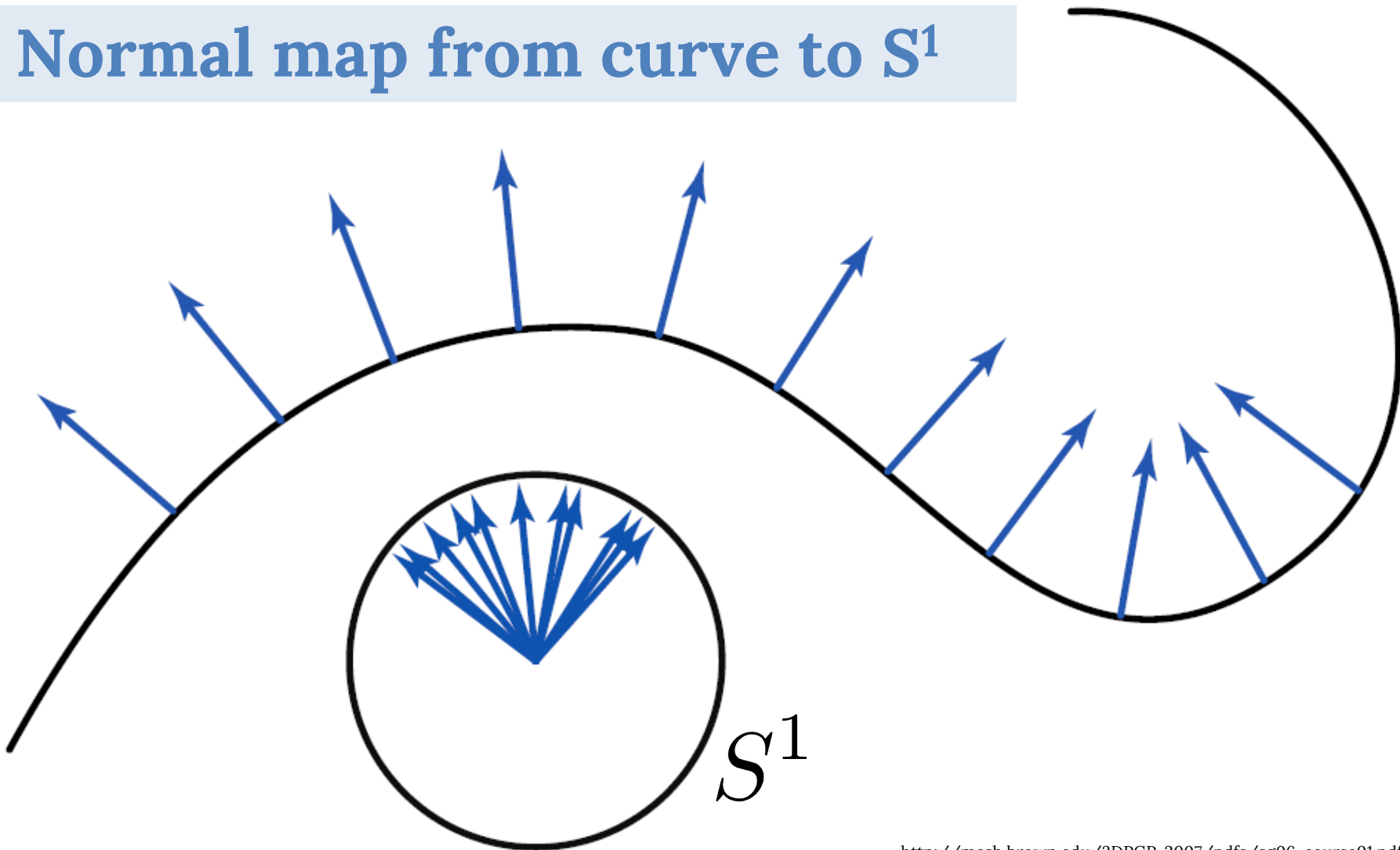
Examine discrete theories
of differentiable curves.

GOAL

Examine discrete theories
of differentiable curves.

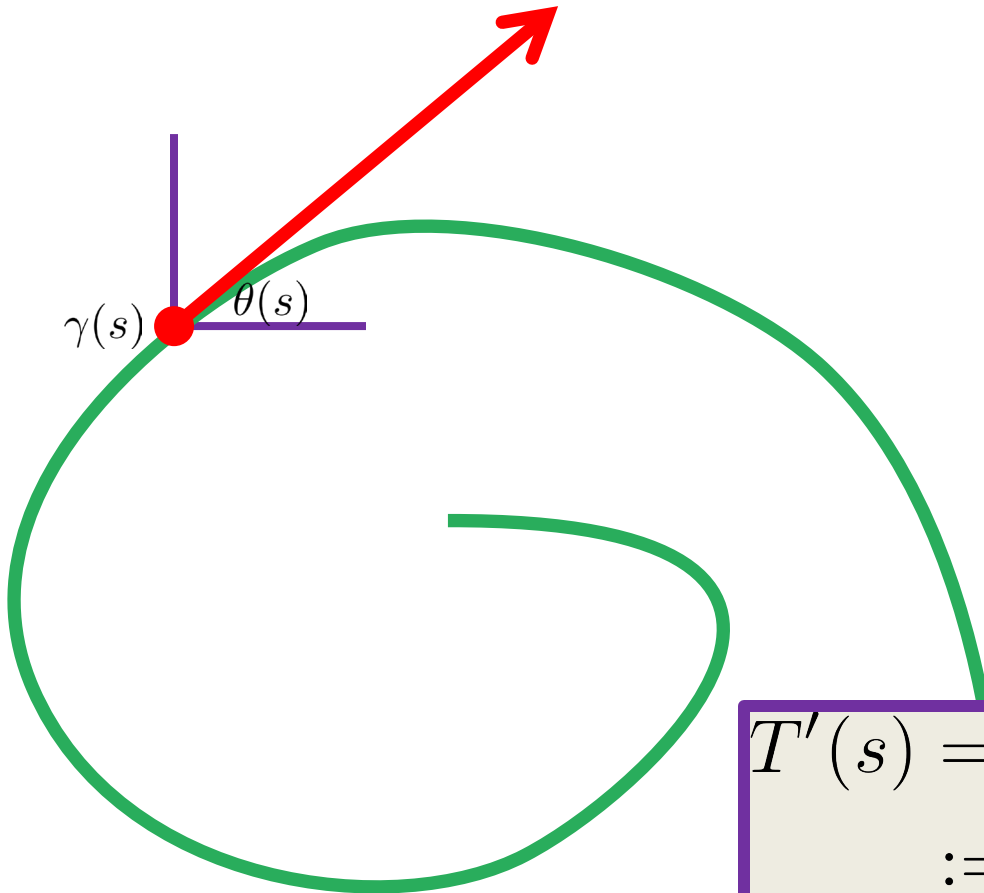
GAUSS MAP

Normal map from curve to S^1



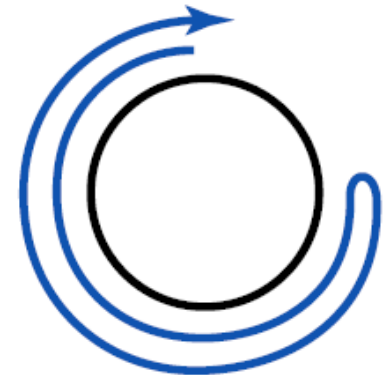
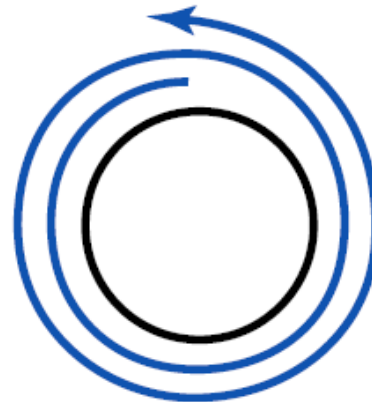
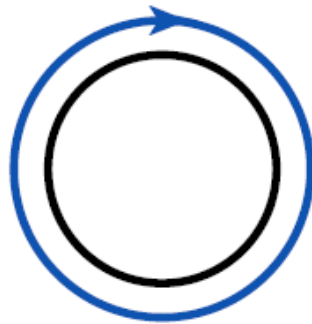
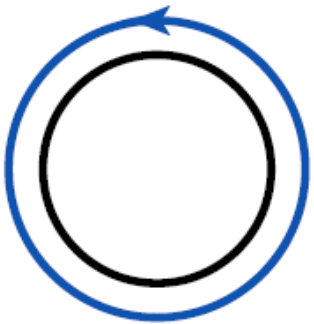
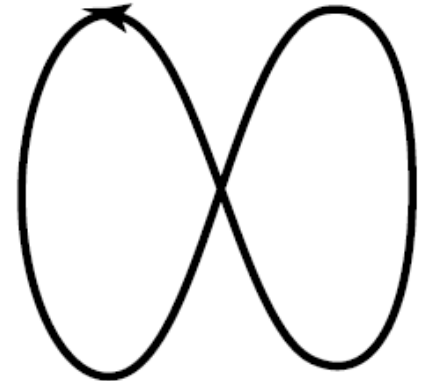
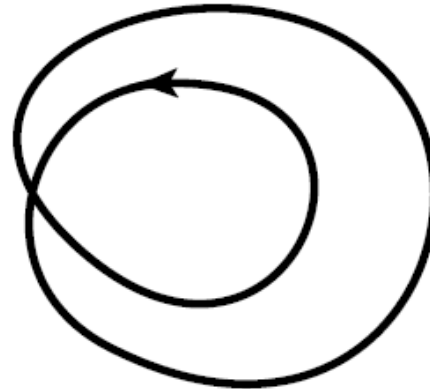
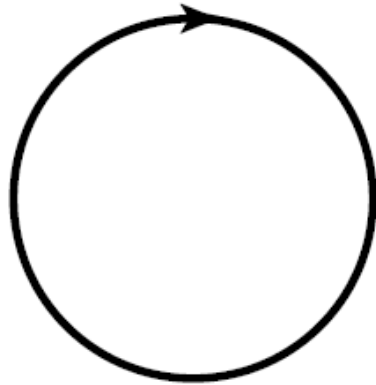
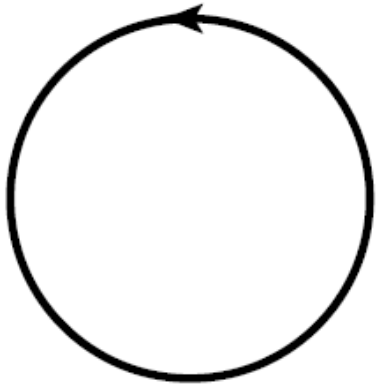
SIGNED CURVATURE ON PLANE CURVES

$$T(s) = (\cos \theta(s), \sin \theta(s))$$



$$\begin{aligned} T'(s) &= \theta'(s)(-\sin \theta(s), \cos \theta(s)) \\ &:= \kappa(s)N(s) \end{aligned}$$

TURNING NUMBERS



+1

-1

+2

0

RECOVERING THETA

$$\theta'(s) = \kappa(s)$$



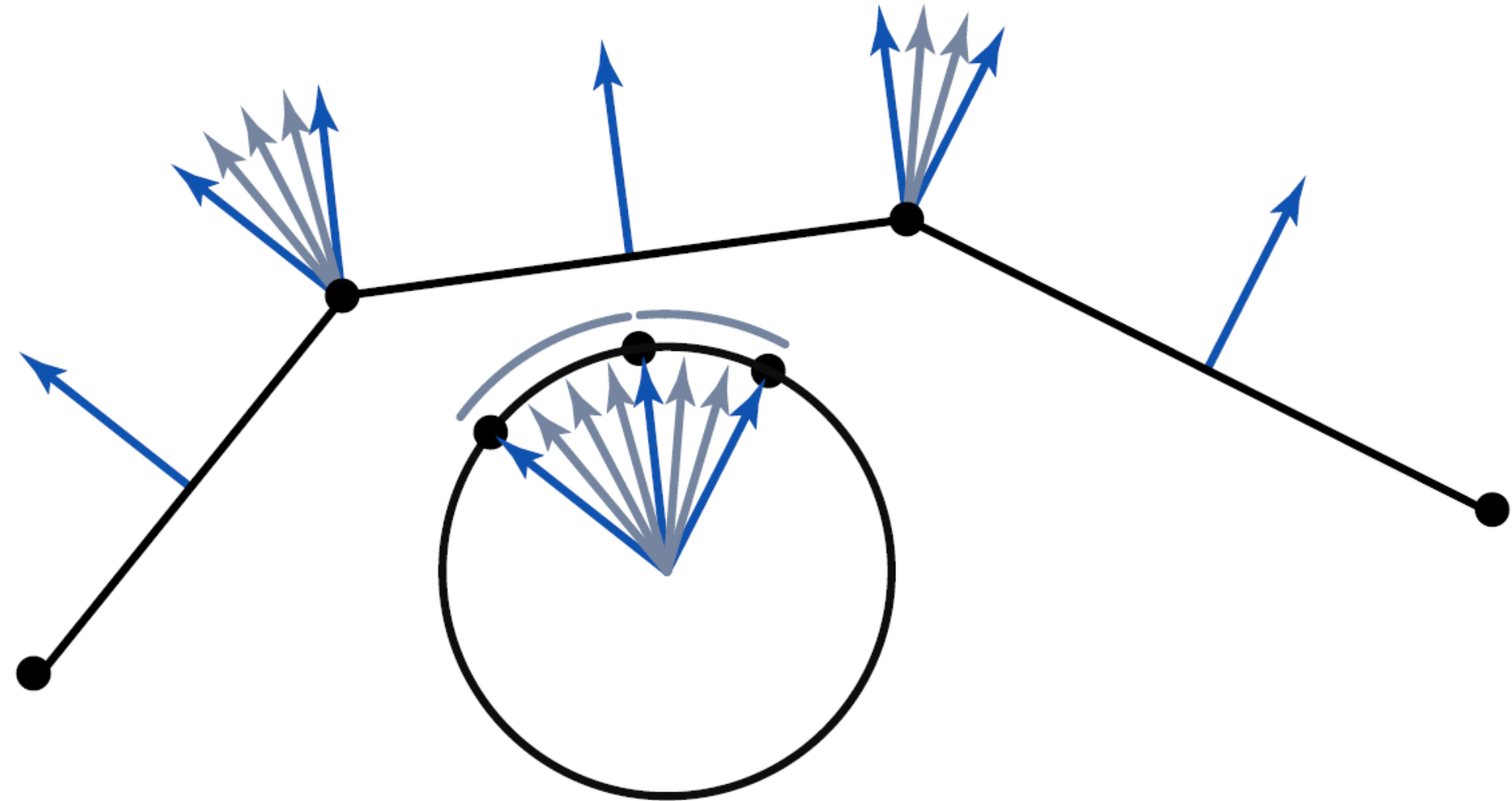
$$\Delta\theta = \int_{s_0}^{s_1} \kappa(s) ds$$

TURNING NUMBER THEOREM

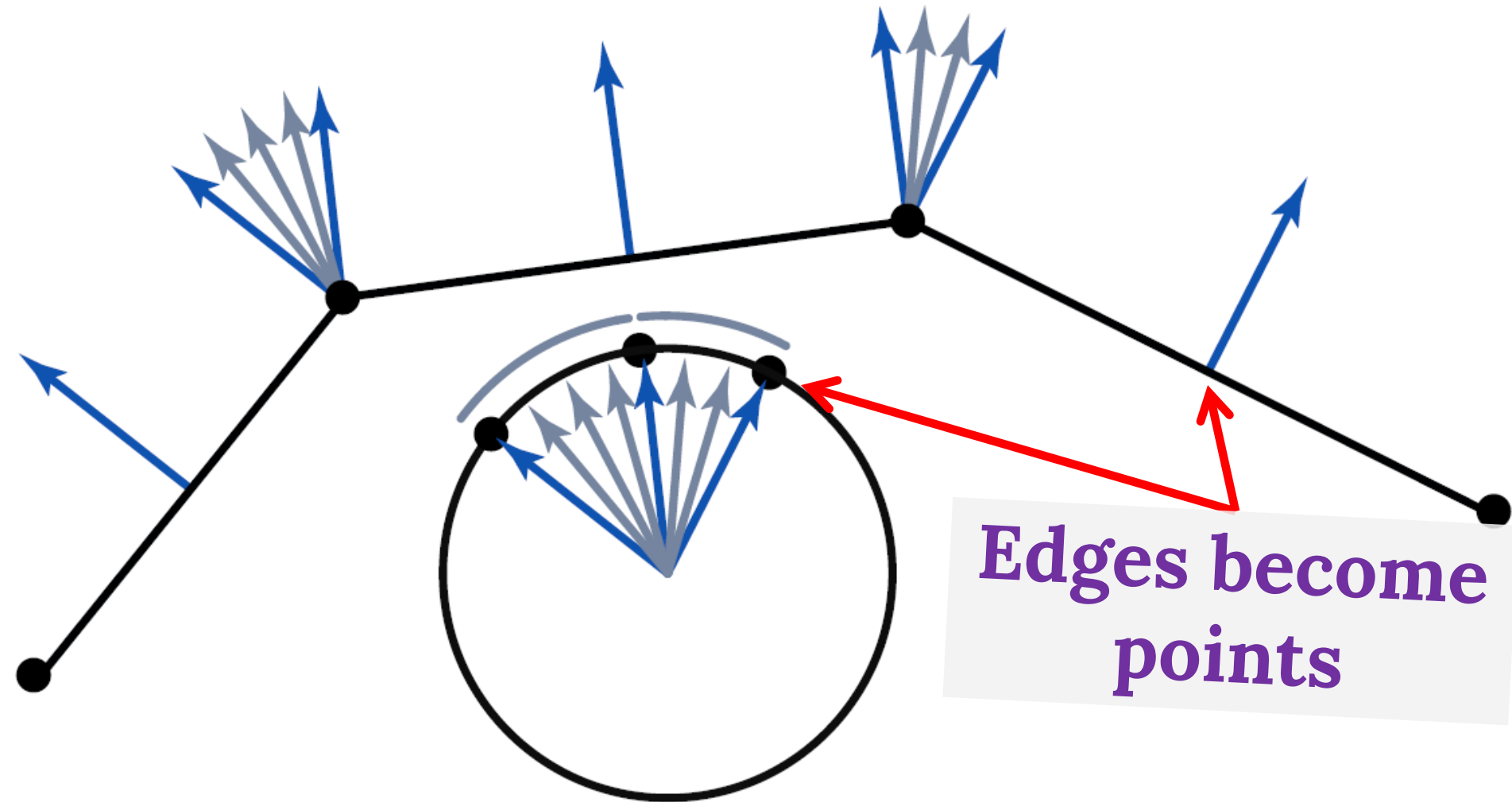
$$\int_{\Omega} \kappa(s) \, ds = 2\pi k$$

A “global” theorem!

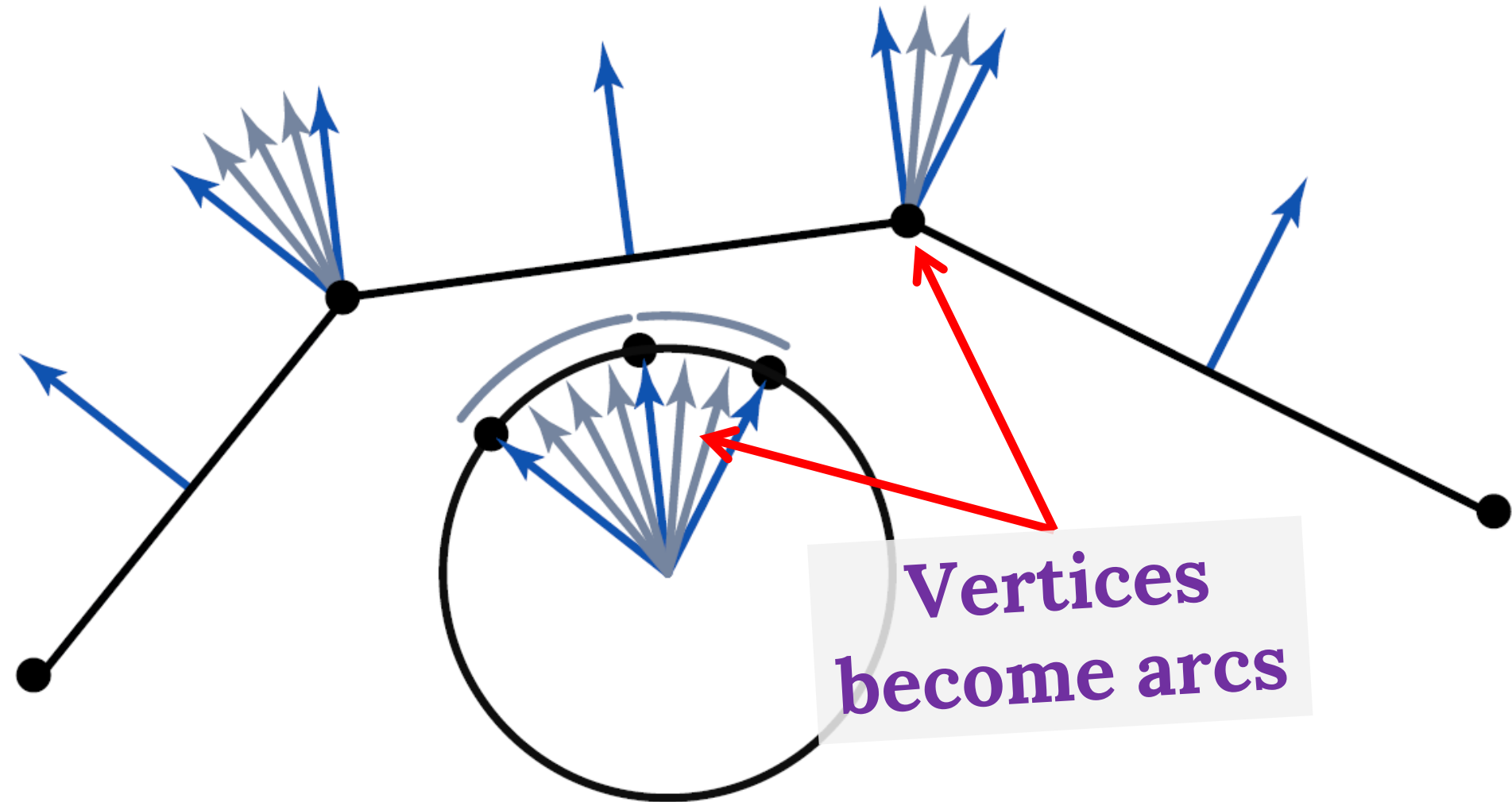
DISCRETE GAUSS MAP



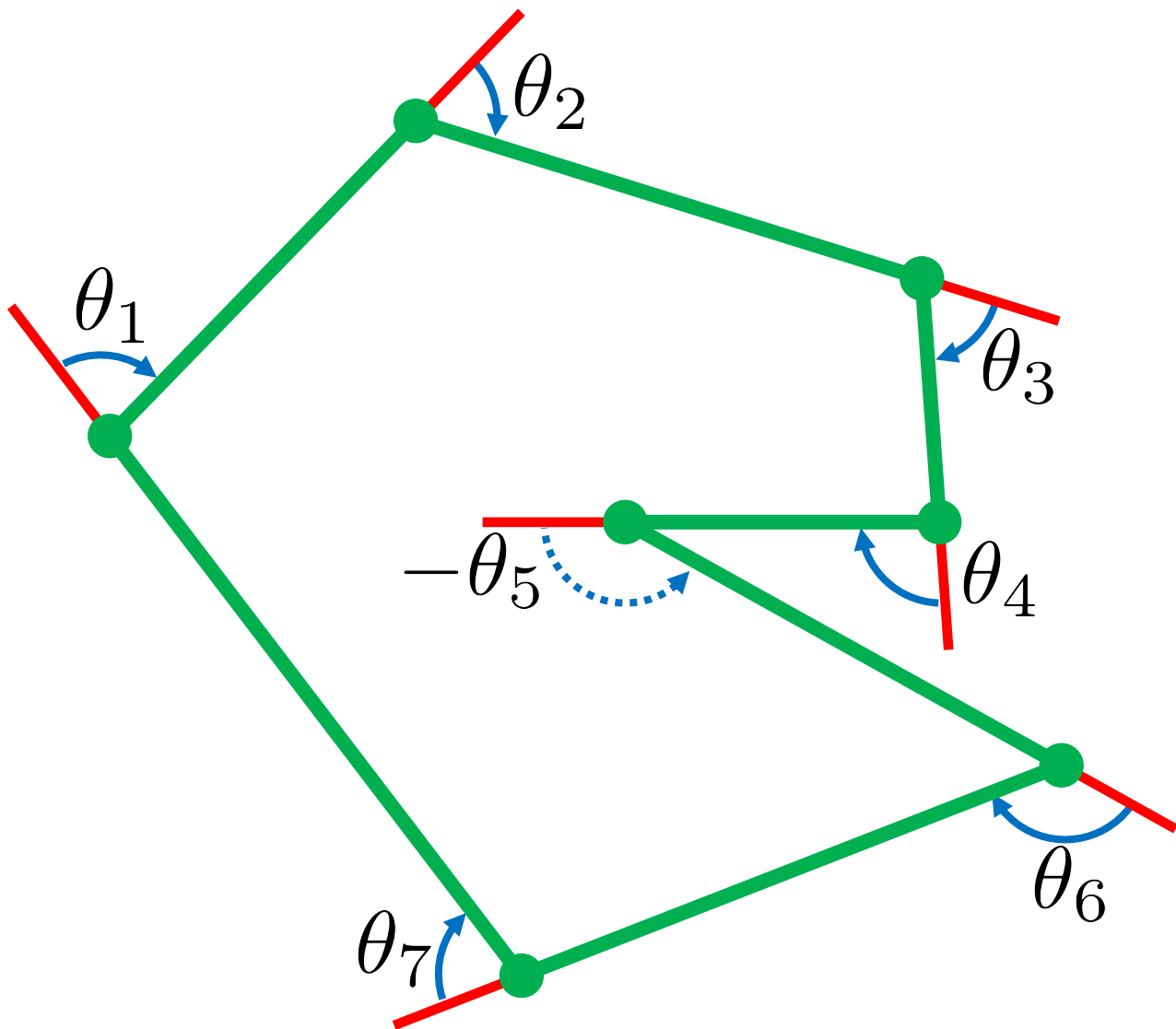
DISCRETE GAUSS MAP



DISCRETE GAUSS MAP

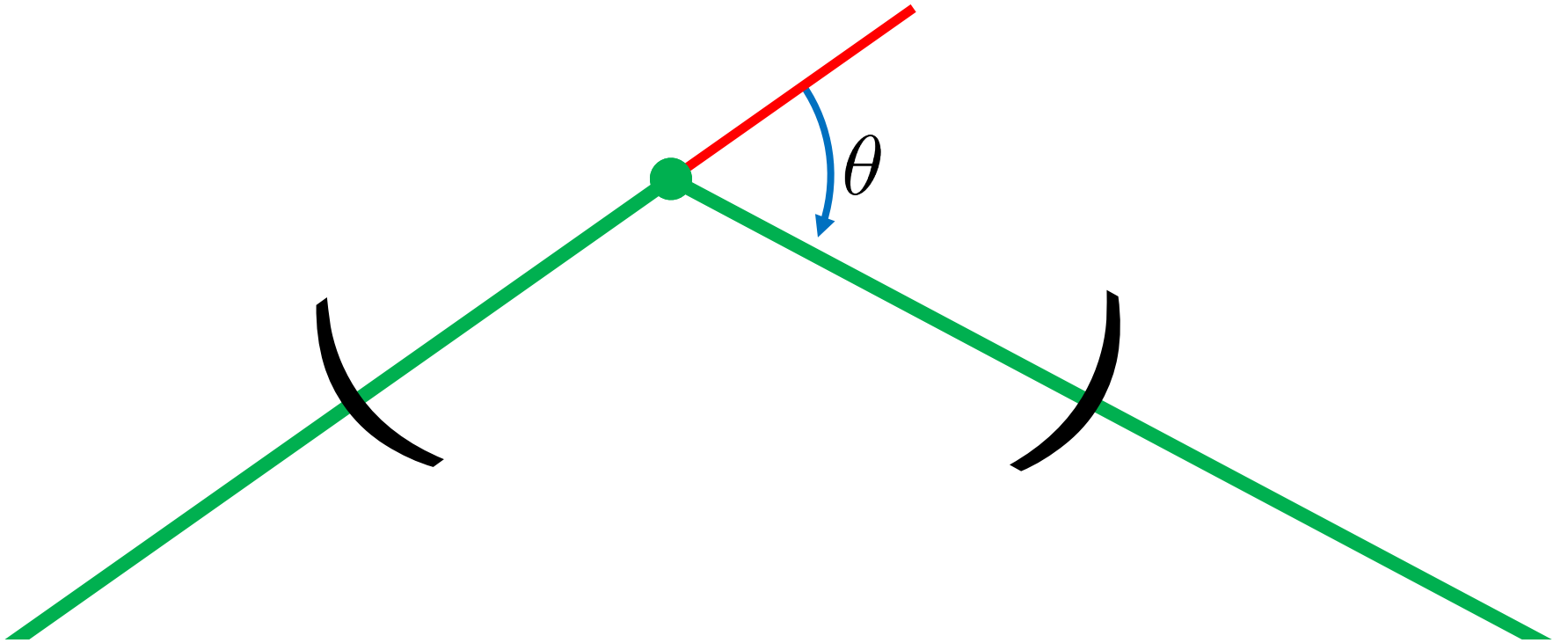


KEY OBSERVATION



$$\sum_i \theta_i = 2\pi k$$

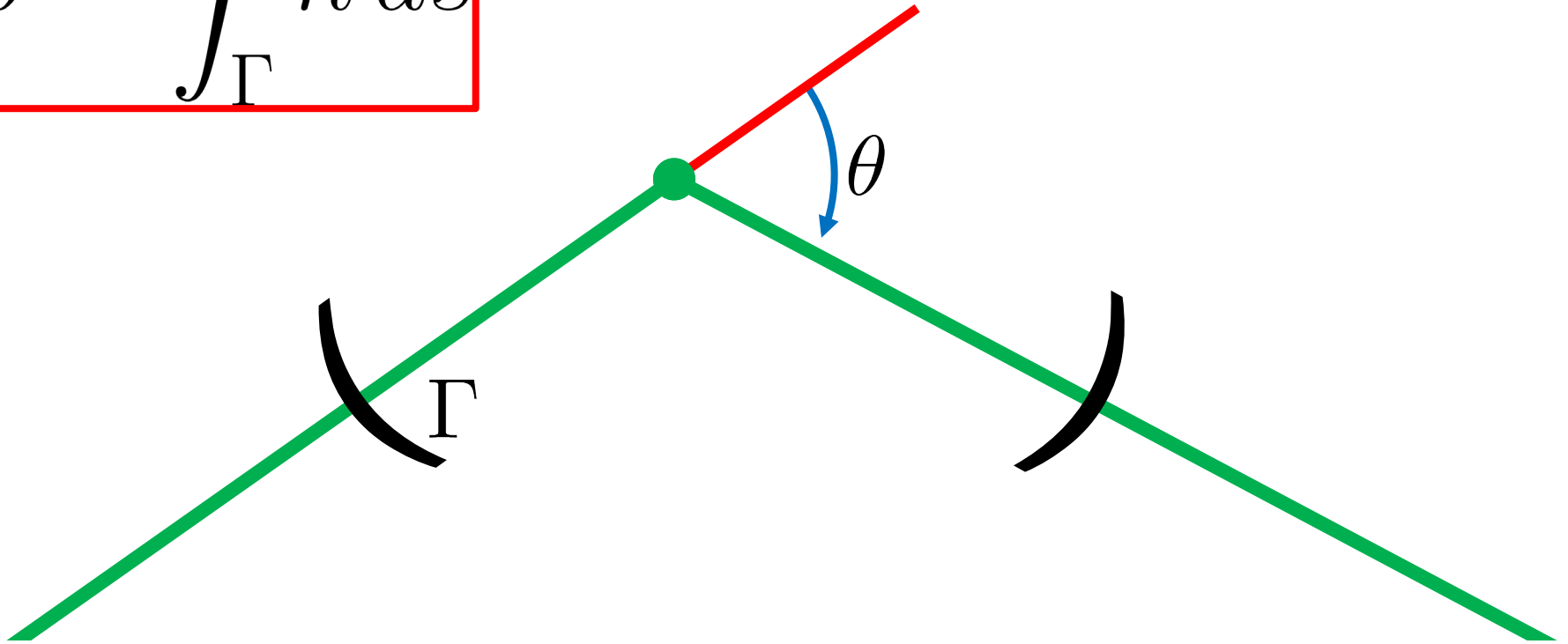
WHAT'S GOING ON?



Total change in curvature

WHAT'S GOING ON?

$$\theta = \int_{\Gamma} \kappa \, ds$$

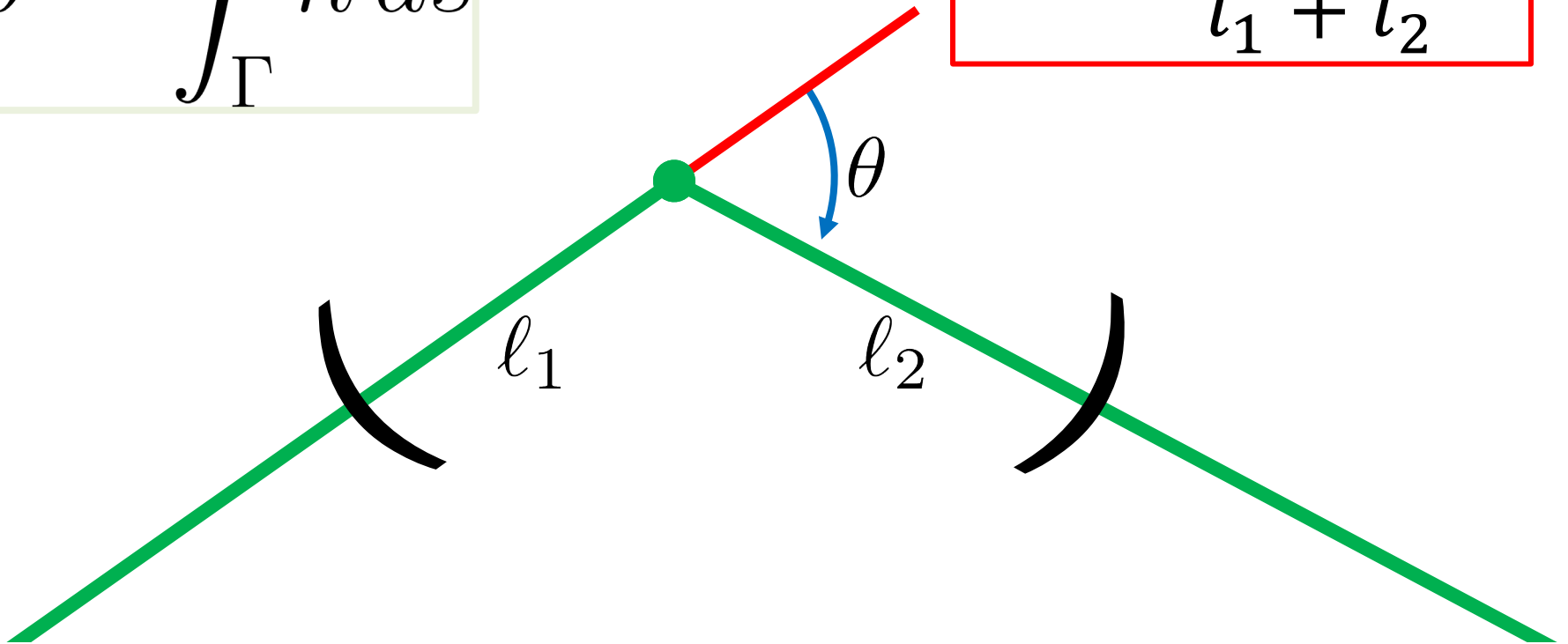


Total change in curvature

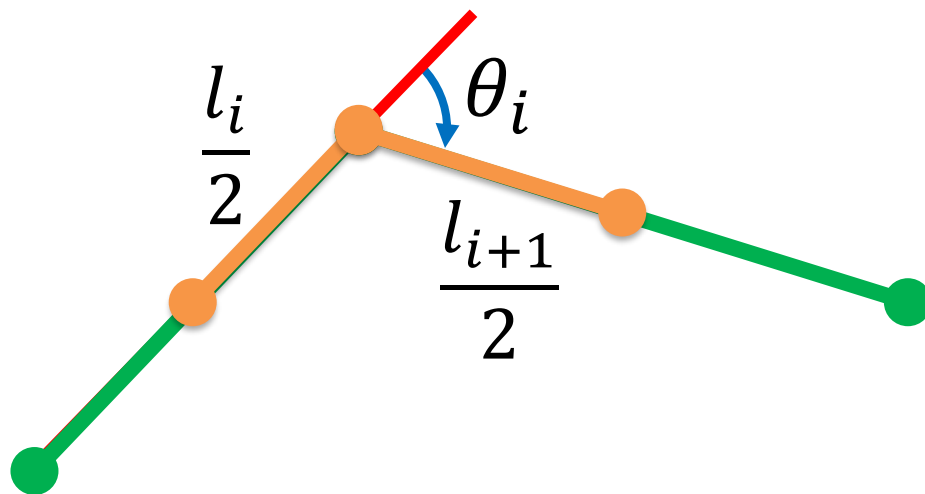
WHAT'S GOING ON?

$$\theta = \int_{\Gamma} \kappa \, ds$$

$$\kappa \approx \frac{2\theta}{l_1 + l_2}$$



Total change in curvature



Continuous:

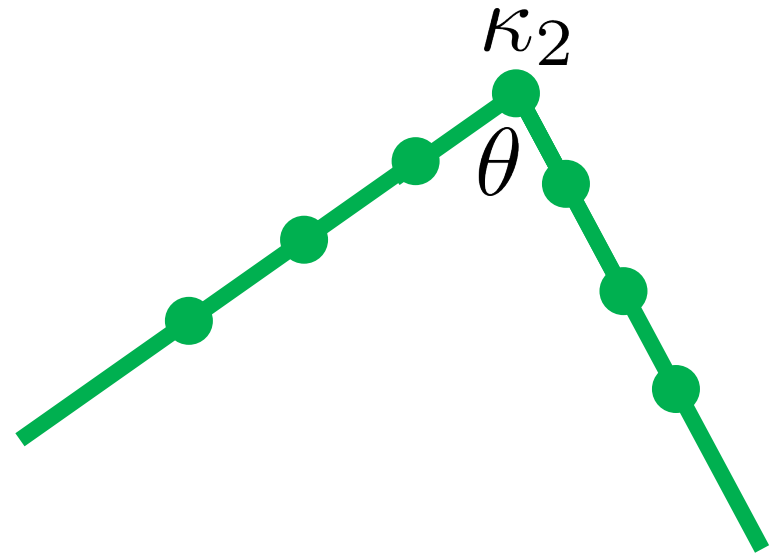
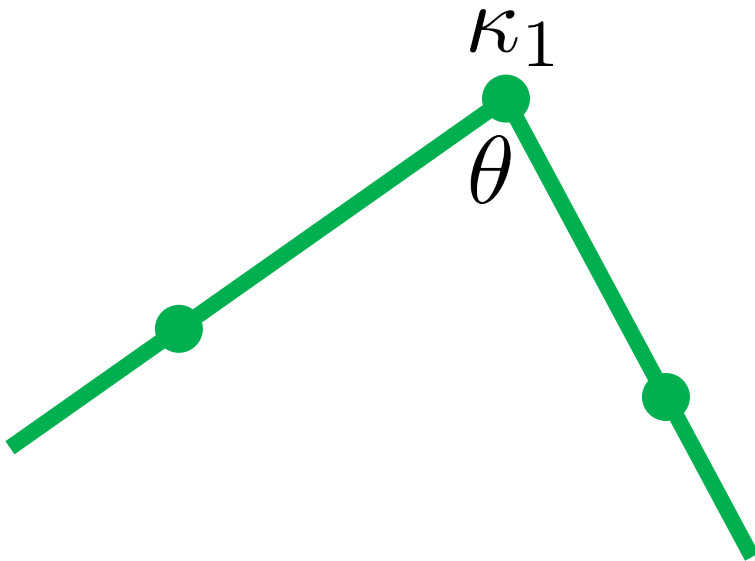
$$k(s) = \theta'(s)$$

Discrete (*finite difference*):

$$k_i = \frac{\theta_i}{(l_i + l_{i+1})/2}$$

INTERESTING DISTINCTION

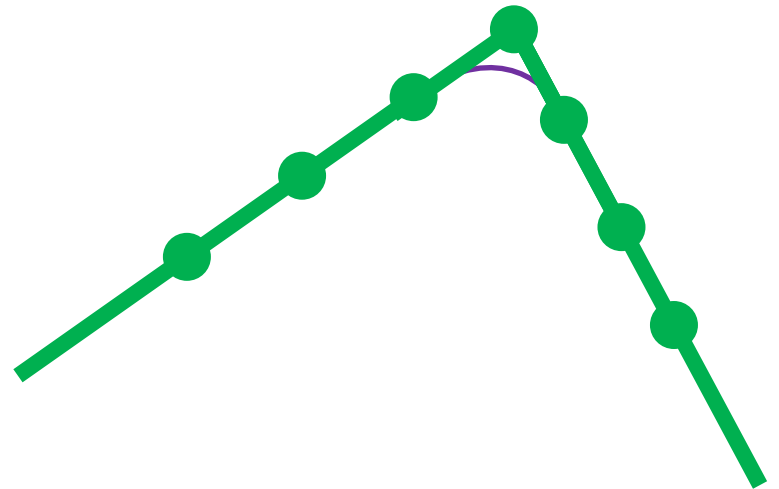
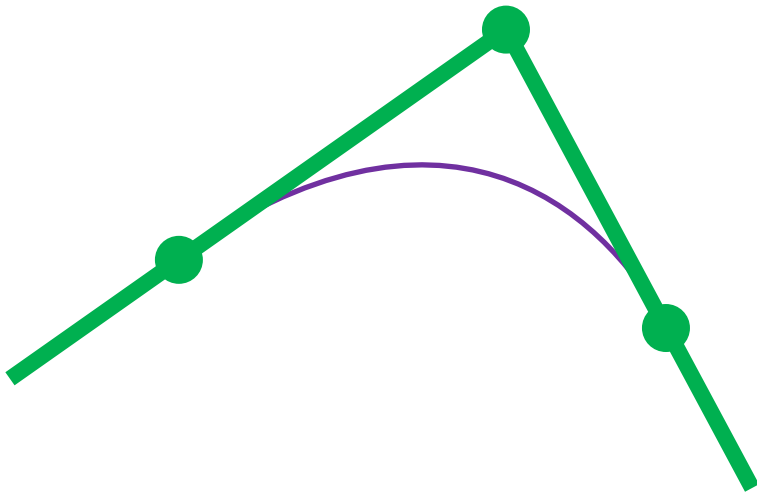
$$\kappa_1 \neq \kappa_2$$



Same integrated curvature

INTERESTING DISTINCTION

$$\kappa_1 \neq \kappa_2$$

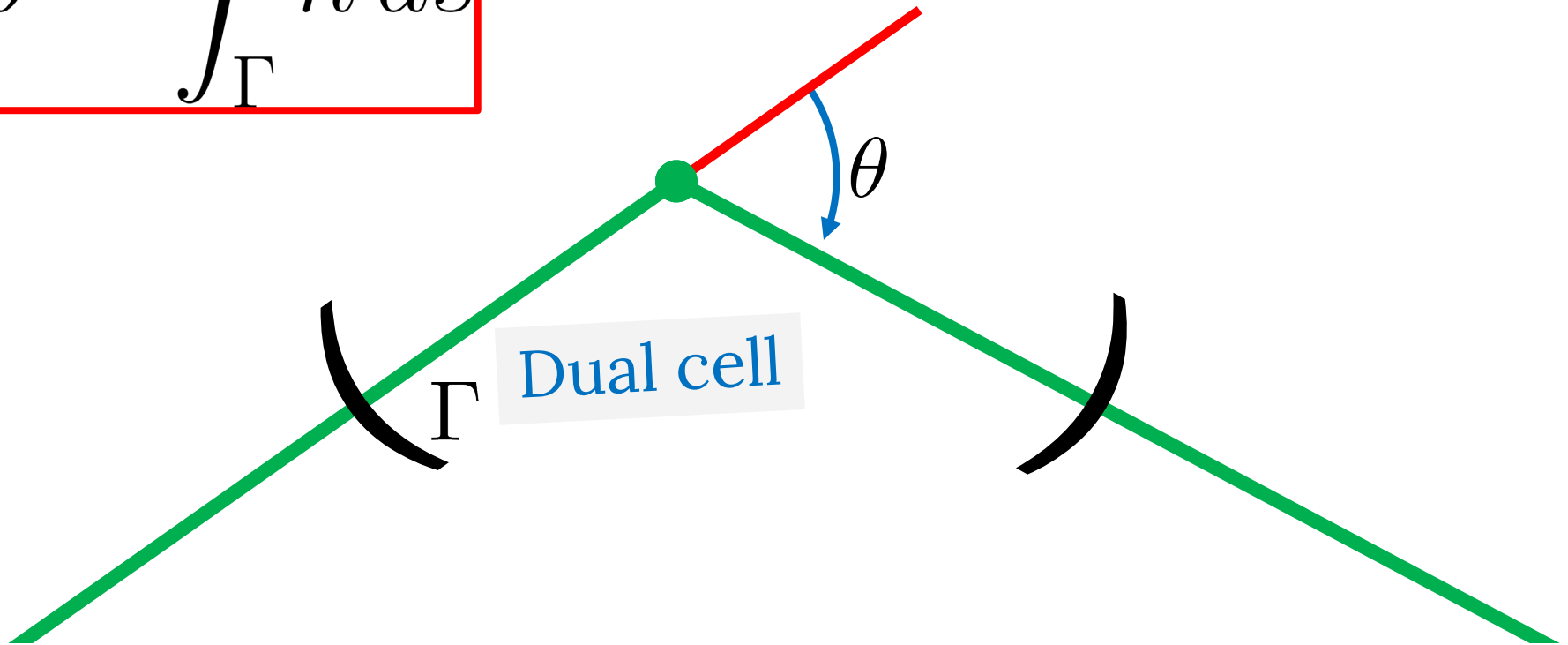


Same integrated curvature

WHAT'S GOING ON?

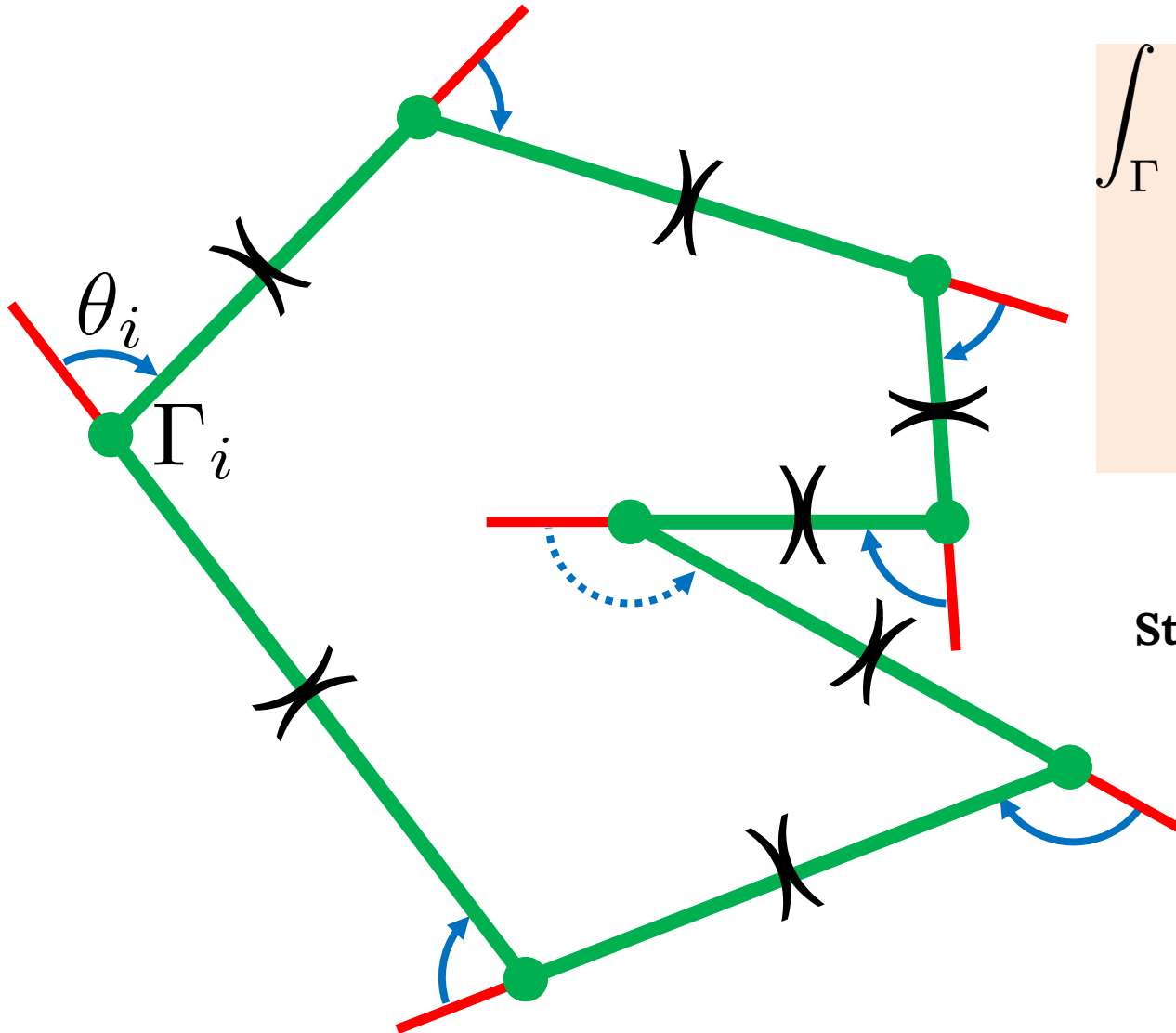
$$\theta = \int_{\Gamma} \kappa \, ds$$

Integrated
quantity



Total change in curvature

DISCRETE TURNING ANGLE THEOREM

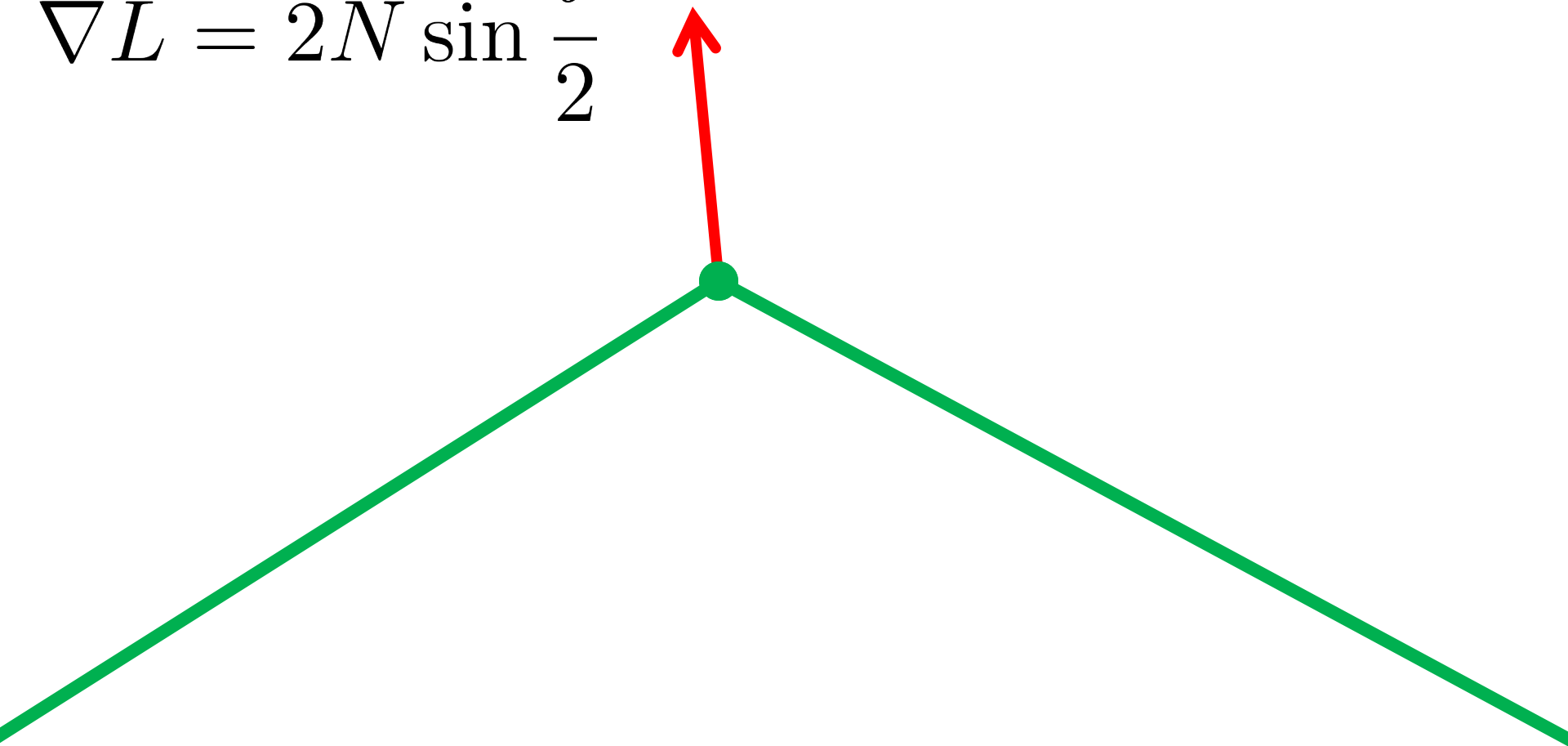


$$\begin{aligned}\int_{\Gamma} \kappa \, ds &= \sum_i \int_{\Gamma_i} \kappa \, ds \\ &= \sum_i \theta_i \\ &= 2\pi k\end{aligned}$$

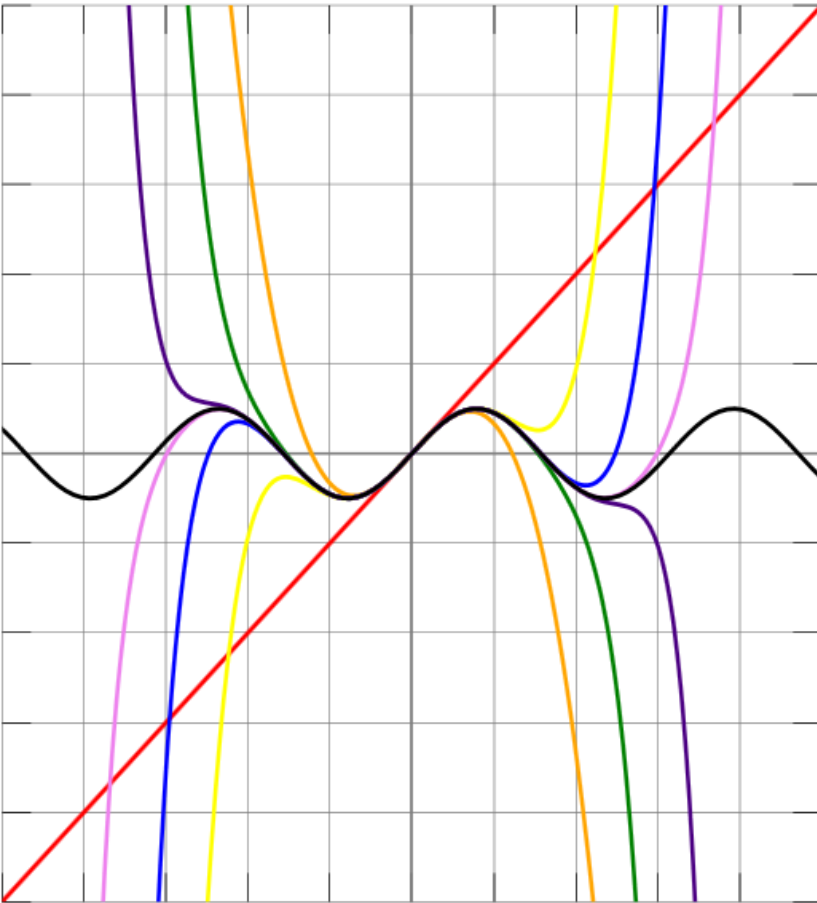
Structure Preservation!

ANOTHER DISCRETIZATION

$$\nabla L = 2N \sin \frac{\theta}{2}$$



FOR SMALL θ



$$2 \sin \frac{\theta}{2} \approx 2 \cdot \frac{\theta}{2} = \theta$$

Same behavior in the limit

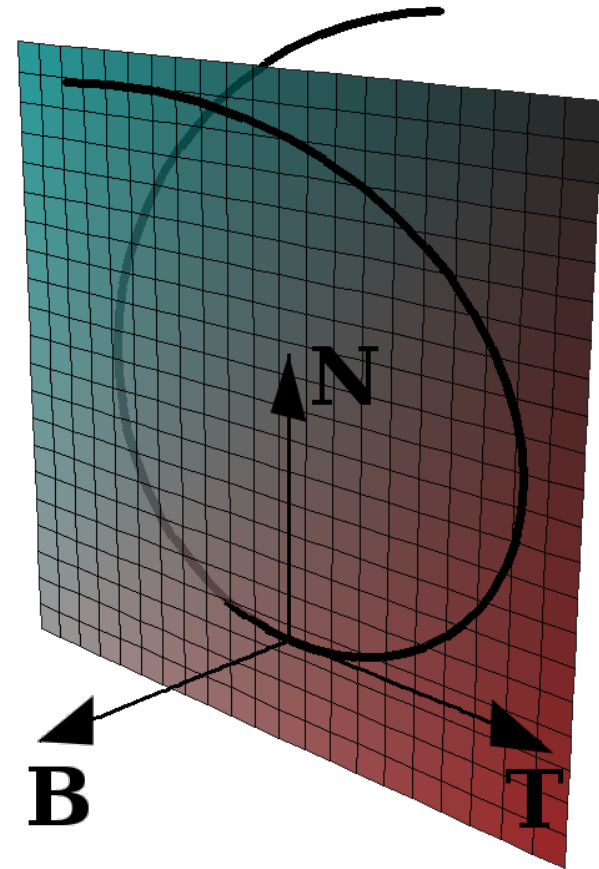
Does discrete curvature
converge in limit?

Yes!

...workflow

Discretizing 3D curves

FRENET FRAME

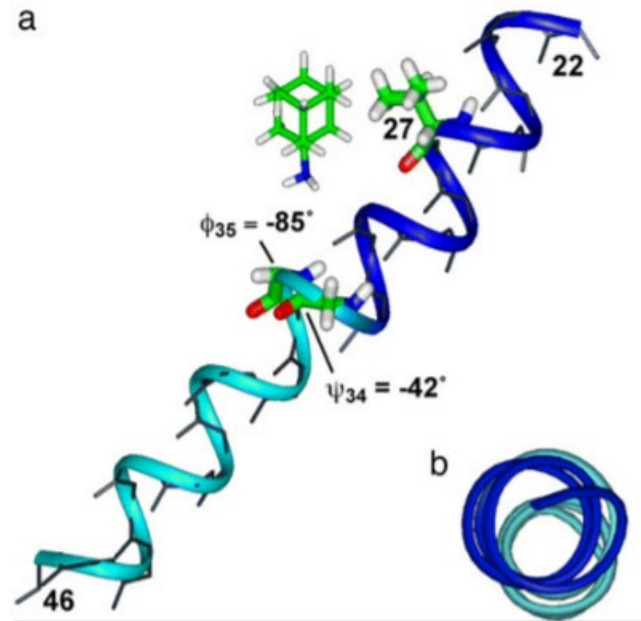


$$\frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

APPLICATION



**NMR
scanner**



Kinked alpha helix

**Structure Determination of Membrane Proteins Using Discrete Frenet Frame
and Solid State NMR Restraints**

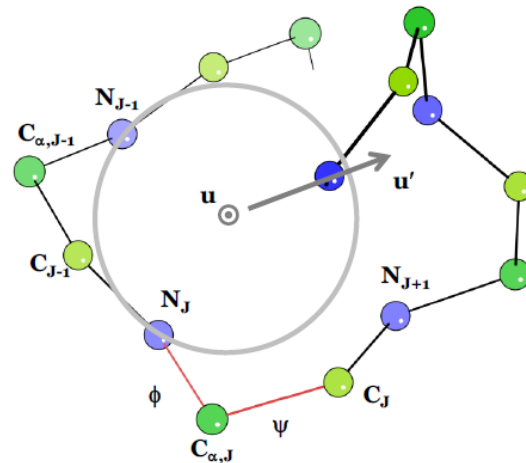
Achuthan and Quine

Discrete Mathematics and its Applications, ed. M. Sethumadhavan (2006)

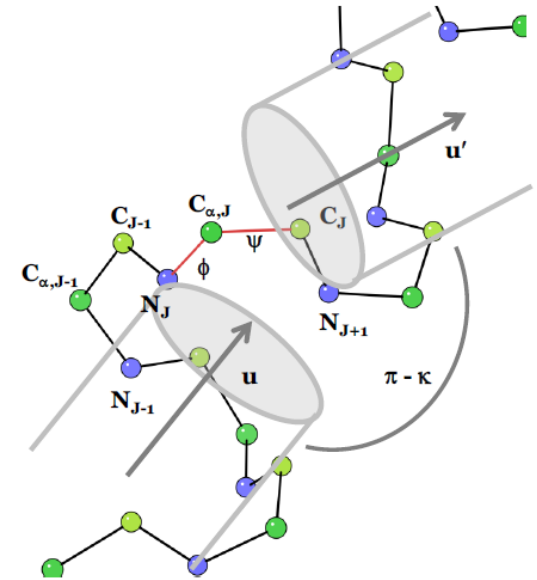
APPLICATION



**NMR
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Kinked alpha helix



**Structure Determination of Membrane Proteins Using Discrete Frenet Frame
and Solid State NMR Restraints**

Achuthan and Quine

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POTENTIAL DISCRETIZATION

$$T_j = \frac{p_{j+1} - p_j}{\|p_{j+1} - p_j\|}$$

$$B_j = t_{j-1} \times t_j$$

$$N_j = b_j \times t_j$$

**Discrete Frenet
frame**

$$T_k = R(B_k, \theta_k) T_{k-1}$$

$$B_{k+1} = R(T_k, \phi_k) B_k$$

**“Bond and torsion
angles”**

**(derivatives converge to
 κ and τ , resp.)**

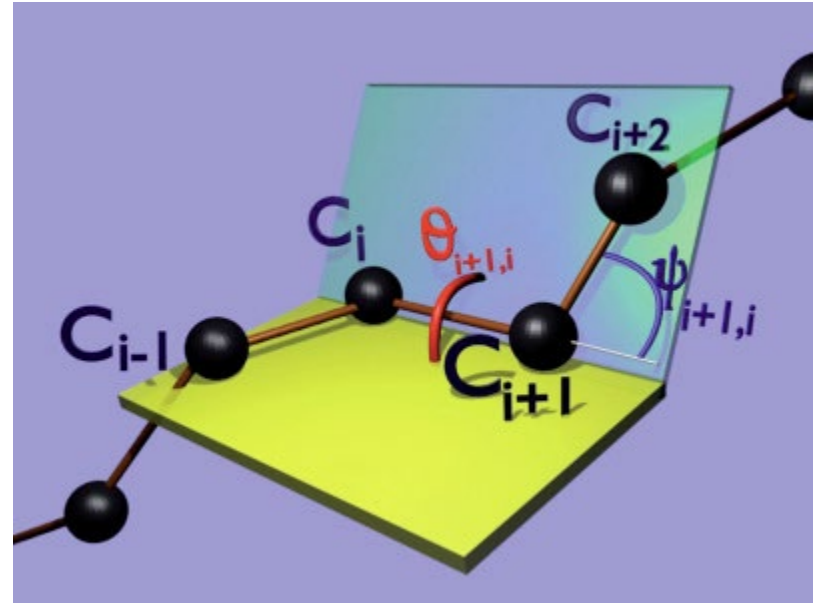
Discrete frame introduced in:

The resultant electric moment of complex molecules

Eyring, Physical Review, 39(4):746–748, 1932.

TRANSFER MATRIX

$$\begin{pmatrix} T_{i+1} \\ N_{i+1} \\ B_{i+1} \end{pmatrix} = R_{i+1,i} \begin{pmatrix} T_i \\ N_i \\ B_i \end{pmatrix}$$



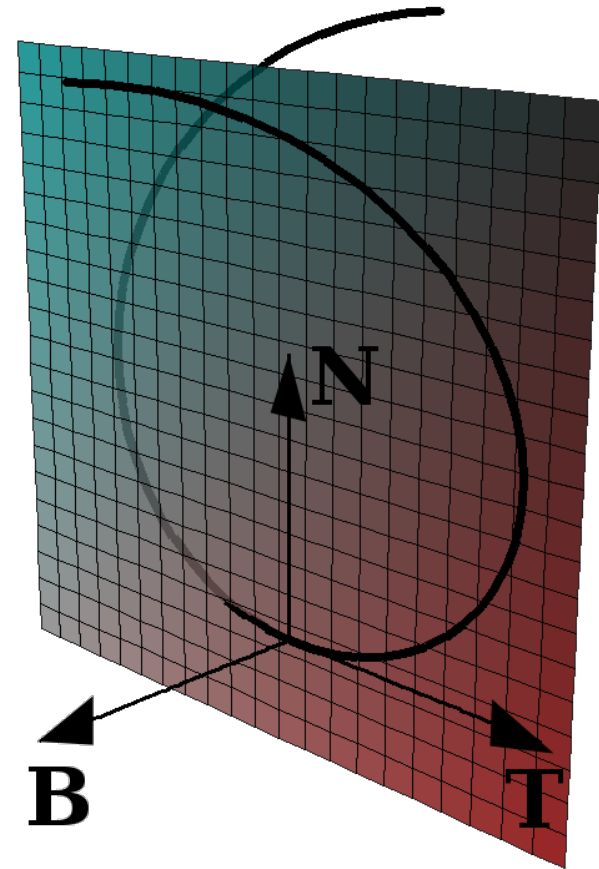
**Discrete construction that works for fractal curves
and converges in continuum limit.**

**Discrete Frenet Frame, Inflection Point Solitons, and Curve Visualization
with Applications to Folded Proteins**

Hu, Lundgren, and Niemi
Physical Review E 83 (2011)

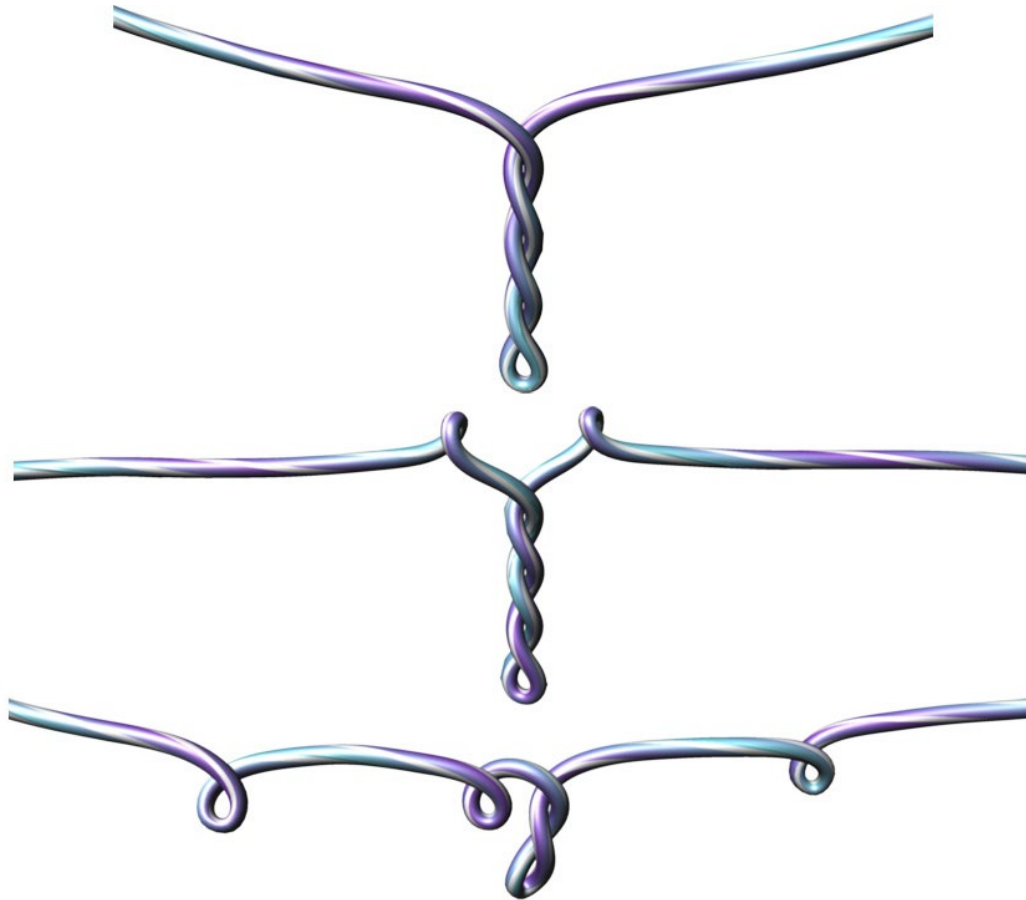
FRENET FRAME: ISSUE

$$\kappa = 0?$$



$$\frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \boxed{\kappa} & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

SEGMENTS NOT ALWAYS ENOUGH



Discrete Elastic Rods

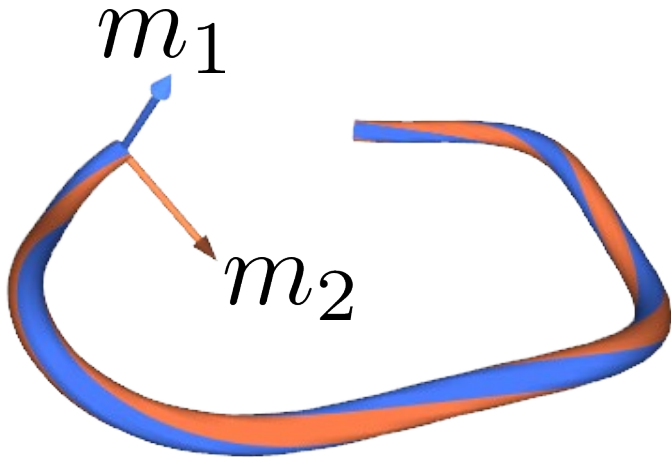
Bergou, Wardetzky, Robinson, Audoly, and Grinspun
SIGGRAPH 2008

SIMULATION GOAL





ADAPTED FRAMED CURVE



$$\Gamma = \{\gamma(s); T, m_1, m_2\}$$

Material
frame

Normal part encodes twist

BENDING ENERGY

$$E_{\text{bend}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \alpha \kappa^2 ds$$

Punish turning the steering wheel

$$\kappa N = T'$$

$$= (T' \cdot T)T + (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2$$

$$= (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2$$

$$:= \omega_1 m_1 + \omega_2 m_2$$

BENDING ENERGY

$$E_{\text{bend}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \alpha(\omega_1^2 + \omega_2^2) ds$$

Punish turning the steering wheel

$$\begin{aligned} \kappa N &= T' \\ &= (T' \cdot T)T + (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \\ &= (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \\ &:= \omega_1 m_1 + \omega_2 m_2 \end{aligned}$$

TWISTING ENERGY

$$E_{\text{twist}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \beta m^2 ds$$

Punish non-tangent change in material frame

$$\begin{aligned} m &:= m'_1 \cdot m_2 \\ &= \frac{d}{dt}(m_1 \cdot m_2) - m_1 \cdot m'_2 \\ &= -m_1 \cdot m'_2 \end{aligned}$$

TWISTING ENERGY

$$E_{\text{twist}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \beta m^2 ds$$

Punish non-tangent change in material frame

Swapping m_1 and m_2 does
not affect E_{twist} !

WHICH BASIS TO USE

THERE IS MORE THAN ONE WAY TO FRAME A CURVE

RICHARD L. BISHOP

The Frenet frame of a 3-times continuously differentiable (that is, C^3) non-degenerate curve in euclidean space has long been the standard vehicle for analysing properties of the curve invariant under euclidean motions. For arbitrary moving frames, that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the curve parameter in terms of the frame itself, and due to orthonormality the coefficient matrix is always skew-symmetric. Thus it generally has three nonzero entries. The Frenet frame gains part of its special significance from the fact that one of the three derivatives is always zero. Another feature of the Frenet frame is that it is *adapted* to the curve: the members are either tangent to or perpendicular to the curve. It is the purpose of this paper to show that there are other frames which have these same advantages and to compare them with the Frenet frame.

1. Relatively parallel fields. We say that a normal vector field M along a curve is *relatively parallel* if its derivative is tangential. Such a field turns only whatever amount is necessary for it to remain normal, so it is as close to being parallel as possible without losing normality. Since its derivative is perpendicular to it, a relatively parallel normal field has constant length. Such fields occur classically in the discussion of curves which are said to be parallel to the given curve. Indeed, if

BISHOP FRAME

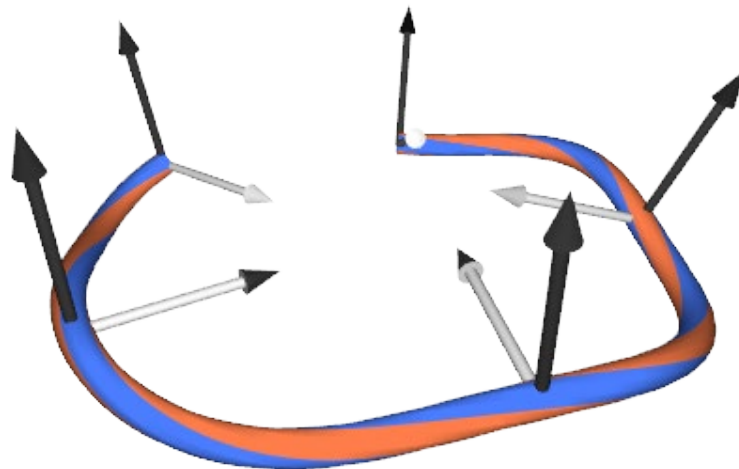
$$T' = \Omega \times T$$

$$u' = \Omega \times u$$

$$v' = \Omega \times v$$

$$u' \cdot v \equiv 0$$

No twist
("parallel
transport")



BISHOP FRAME

$$T' = \Omega \times T$$

$$u' = \Omega \times u$$

$$v' = \Omega \times v$$

$$\Omega := \kappa B \text{ (“curvature binormal”)}$$

Darboux vector

$$u' \cdot v \equiv 0$$

No twist
 (“parallel
 transport”)

CURVE-ANGLE REPRESENTATION

$$m_1 = u \cos \theta + v \sin \theta$$

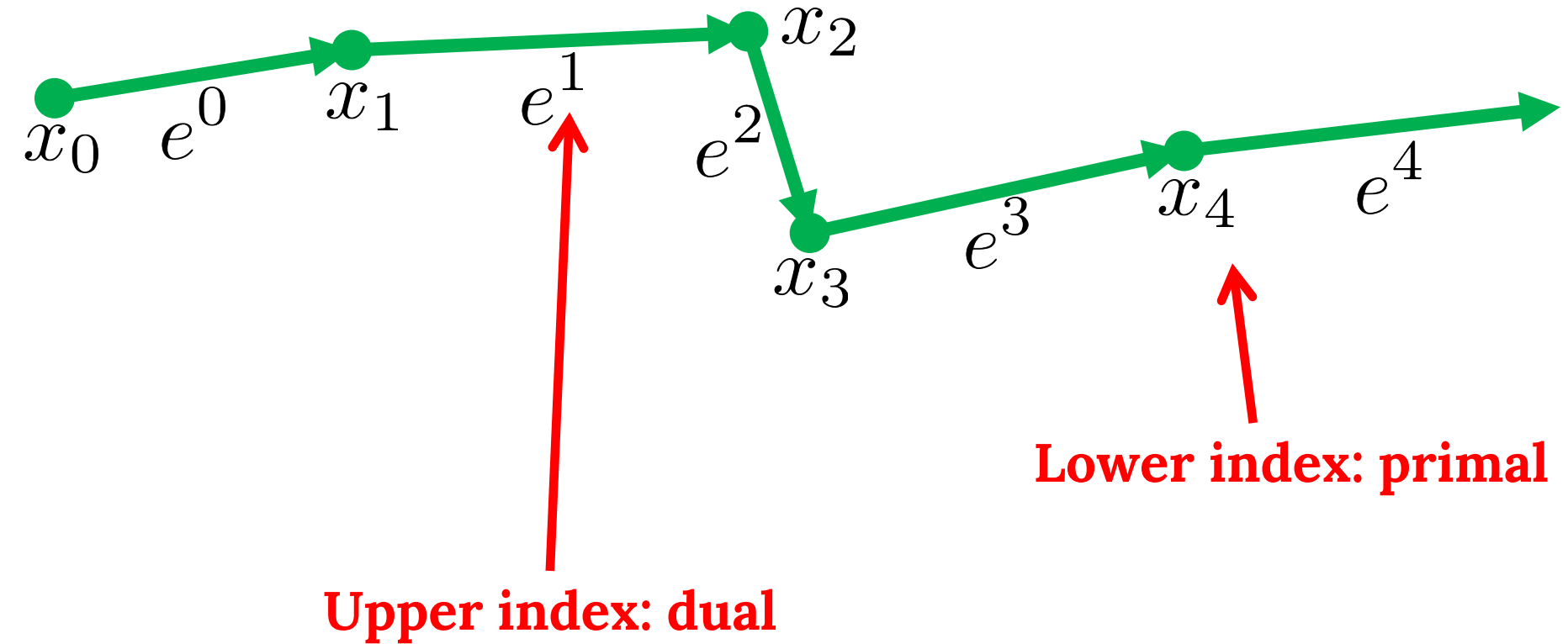
$$m_2 = -u \sin \theta + v \cos \theta$$

$$E_{\text{twist}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \beta(\theta')^2 ds$$

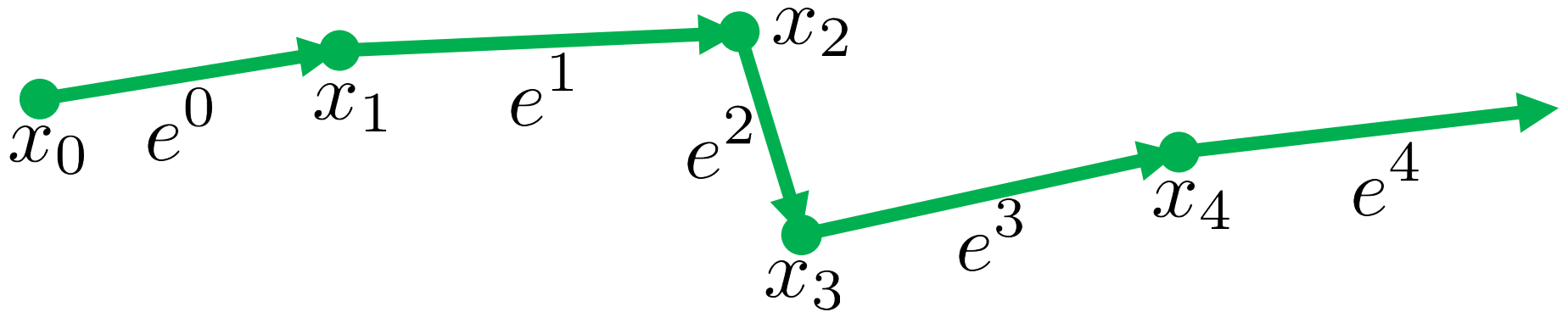
Degrees of freedom for elastic energy:

- Shape of curve
- Twist angle θ

DISCRETE KIRCHOFF RODS



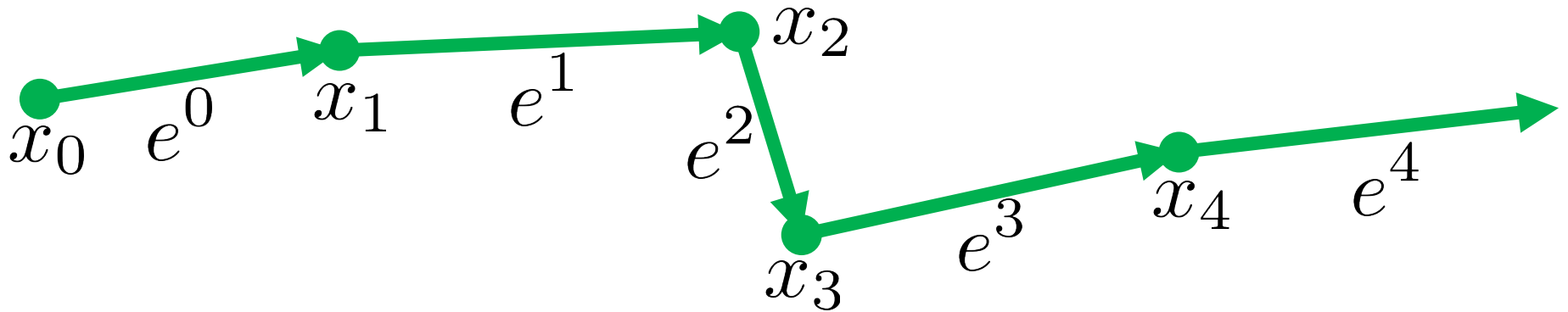
DISCRETE KIRCHOFF RODS



$$T^i := \frac{e^i}{\|e^i\|}$$

Tangent unambiguous on edge

DISCRETE KIRCHOFF RODS



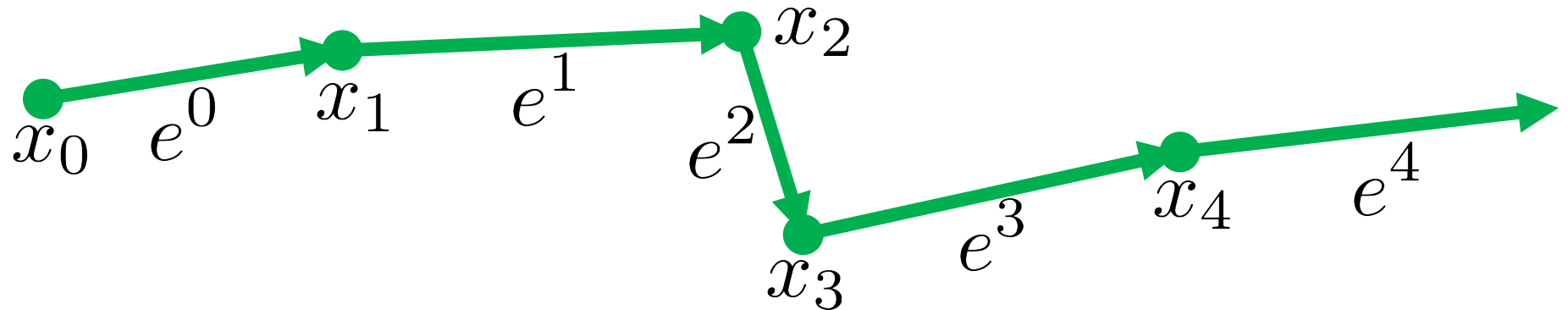
$$\kappa_i := 2 \tan \frac{\phi_i}{2}$$

Turning angle

**Yet another
curvature!**

Integrated curvature

DISCRETE KIRCHHOFF RODS



$$\kappa_i := 2 \tan \frac{\phi_i}{2}$$

Yet another
curvature!

$$(\kappa B)_i := \frac{2e^{i-1} \times e^i}{\|e^{i-1}\| \|e^i\| + e^{i-1} \cdot e^i}$$

Orthogonal to osculating plane,
norm κ_i

Darboux vector

BENDING ENERGY

$$\begin{aligned} E_{\text{bend}}(\Gamma) &:= \frac{\alpha}{2} \sum_i \left(\frac{(\kappa B)_i}{\ell_i/2} \right)^2 \frac{\ell_i}{2} \\ &= \alpha \sum_i \frac{\|(\kappa B)_i\|^2}{\ell_i} \end{aligned}$$

Can extend for
natural bend

Convert to pointwise and integrate

DISCRETE PARALLEL TRANSPORT

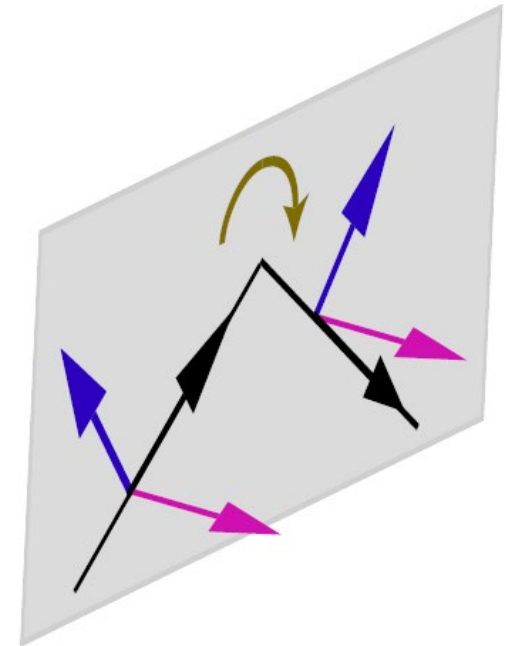
$$P_i(T^{i-1}) = T^i$$

$$P_i(T^{i-1} \times T^i) = T^{i-1} \times T^i$$

- Map tangent to tangent
- Preserve binormal
- Orthogonal

$$u^i = P_i(u^{i-1})$$

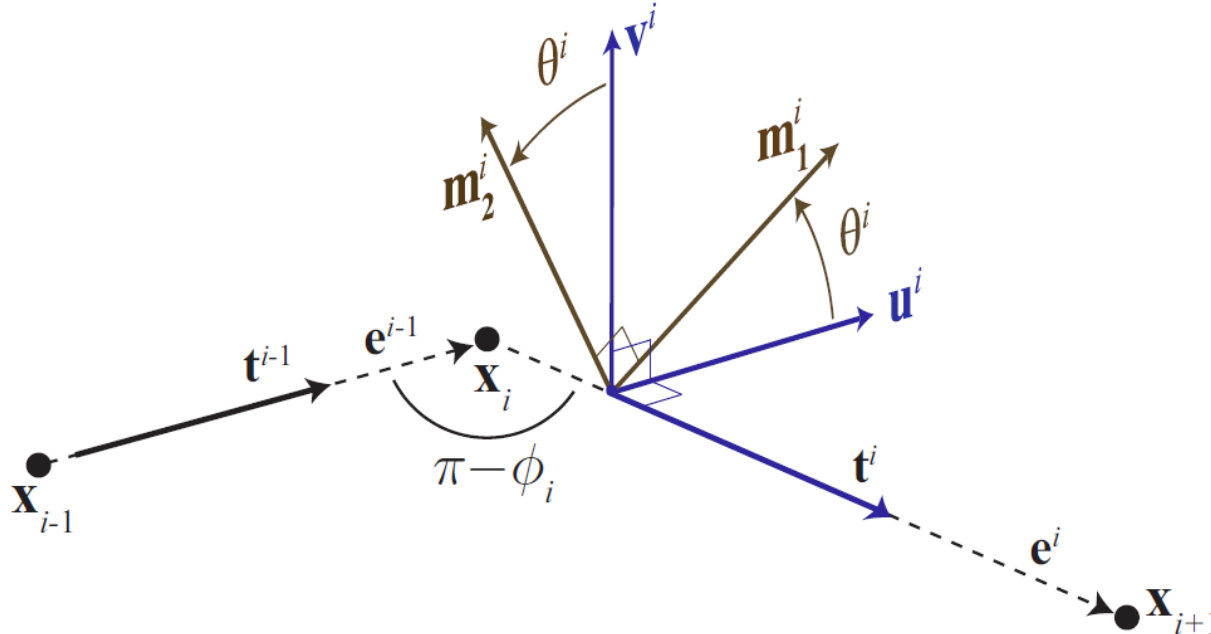
$$v^i = T^i \times u^i$$




DISCRETE MATERIAL FRAME

$$m_1^i = u^i \cos \theta^i + v^i \sin \theta^i$$

$$m_2^i = -u^i \sin \theta^i + v^i \cos \theta^i$$



DISCRETE TWISTING ENERGY

$$E_{\text{twist}}(\Gamma) := \beta \sum_i \frac{(\theta^i - \theta^{i-1})^2}{\ell_i}$$


Note θ_0 can be arbitrary

SIMULATION

`\omit{physics}`

Worth reading!

EXTENSION AND SPEEDUP

Discrete Viscous Threads

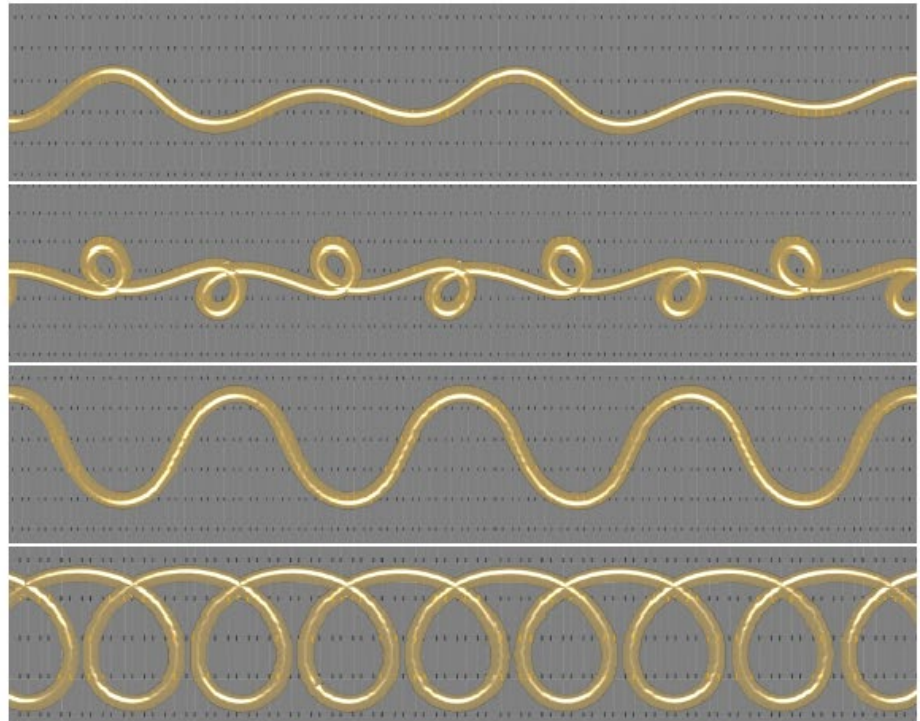
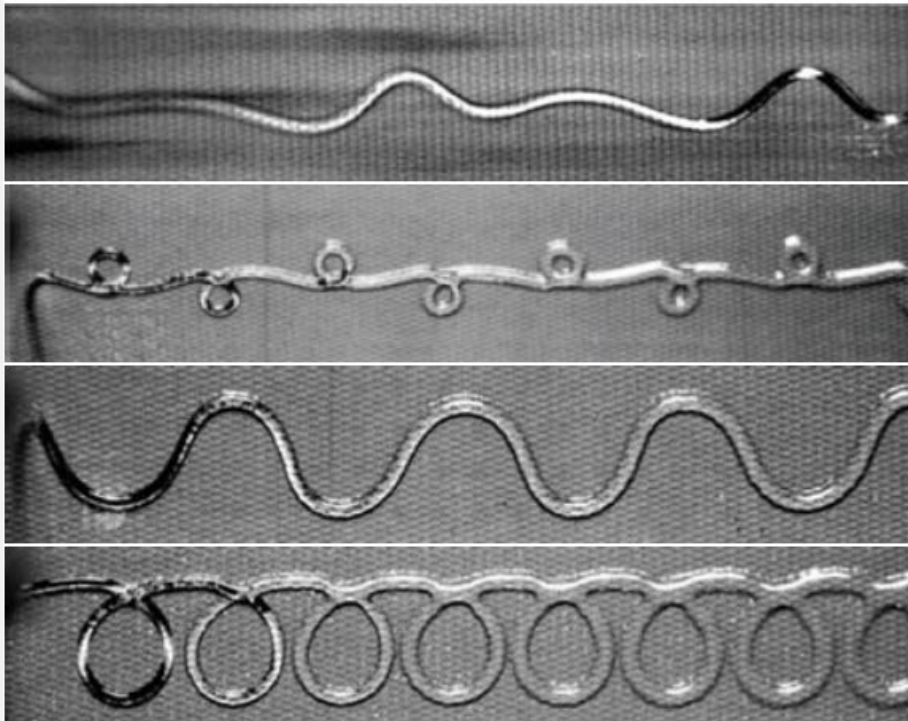
Miklós Bergou
Columbia University

Basile Audoly
UPMC Univ. Paris 06 & CNRS

Etienne Vouga
Columbia University

Max Wardetzky
Universität Göttingen

Eitan Grinspun
Columbia University



EXTENSION AND SPEEDUP

Discrete Viscous Threads

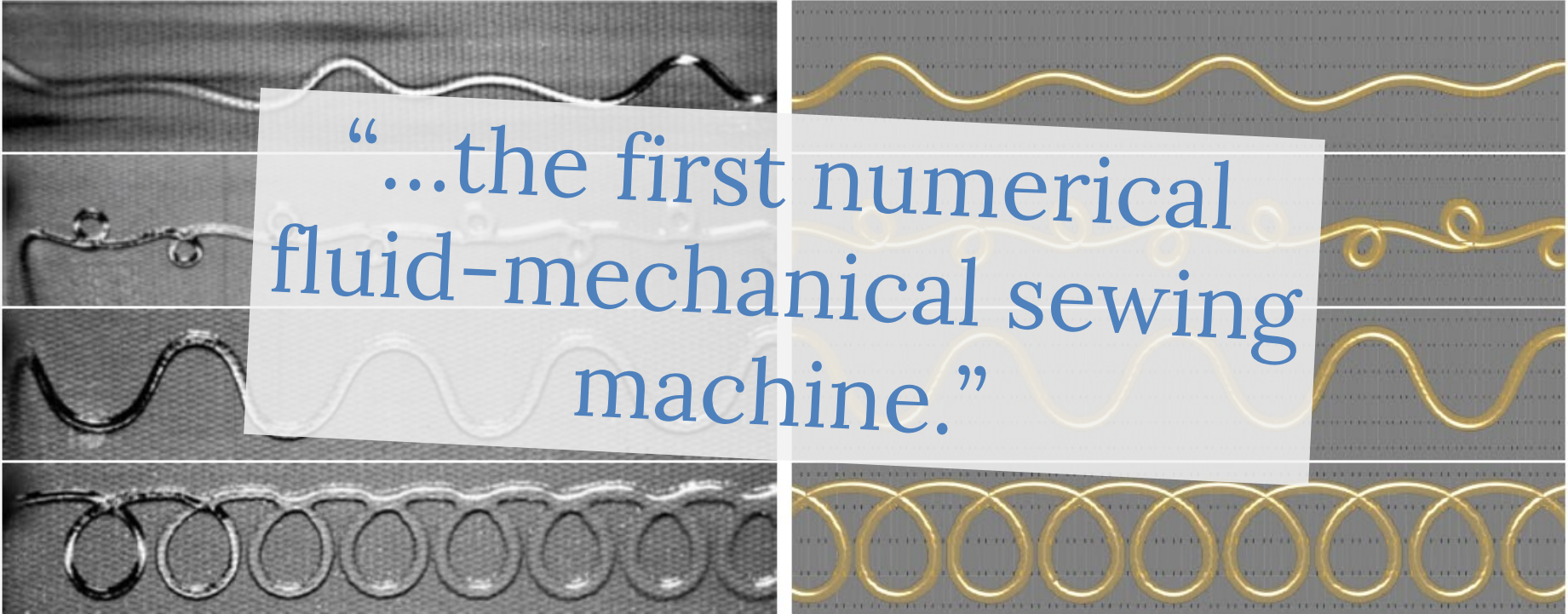
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Columbia University



“...the first numerical
fluid-mechanical sewing
machine.”

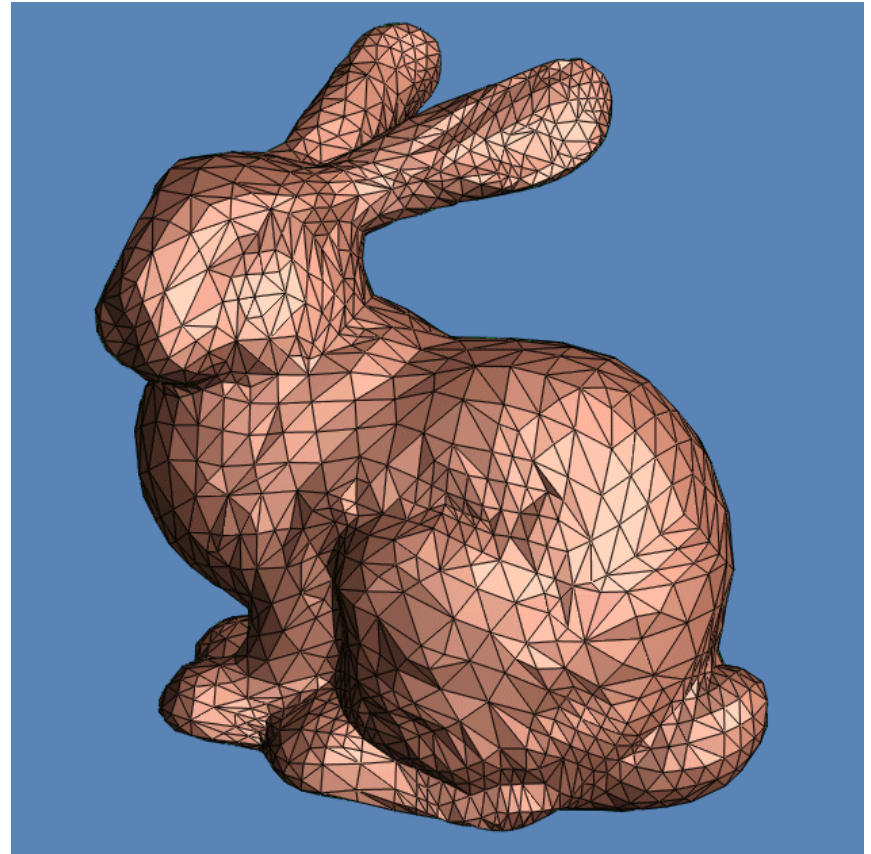
Three different curvature discretizations

$$\theta \qquad 2 \sin \frac{\theta}{2} \qquad 2 \tan \frac{\theta}{2}$$

Easy theoretical object,
hard to use.

$$\frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

NEXT



<http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg>
<http://www.stat.washington.edu/wxs/images/BUNMID.gif>

Surfaces