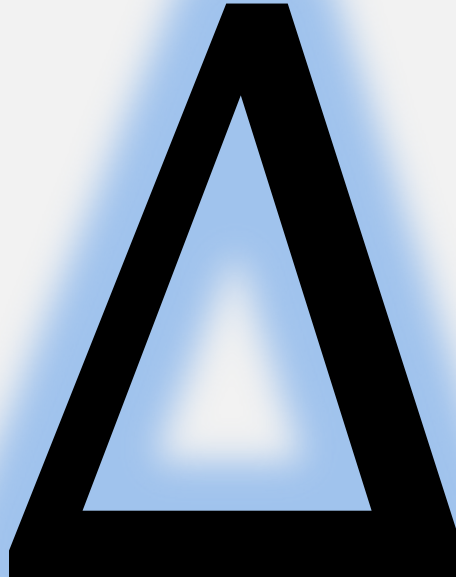


IFT 6113

DISCRETE LAPLACIAN

<http://tiny.cc/ift6113>



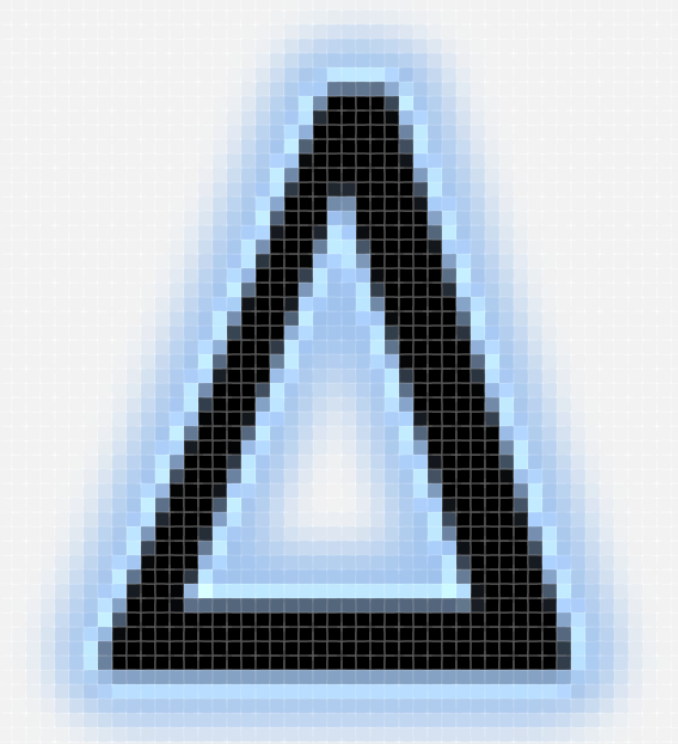
Mikhail Bessmeltsev

Some slides by Justin Solomon

IFT 6113

DISCRETE LAPLACIAN

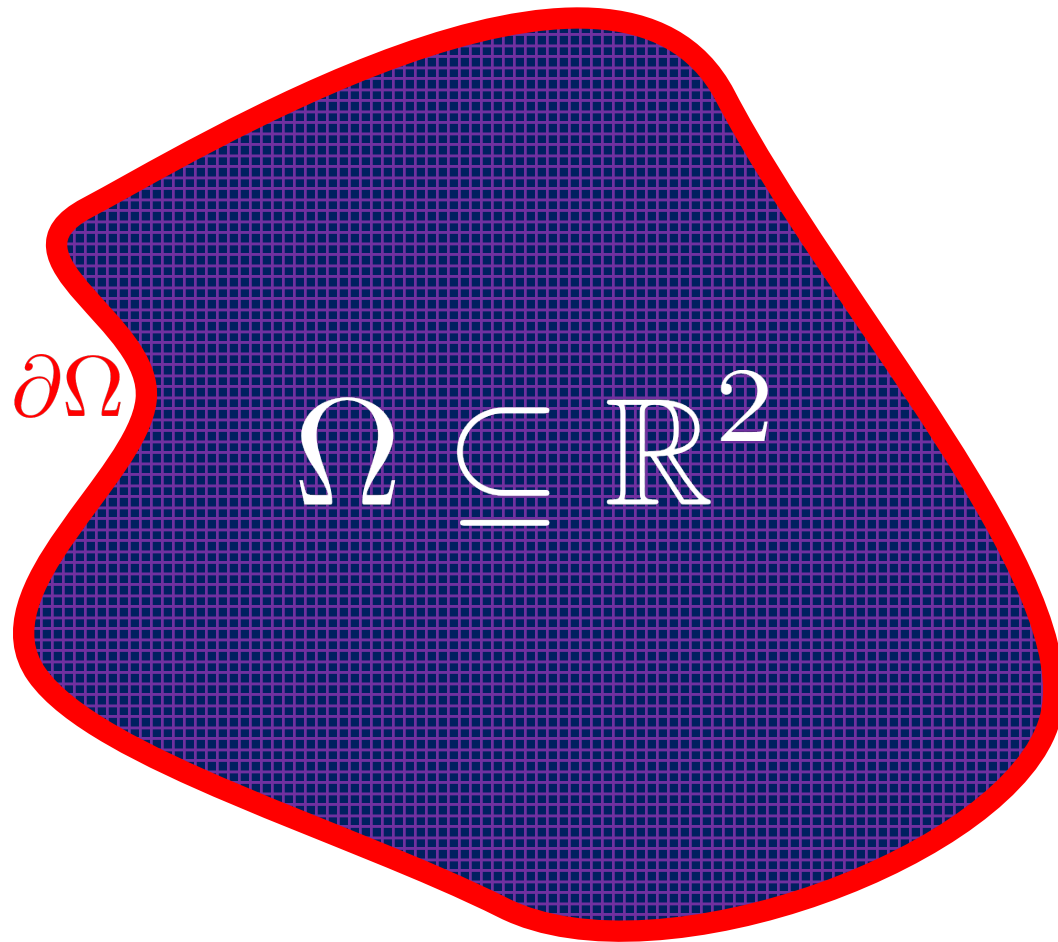
<http://tiny.cc/ift6113>



Mikhail Bessmeltsev

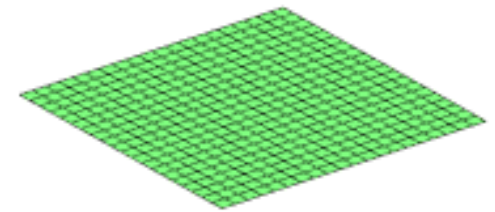
Recall:

Planar Region



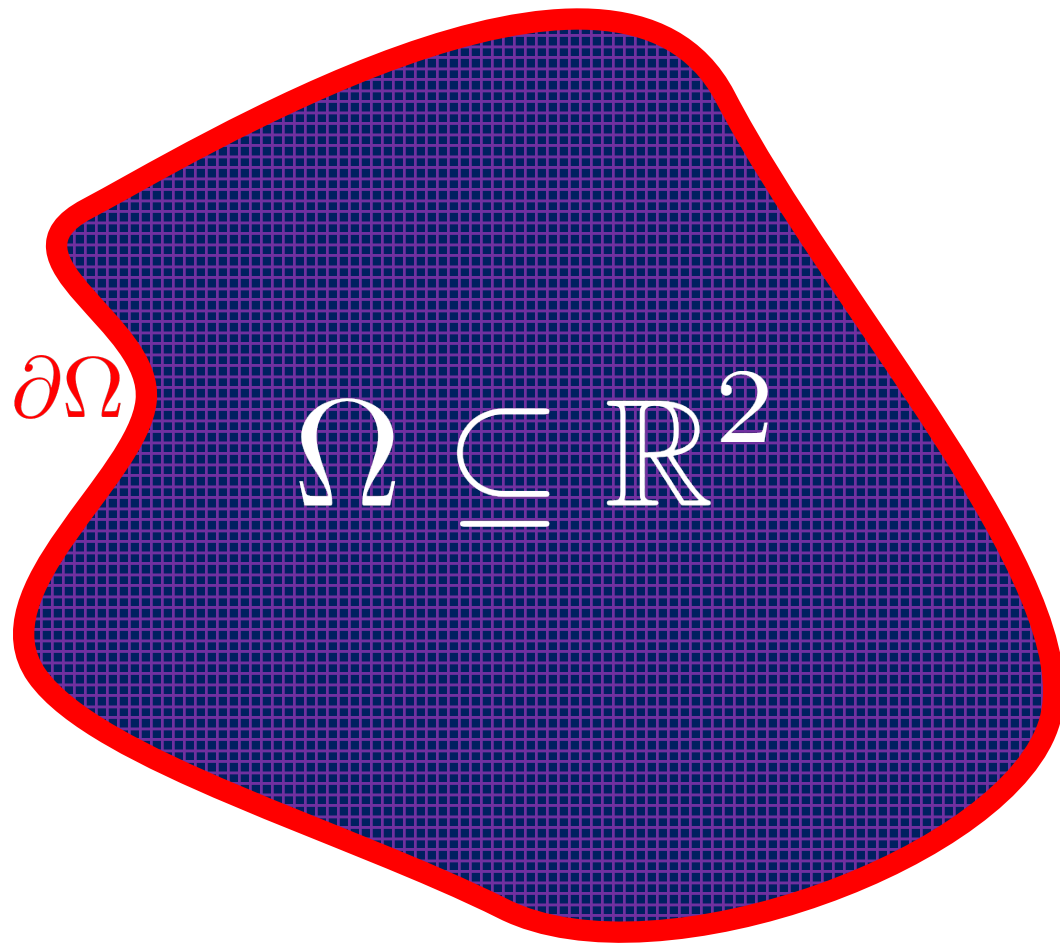
Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$
$$\Delta := \sum_i \frac{\partial^2}{\partial x_i^2}$$



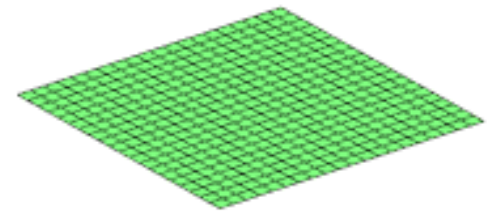
Recall:

Planar Region



How to
discretize on a
mesh?

$$\Delta := \sum_i \frac{\partial^2}{\partial x_i^2}$$



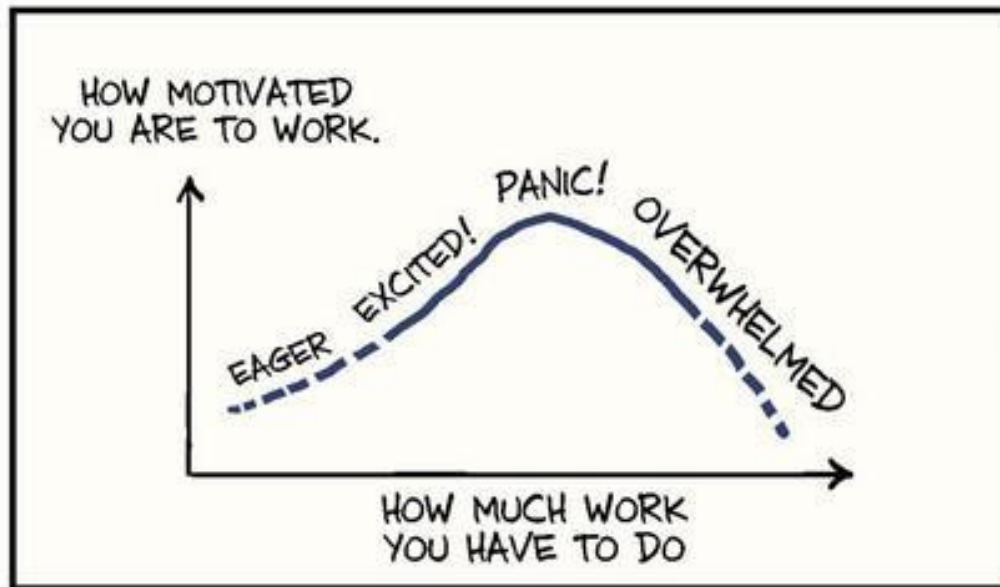
Discretizing the Laplacian

$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right)$$

?!

BRACE YOURSELVES

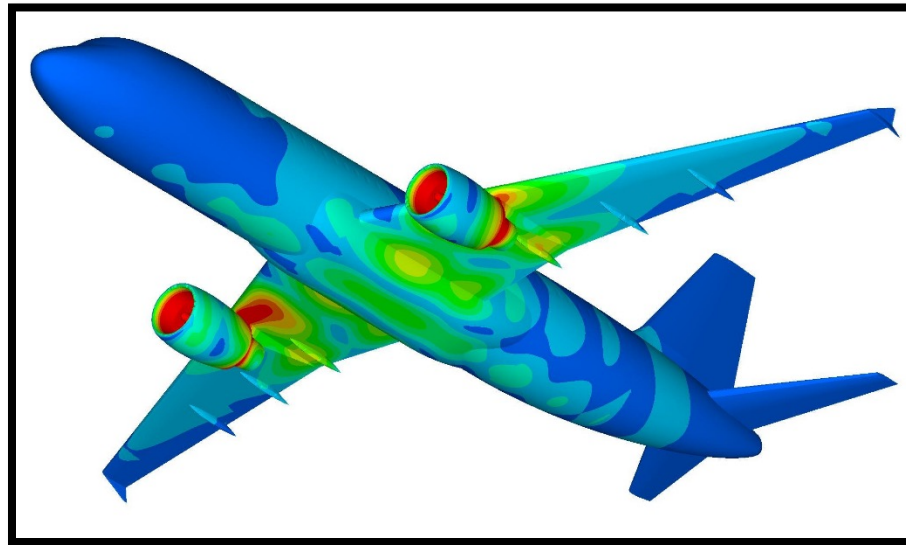
- A2 out, due **Oct 21st**
- Project proposal due: ~~Oct 20th~~ **Oct 25th**
- Paper title due: **Oct 29th**



Today's Approach

First-order Galerkin

Finite element method (FEM)

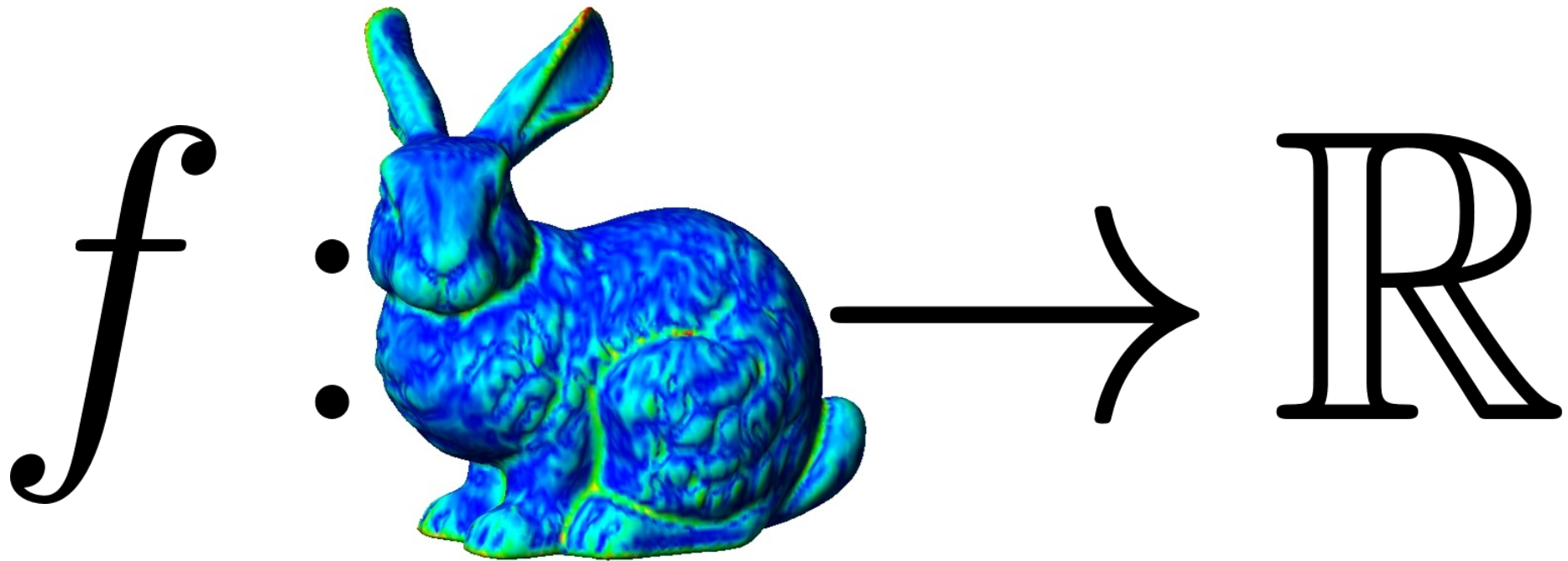


TODO: **Discretize**

- ❑ A function on a mesh
- ❑ Gradient of a function
- ❑ Laplacian operator

Recall:

Scalar Functions

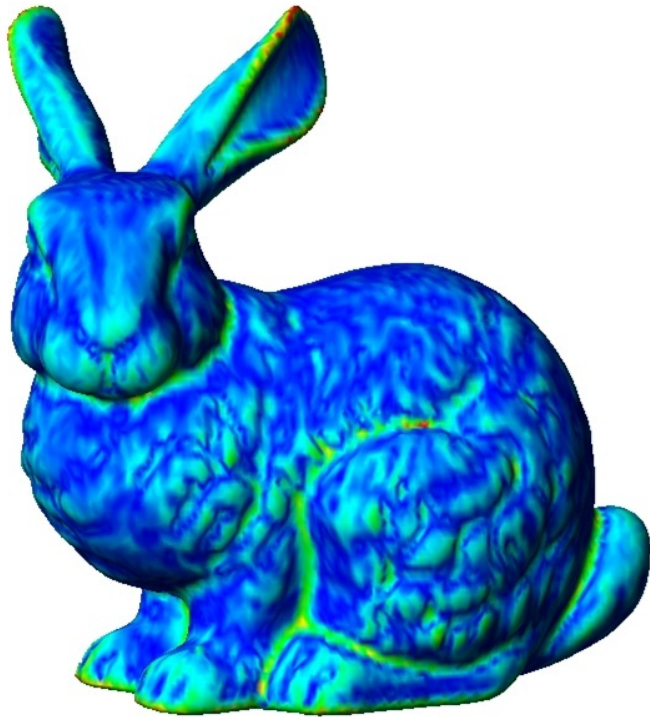


http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

Map points to real numbers

Recall:

Discrete Scalar Functions



$$f \in \mathbb{R}^{|V|}$$

http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

Map vertices to real numbers

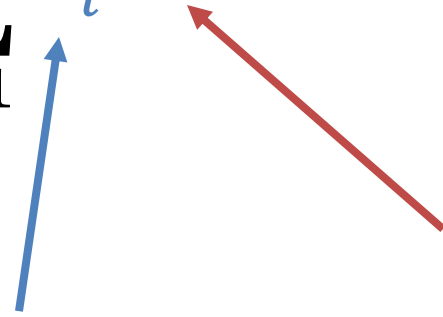
Functional basis

In vector space we can find

$$v = \sum_{i=1}^n v_i e^i$$

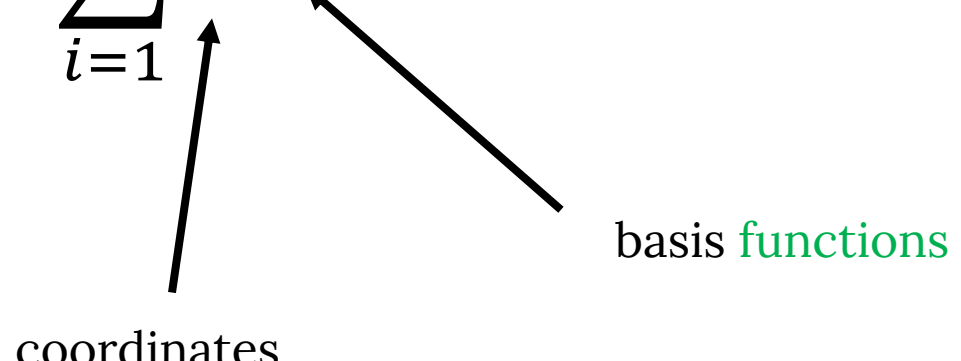
coordinates

basis vectors



Functional basis

Can we do the same for functions?

$$f(x) = \sum_{i=1}^{\infty} f_i e^i(x)$$


The diagram illustrates the components of the functional basis expansion. An arrow points from the word "coordinates" to the coefficient f_i in the summation. Another arrow points from the phrase "basis functions" to the term $e^i(x)$ in the summation.

Functional basis

For example, Fourier basis for $f: [0,1] \rightarrow \mathbb{R}$

$$f(x) = \sum_{i=1}^{\infty} A_i \sin(2\pi n x) + B_i \cos(2\pi n x)$$

Diagram illustrating the components of the Fourier basis expansion:

- The term $A_i \sin(2\pi n x)$ is associated with the label "coordinates" via an arrow.
- The term $B_i \cos(2\pi n x)$ is associated with the label "basis functions" via an arrow.

Functional basis

$$f \in \mathbb{R}^{|V|}$$

$$f(x) = \sum ?$$

First Order Finite Elements

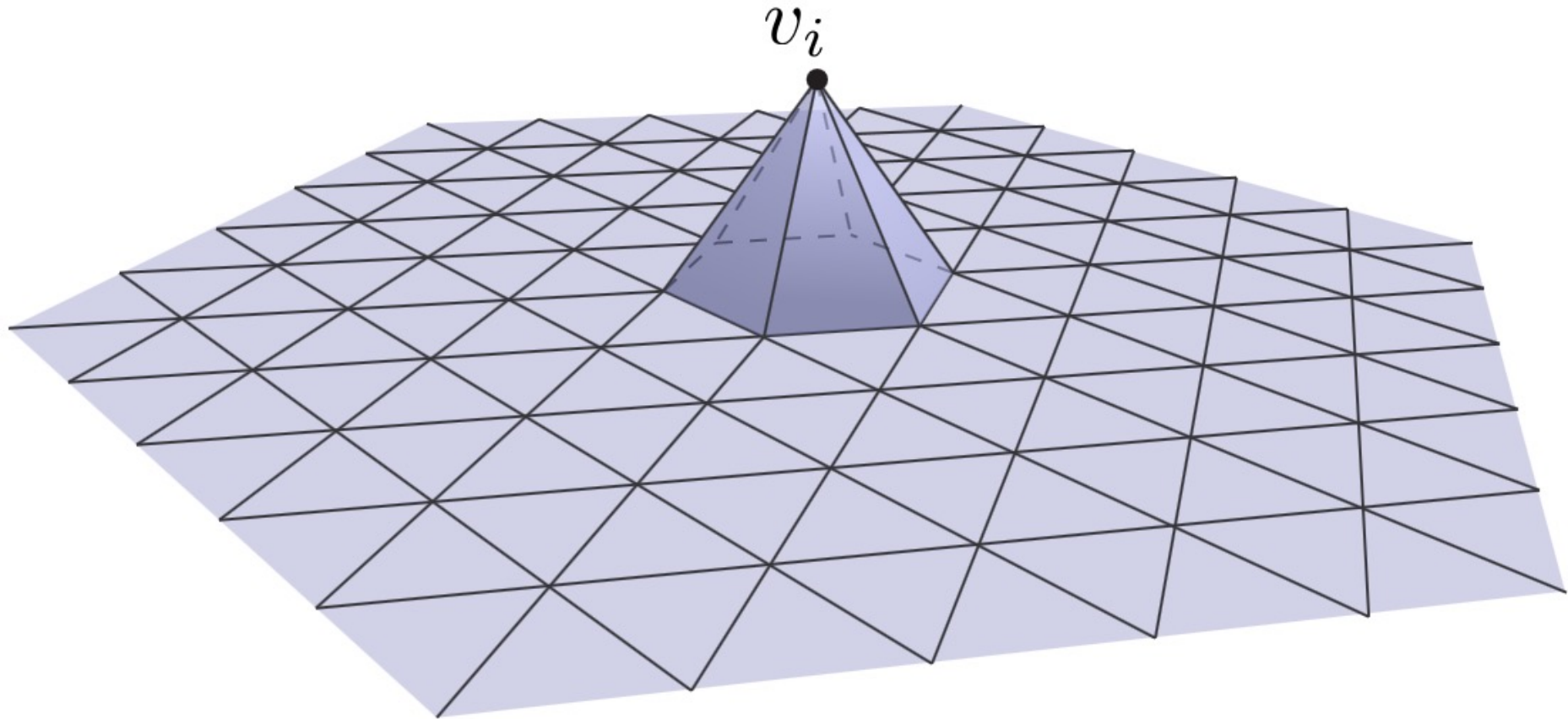
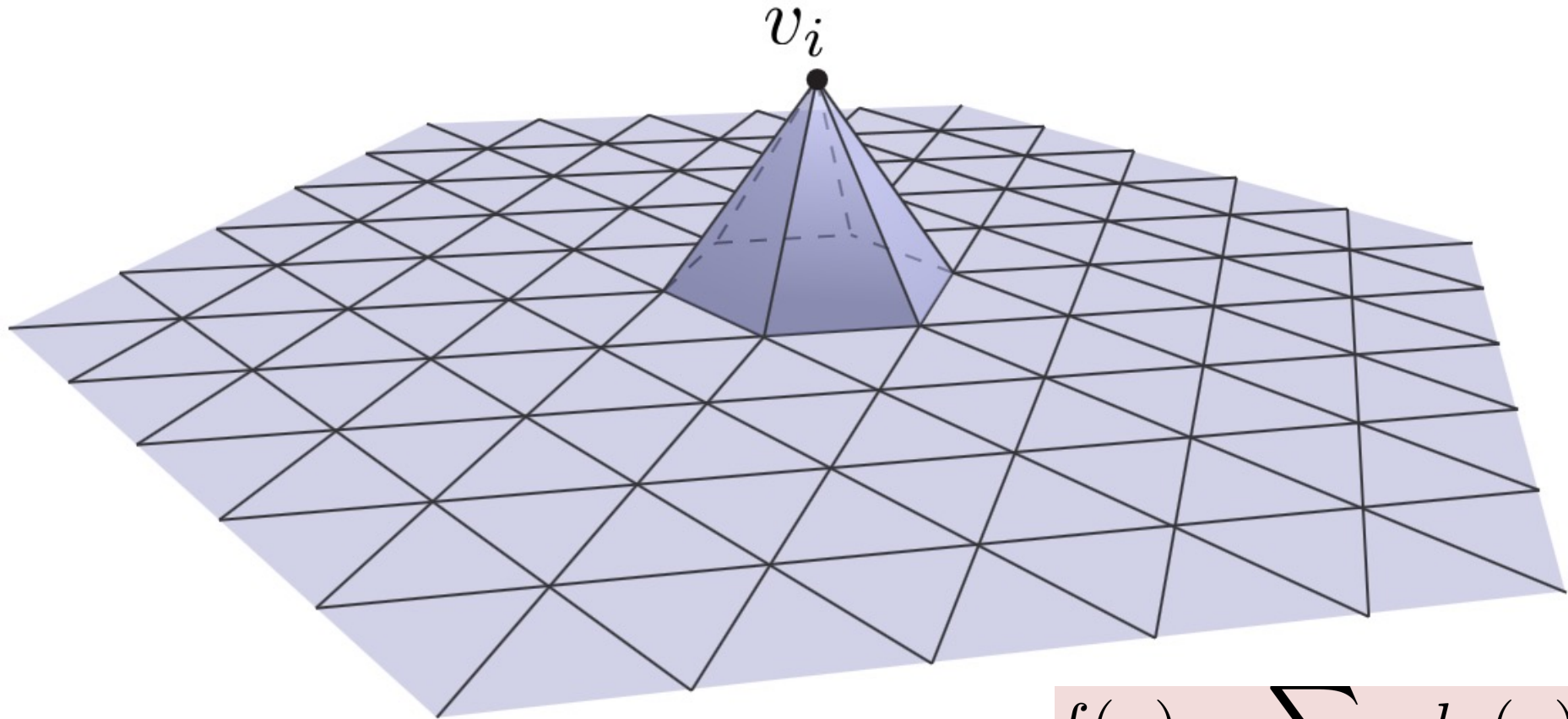


Image courtesy K. Crane, CMU

One “hat function” per vertex

Representing Functions

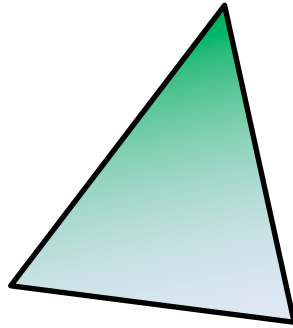


$$f(x) = \sum_i a_i h_i(x)$$
$$a \in \mathbb{R}^{|V|}$$

TODO: **Discretize**

- ~~✓ A function on a mesh~~
- ☐ Gradient of a function
- ☐ Laplacian operator

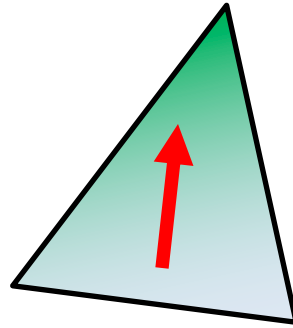
Gradient discretization



$$f(x) = \sum_{i=1}^n a_i h_i(x)$$

$$\nabla f?$$

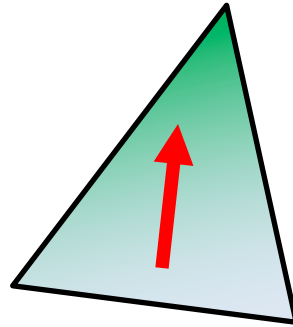
Gradient discretization



$$\nabla f$$

One vector per face

Gradient discretization



$$f(x) = \sum_{i=1}^n a_i h_i(x)$$

$$\nabla f(x) = \sum_{i=1}^n a_i \nabla h_i(x)$$

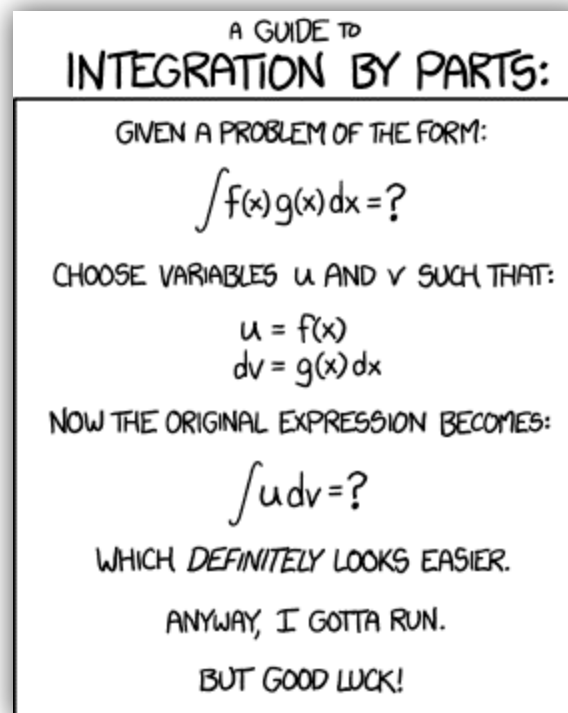
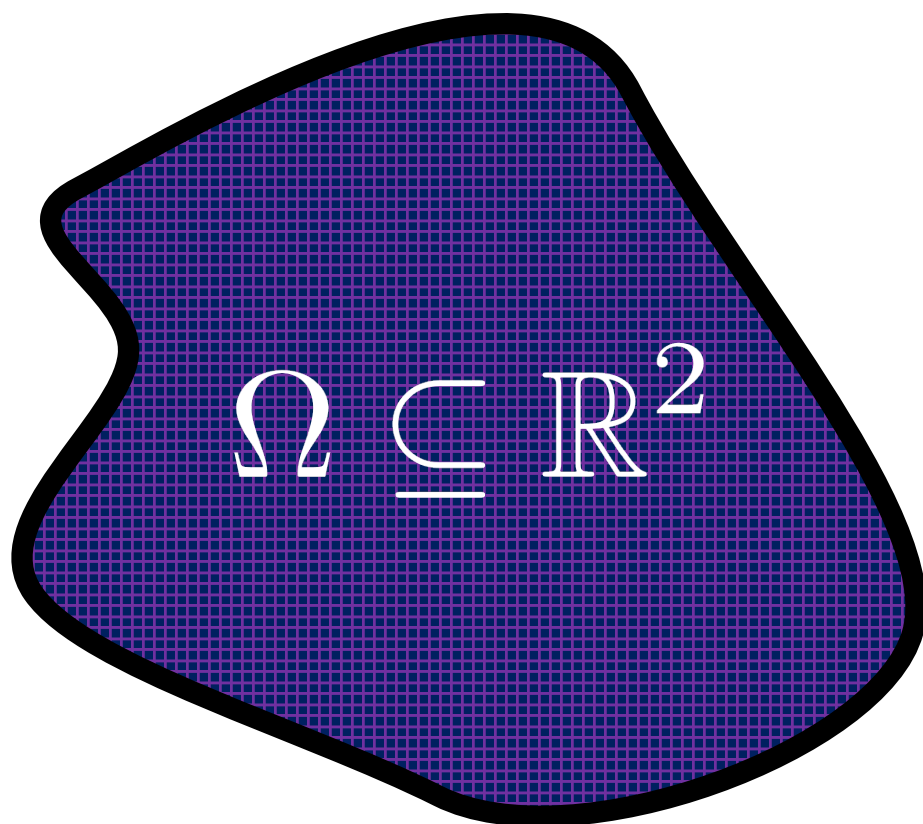
One vector per face

TODO: **Discretize**

- ~~✓ A function on a mesh~~
- ~~✓ Gradient of a function~~
- ☐ Laplacian operator

Integration by Parts to the Rescue

$$\int_{\Omega} f \Delta g \, dA = \text{boundary terms} - \int_{\Omega} \nabla f \cdot \nabla g \, dA$$




Slightly Easier?

$$\int_{\Omega} f \Delta g \, dA = \text{boundary terms} - \int_{\Omega} \nabla f \cdot \nabla g \, dA$$



Laplacian
(second derivative)



Gradient
(first derivative)

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

Overview:

Galerkin FEM Approach

$$g = \Delta f \quad \Rightarrow \quad \langle g, \psi \rangle = \langle \Delta f, \psi \rangle$$

For any **test
function** ψ

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

$$\Rightarrow \int \psi g \, dA = \int \psi \Delta f \, dA$$

For any **test
function** ψ

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

$$\Rightarrow \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

For any **test
function** ψ

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

$$\Rightarrow \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

If we have a function basis $\{\psi_i\}$

$$f = \sum_{i=0}^{\infty} a_i \psi_i$$

$$g = \sum_{i=0}^{\infty} b_i \psi_i$$

Overview:

Galerkin FEM Approach

We only need to test $\psi = \psi_i$

$$\Rightarrow \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

If we have a function basis $\{\psi_i\}$

$$f = \sum_{i=0}^{\infty} a_i \psi_i$$

$$g = \sum_{i=0}^{\infty} b_i \psi_i$$

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

Approximate $f \approx \sum_i a_i \psi_i$ and $g \approx \sum_i b_i \psi_i$

$$\implies \text{Linear system } \sum_i b_i \langle \psi_i, \psi_j \rangle = - \sum_i a_i \langle \nabla \psi_i, \nabla \psi_j \rangle$$

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = - \int (\nabla \psi \cdot \nabla f) \, dA$$

Approximate $f \approx \sum_i a_i \psi_i$ and $g \approx \sum_i b_i \psi_i$

$$\implies \text{Linear system } \sum_i b_i \langle \psi_i, \psi_j \rangle = - \sum_i a_i \langle \nabla \psi_i, \nabla \psi_j \rangle$$

Mass matrix: $M_{ij} := \langle \psi_i, \psi_j \rangle$

Stiffness matrix: $L_{ij} := \langle \nabla \psi_i, \nabla \psi_j \rangle$

$$\implies Mb = La$$

Which
basis?

Important to Note

Not the only way

to approximate the Laplacian operator.

- Divided differences
- Higher-order elements
- Boundary element methods
- Discrete exterior calculus
- ...

L^2 Dual of a Function

Function $f : M \rightarrow \mathbb{R}$



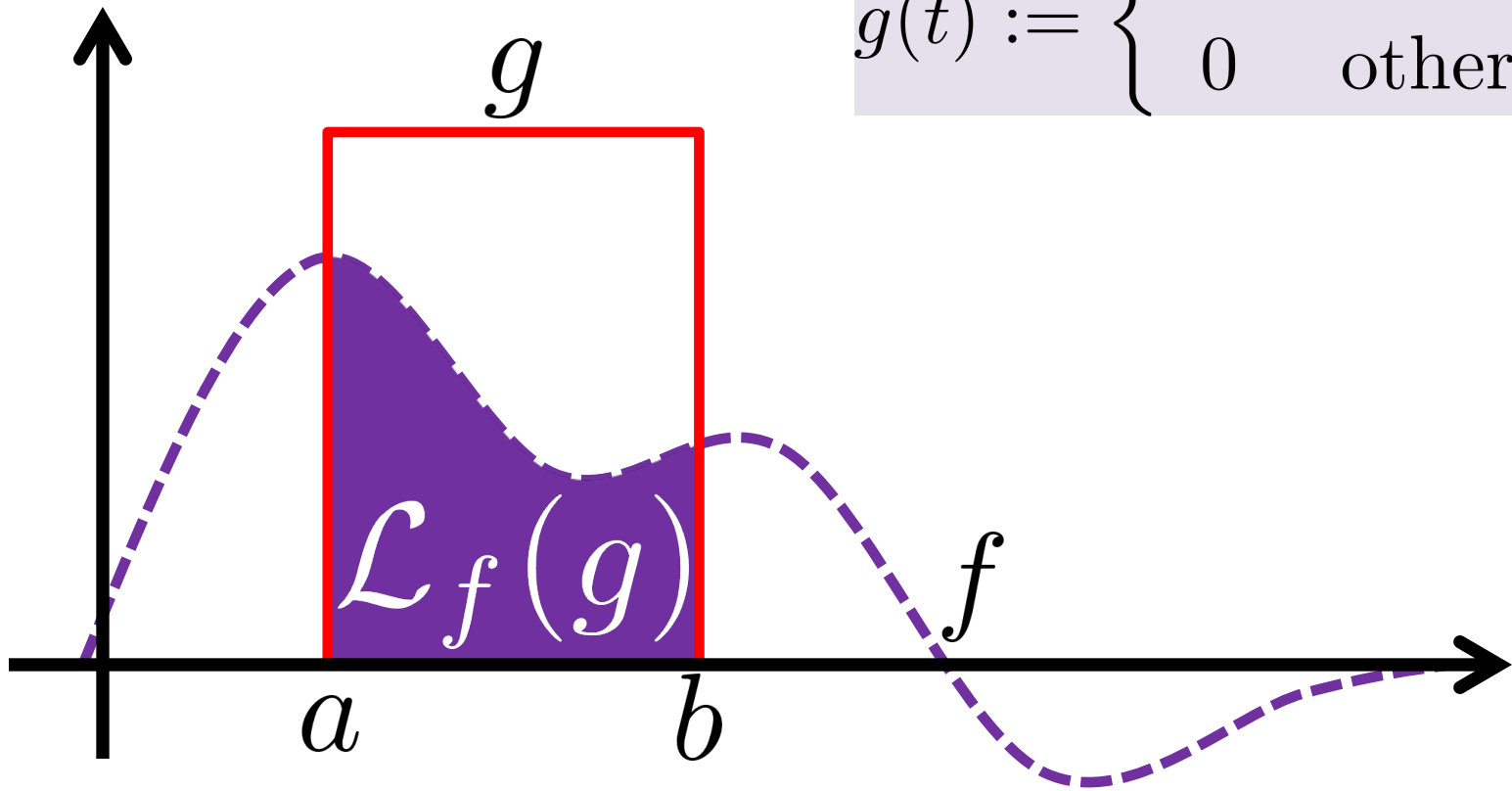
Operator $\mathcal{L}_f : L^2(M) \rightarrow \mathbb{R}$

$$\mathcal{L}_f[g] := \int_M f(x)g(x) dA$$


Test function

Observation

$$g(t) := \begin{cases} 1 & \text{if } t \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$



Can recover function from dual

Dual of Laplacian

Space of test functions (no boundary!):

$$\{g \in L^\infty(M) : g|_{\partial M} \equiv 0\}$$

$$\begin{aligned}\mathcal{L}_{\Delta f}[g] &= \int_M g \Delta f \, dA \\ &= - \int_M \nabla g \cdot \nabla f \, dA\end{aligned}$$

Use Laplacian without evaluating it!

Galerkin's Approach

Choose one of each:

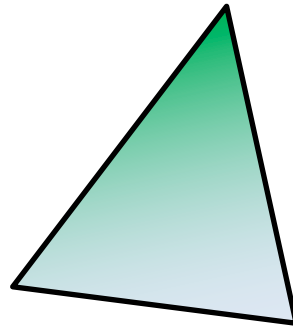
- Function space
- Test functions

Often the same!

One Derivative is Enough

$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

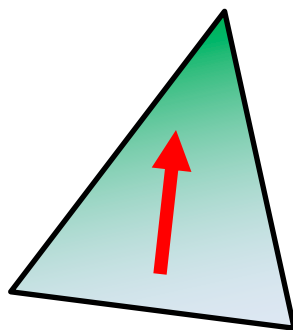
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

*Linear combination of hats
(piecewise linear)*

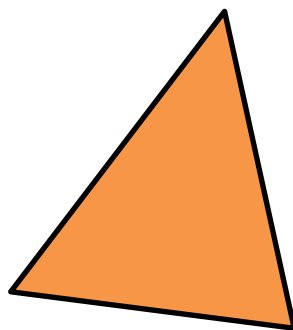
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

One vector per face

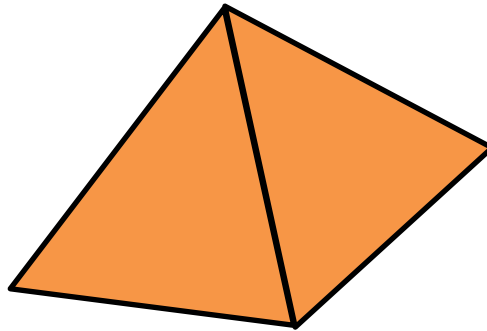
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$

One scalar per face

What Do We Need

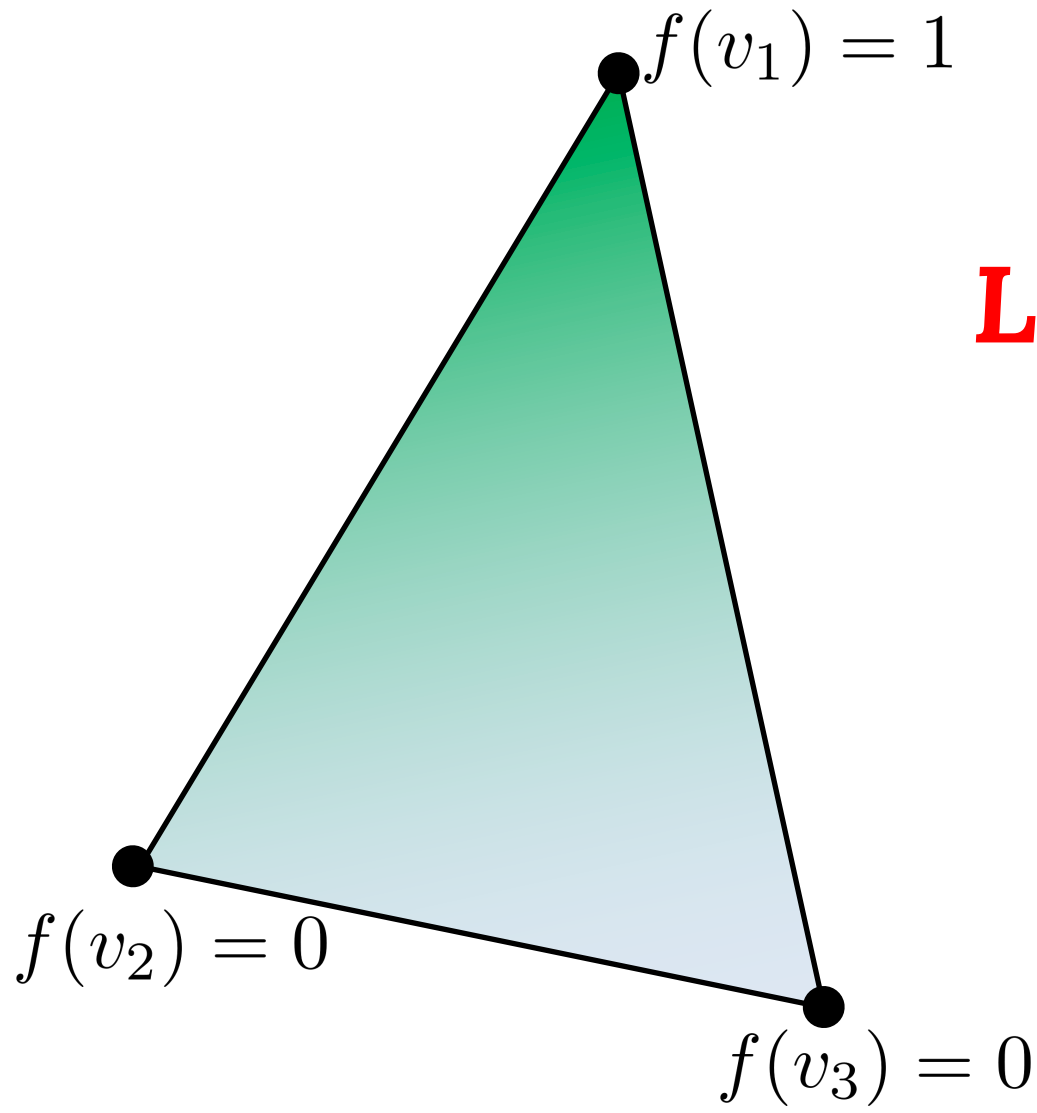


$$\mathcal{L}_{\Delta f}[g] = - \int_M \nabla g \cdot \nabla f \, dA$$



*Sum scalars per face
multiplied by face areas*

Gradient of a Hat Function



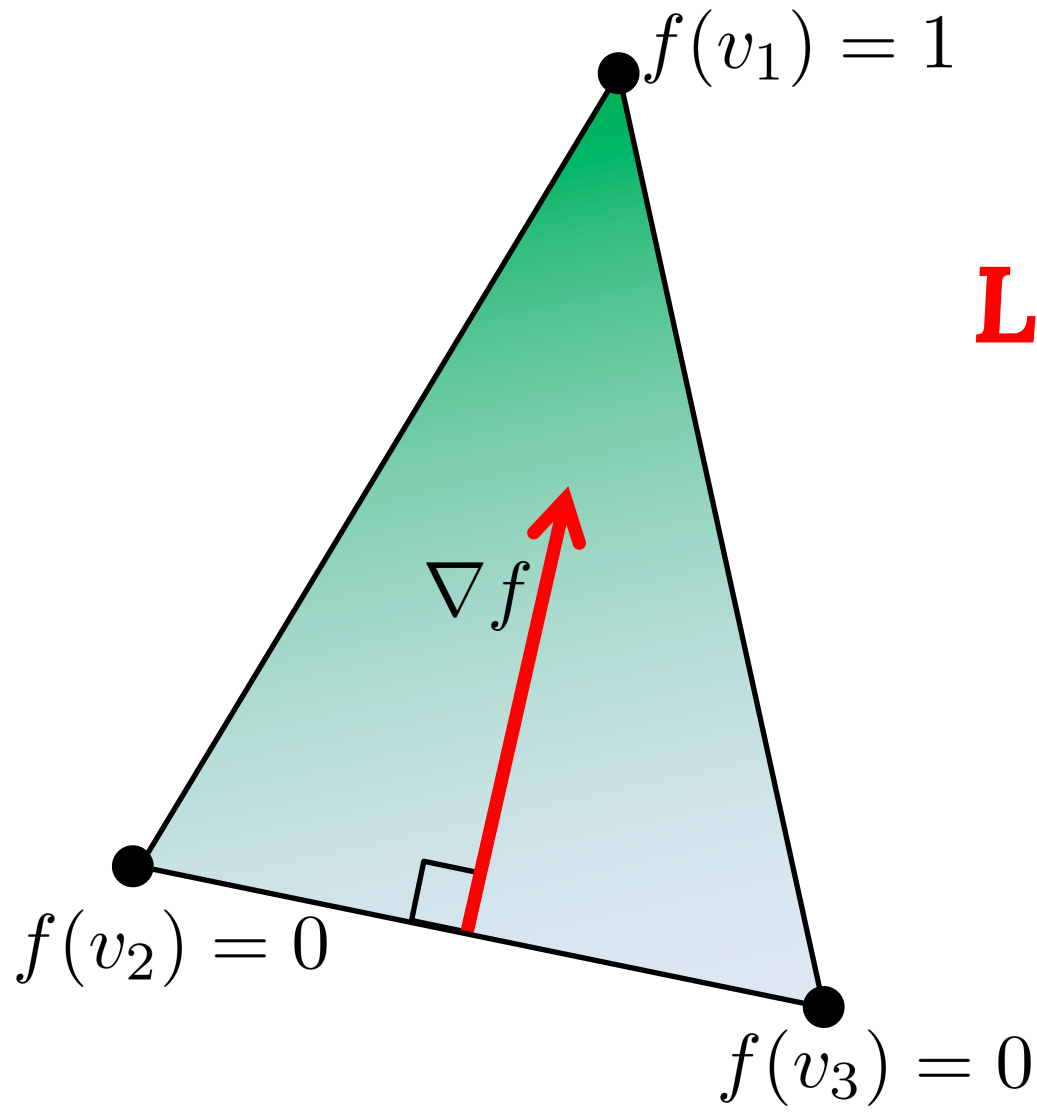
Linear along edges

$$\nabla f \cdot (v_1 - v_3) = 1$$

$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$

Gradient of a Hat Function



Linear along edges

$$\nabla f \cdot (v_1 - v_3) = 1$$

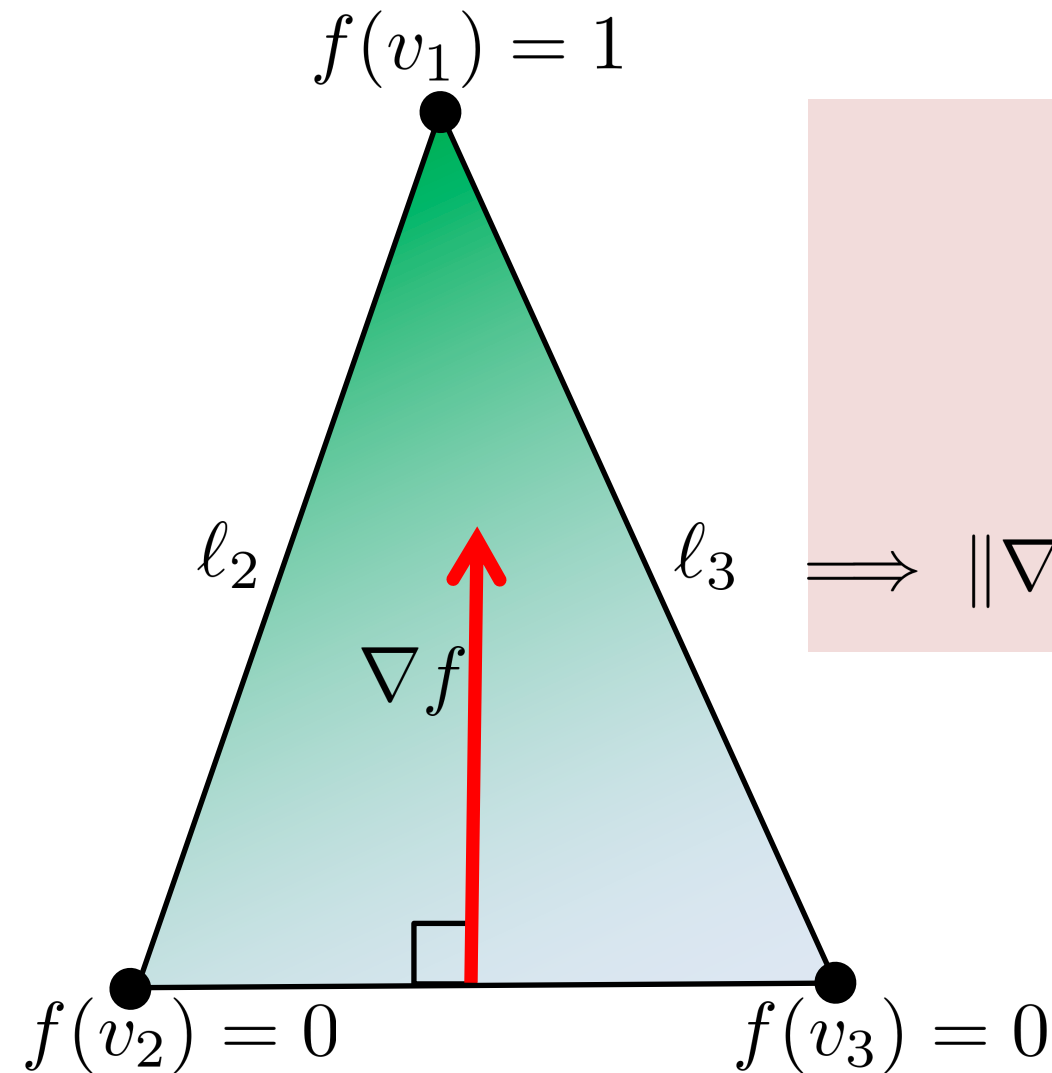
$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$



$$\nabla f \cdot (v_2 - v_3) = 0$$

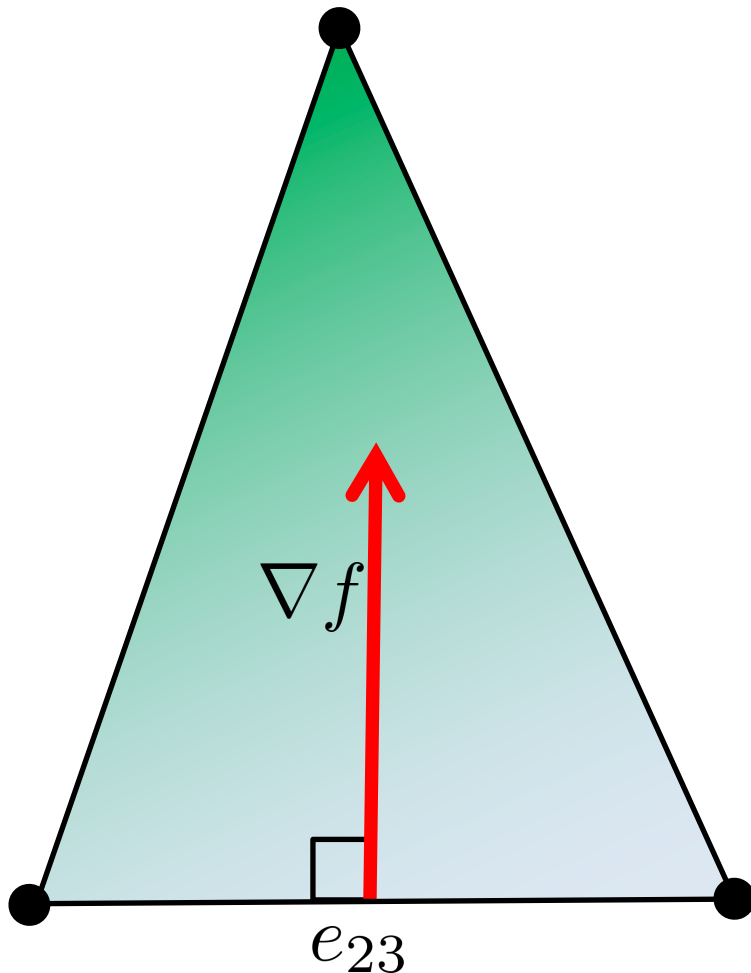
Gradient of a Hat Function



$$\begin{aligned} 1 &= \nabla f \cdot (v_1 - v_3) \\ &= \|\nabla f\| \ell_3 \cos \left(\frac{\pi}{2} - \theta_3 \right) \\ &= \|\nabla f\| \ell_3 \sin \theta_3 \\ \implies \|\nabla f\| &= \frac{1}{\ell_3 \sin \theta_3} = \frac{1}{h} \end{aligned}$$

Gradient of a Hat Function

$$\|\nabla f\| = \frac{1}{\ell_3 \sin \theta_3} = \frac{1}{h}$$

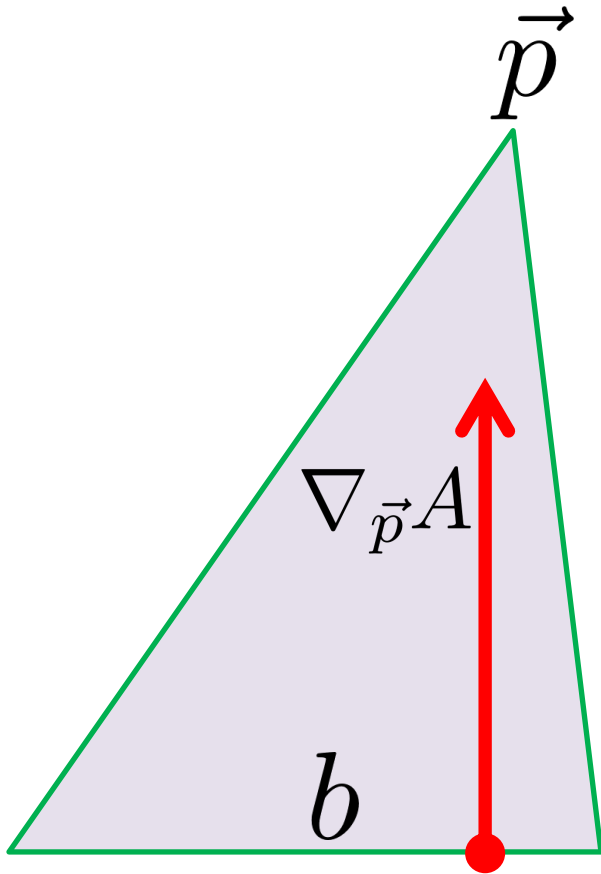


$$\nabla f = \frac{e_{23}^\perp}{2A}$$

*Length of e_{23} cancels
“base” in A*

Recall:

Single Triangle: Complete

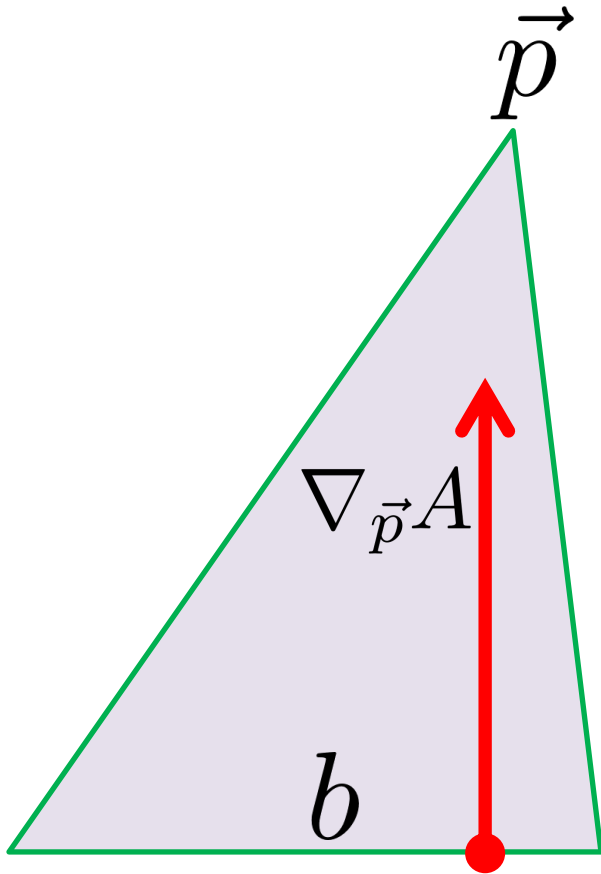


$$\vec{p} = p_n \vec{n} + p_e \vec{e} + p_{\perp} \vec{e}_{\perp}$$
$$A = \frac{1}{2} b \sqrt{p_n^2 + p_{\perp}^2}$$
$$\nabla_{\vec{p}} A = \frac{1}{2} b \vec{e}_{\perp}$$

Similar expression

Recall:

Single Triangle: Complete



$$\vec{p} = p_n \vec{n} + p_e \vec{e} + p_{\perp} \vec{e}_{\perp}$$

$$A = \frac{1}{2} b \sqrt{p_n^2 + p_{\perp}^2}$$

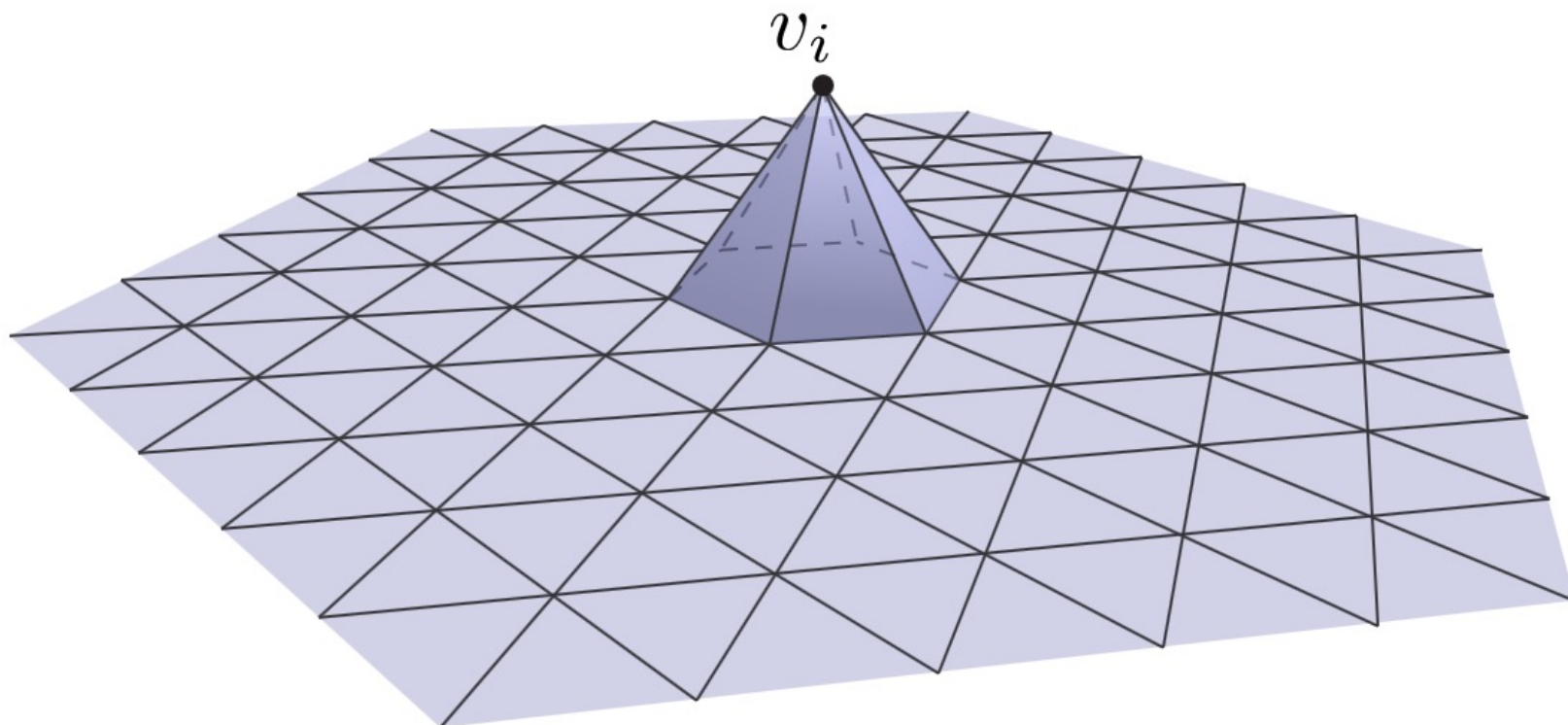
$$\nabla_{\vec{p}} A = \frac{1}{2} b \vec{e}_{\perp}$$

$$\nabla f = \frac{e_{23}^{\perp}}{2A} = \frac{\vec{e}_{\perp}}{h} = \frac{\nabla_{\vec{p}} A}{A}$$

Similar expression

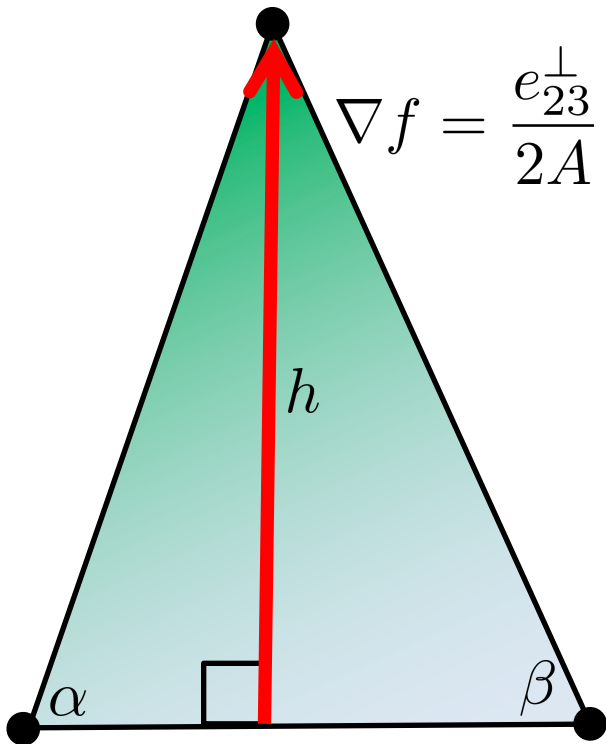
What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = - \int_M \boxed{\nabla g \cdot \nabla f} dA$$



What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = - \int_M \boxed{\nabla g \cdot \nabla f} dA$$



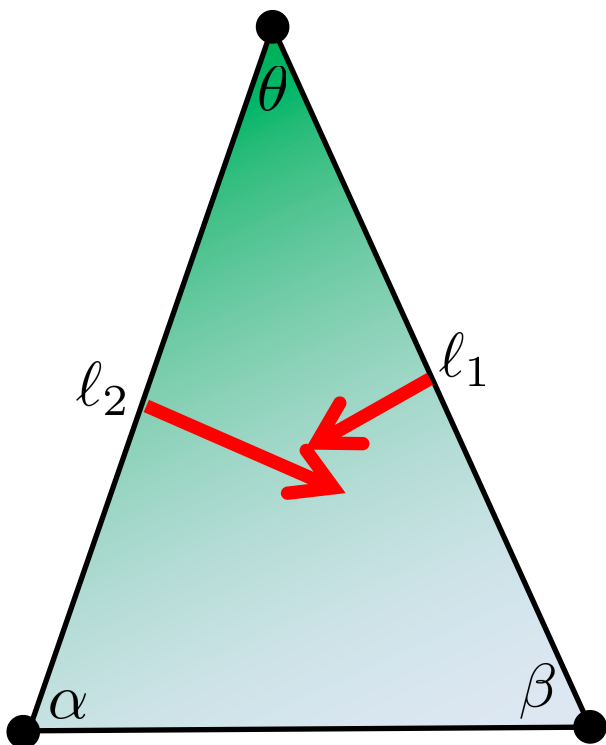
Case 1: Same vertex

$$\begin{aligned} \int_T \langle \nabla f, \nabla f \rangle dA &= A \|\nabla f\|^2 \\ &= \frac{A}{h^2} = \frac{b}{2h} \\ &= \frac{1}{2} (\cot \alpha + \cot \beta) \end{aligned}$$

What We Actually Need

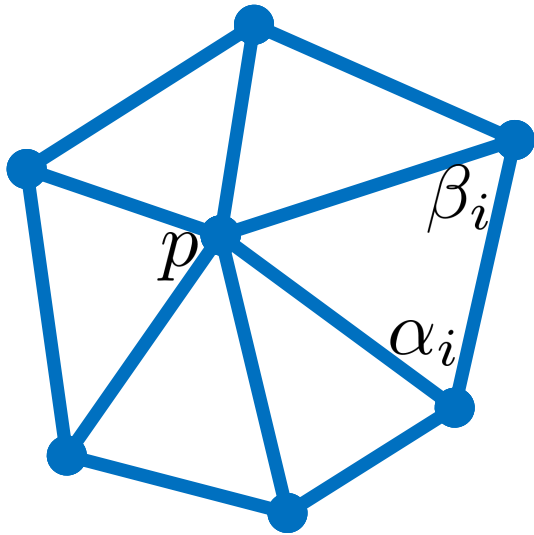
$$\mathcal{L}_{\Delta f}[g] = - \int_M \boxed{\nabla g \cdot \nabla f} dA$$

Case 2: Different vertices

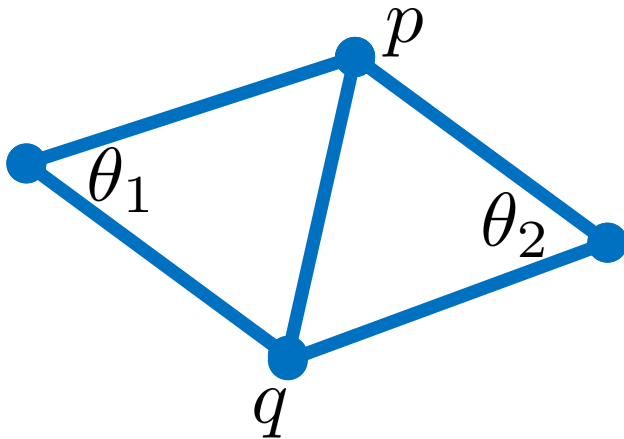


$$\begin{aligned} \int_T \langle \nabla f_\alpha, \nabla f_\beta \rangle dA &= A \langle \nabla f_\alpha, \nabla f_\beta \rangle \\ &= \frac{1}{4A} \langle e_{31}^\perp, e_{12}^\perp \rangle = -\frac{\ell_1 \ell_2 \cos \theta}{4A} \\ &= \frac{-h^2 \cos \theta}{4A \sin \alpha \sin \beta} = \frac{-h \cos \theta}{2b \sin \alpha \sin \beta} \\ &= -\frac{\cos \theta}{2 \sin(\alpha + \beta)} = -\frac{1}{2} \cot \theta \end{aligned}$$

Summing Around a Vertex



$$\langle \nabla h_p, \nabla h_p \rangle = \frac{1}{2} \sum_i (\cot \alpha_i + \cot \beta_i)$$

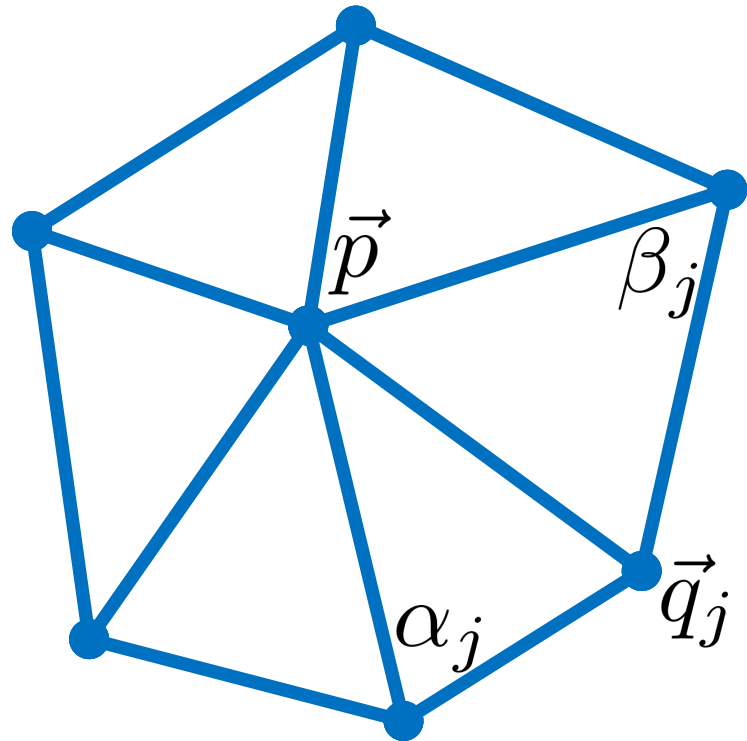


$$\langle \nabla h_p, \nabla h_q \rangle = -\frac{1}{2} (\cot \theta_1 + \cot \theta_2)$$

Recall:

Summing Around a Vertex

$$\nabla_{\vec{p}} A = \frac{1}{2} \sum_j (\cot \alpha_j + \cot \beta_j) (\vec{p} - \vec{q}_j)$$

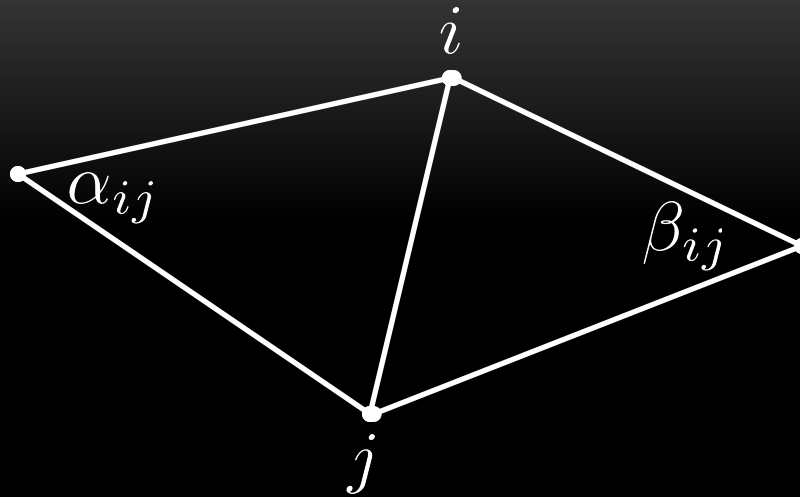


$$\nabla_{\vec{p}} A = \frac{1}{2} ((\vec{p} - \vec{r}) \cot \alpha + (\vec{p} - \vec{q}) \cot \beta)$$

Same weights up
to sign!

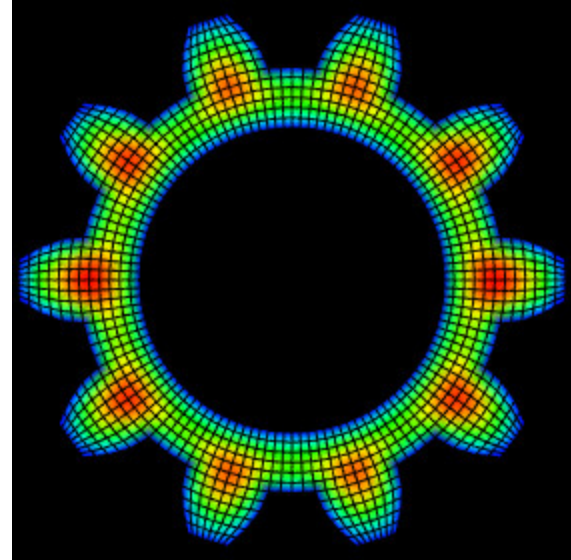
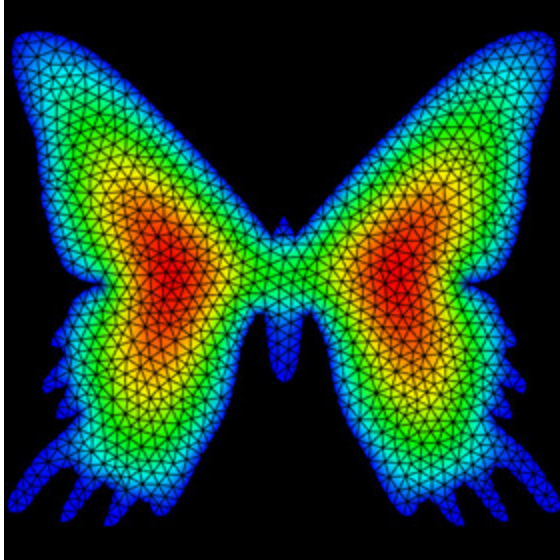
THE COTANGENT LAPLACIAN

$$L_{ij} = \begin{cases} \frac{1}{2} \sum_{i \sim k} (\cot \alpha_{ik} + \cot \beta_{ik}) & \text{if } i = j \\ -\frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}) & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$



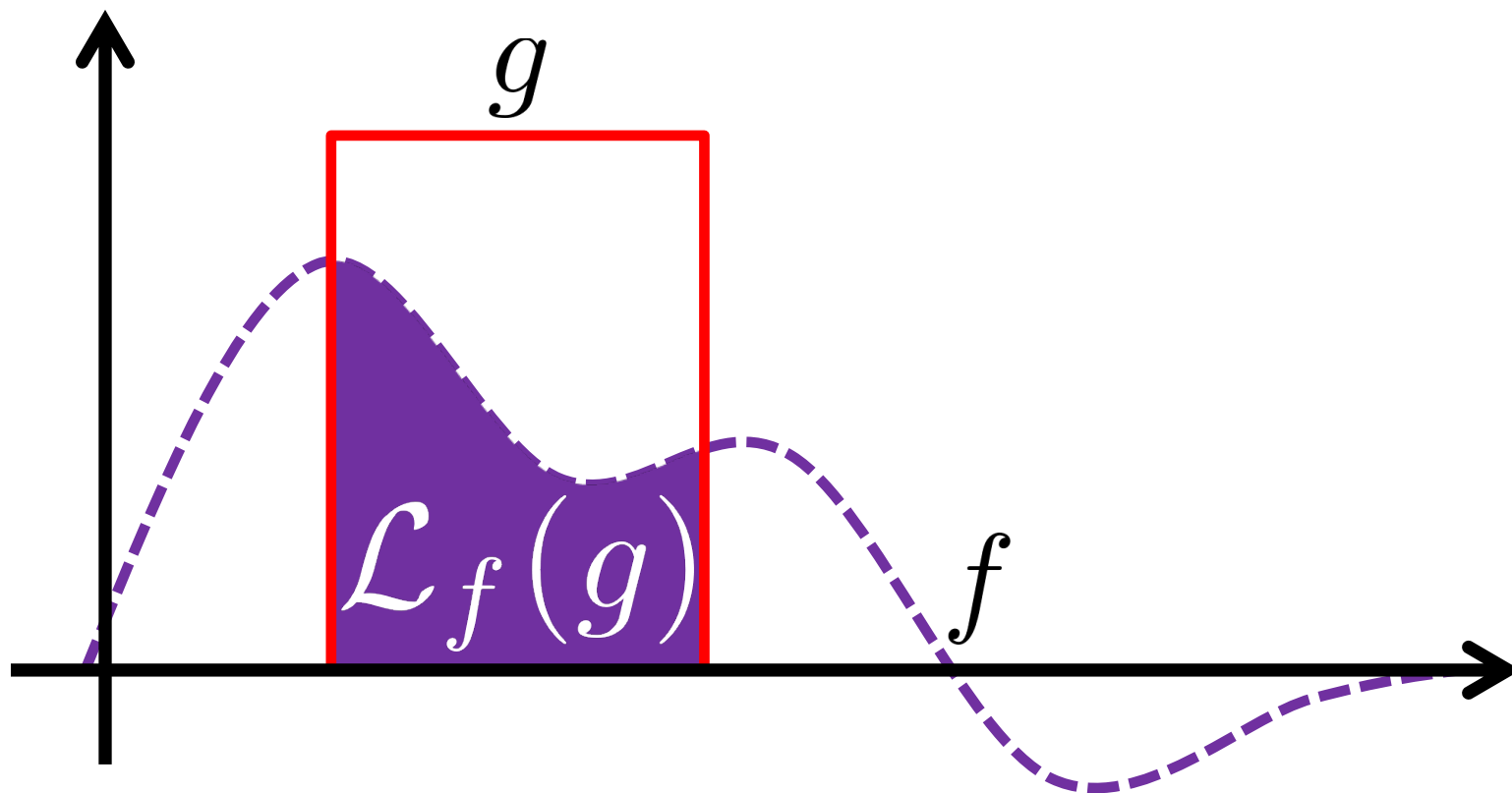
Poisson Equation

$$\Delta f = g$$



Weak Solutions

$$\int_M \phi \Delta f \, dA = \int_M \phi g \, dA \quad \forall \text{ test functions } \phi$$



FEM Hat Weak Solutions

$$\int_M h_i \Delta f \, dA = \int_M h_i g \, dA \quad \forall \text{ hat functions } h_i$$

$$\begin{aligned} \int_M h_i \Delta f \, dA &= - \int_M \nabla h_i \cdot \nabla f \, dA \\ &= - \int_M \nabla h_i \cdot \nabla \sum_j a_j h_j \, dA \\ &= - \sum_j a_j \int_M \nabla h_i \cdot \nabla h_j \, dA \\ &= \sum_j L_{ij} a_j \end{aligned}$$

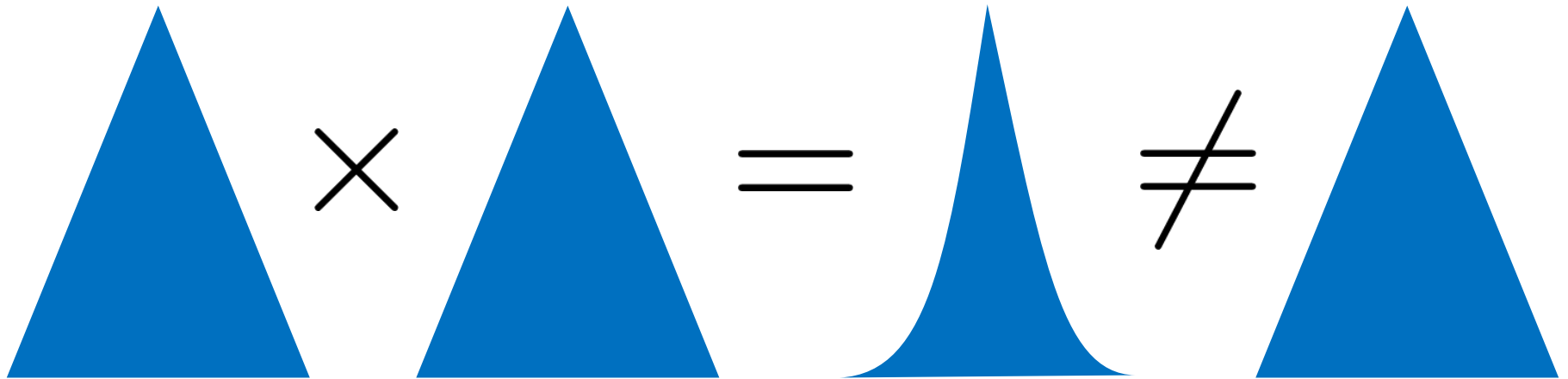
Stacking Integrated Products

$$\begin{pmatrix} \int_M h_1 \Delta f \, dA \\ \int_M h_2 \Delta f \, dA \\ \vdots \\ \int_M h_{|V|} \Delta f \, dA \end{pmatrix} = \begin{pmatrix} \sum_j L_{1j} a_j \\ \sum_j L_{2j} a_j \\ \vdots \\ \sum_j L_{|V|j} a_j \end{pmatrix} = L \vec{a}$$

Multiply by Laplacian matrix!

Problematic Right-Hand Side

$$\int_M h_i \Delta f \, dA = \int_M h_i g \, dA \quad \forall \text{ hat functions } h_i$$



Product of hats is quadratic

A Few Ways Out

- ***Just do the integral***

“Consistent” approach

- ***Approximate some more***

- ***Redefine g***

A Few Ways Out

- ***Just do the integral***

“Consistent” approach

- ***Approximate some more***

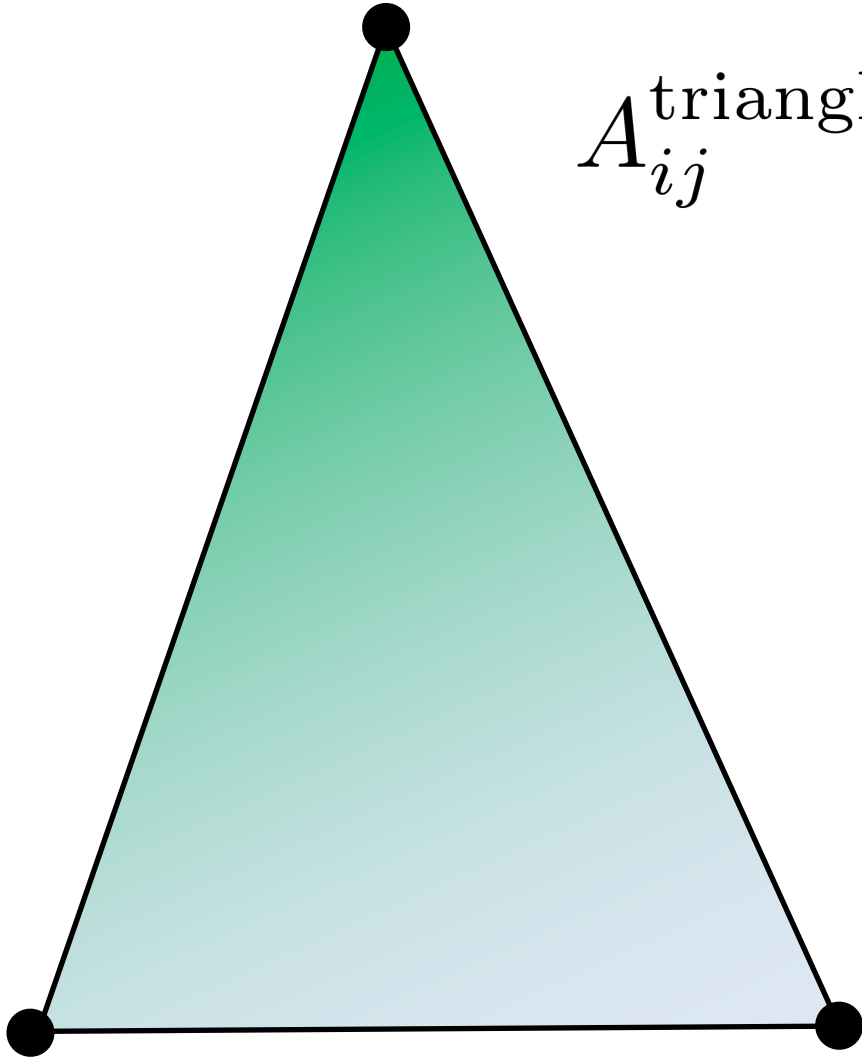
- ***Redefine g***

The Mass Matrix

$$A_{ij} := \int_M h_i h_j dA$$

- **Diagonal** elements:
Norm of h_i
- **Off-diagonal** elements:
Overlap between h_i and h_j

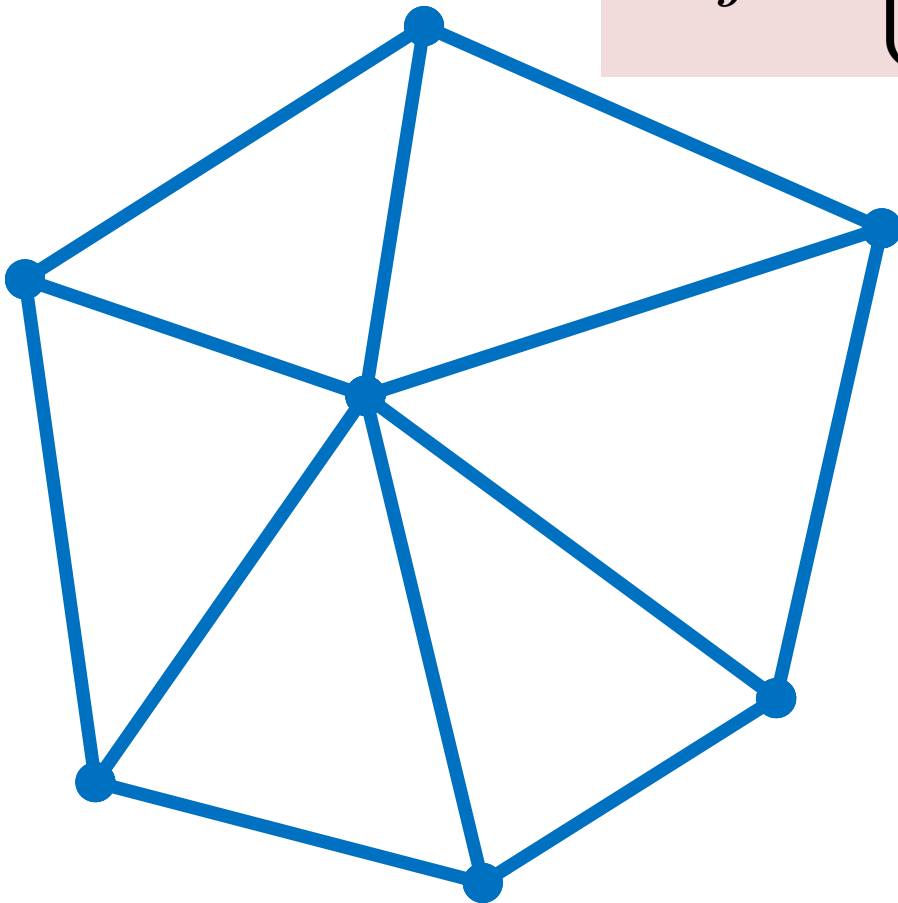
Consistent Mass Matrix



$$A_{ij}^{\text{triangle}} = \begin{cases} \frac{\text{area}}{6} & \text{if } i = j \\ \frac{\text{area}}{12} & \text{if } i \neq j \end{cases}$$

Non-Diagonal Mass Matrix

$$M_{ij} = \begin{cases} \frac{\text{one-ring area}}{6} & \text{if } i = j \\ \frac{\text{adjacent area}}{12} & \text{if } i \neq j \end{cases}$$



Properties of Mass Matrix

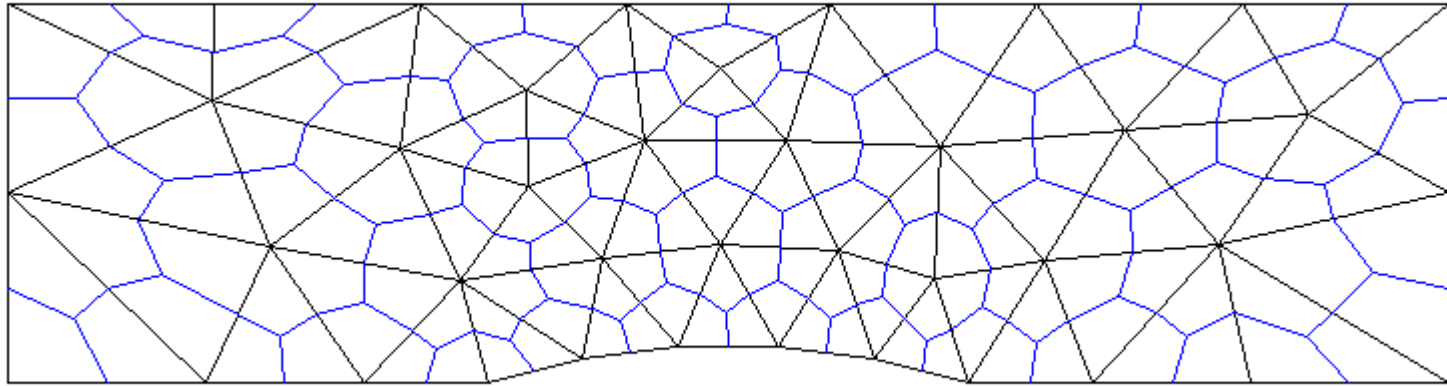
- Rows sum to one ring area / 3
- Involves only vertex and its neighbors
- Partitions surface area

Issue: Not trivial to invert!

Use for Integration

$$\begin{aligned}\int_M f &= \int_M \sum_j a_j h_j(\cdot) 1 \\ &= \int_M \sum_j a_j h_j \sum_i h_i \\ &= \sum_{ij} A_{ij} a_j \\ &= \mathbf{1}^\top A \vec{a}\end{aligned}$$

Lumped Mass Matrix



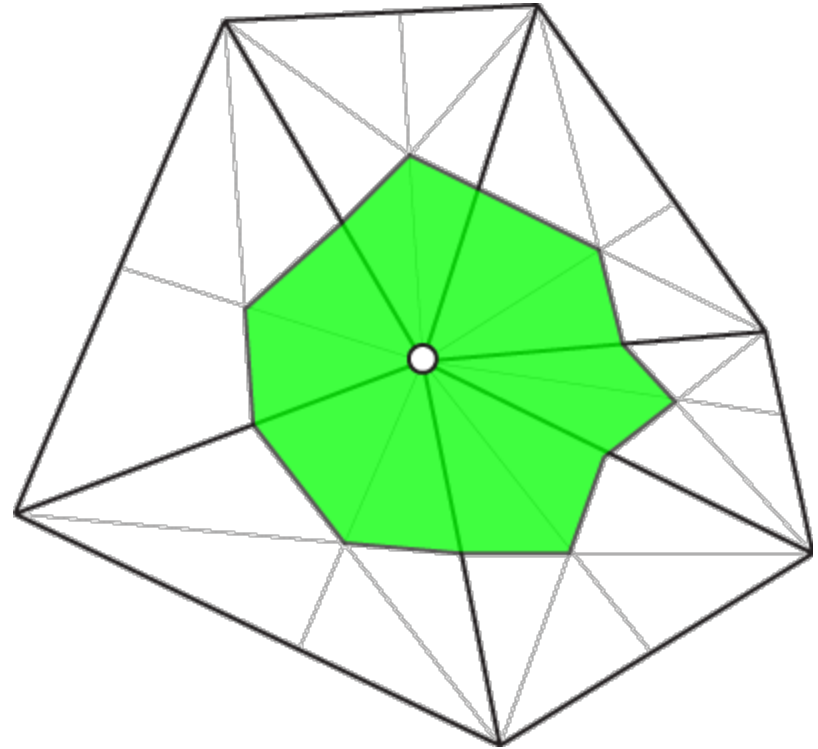
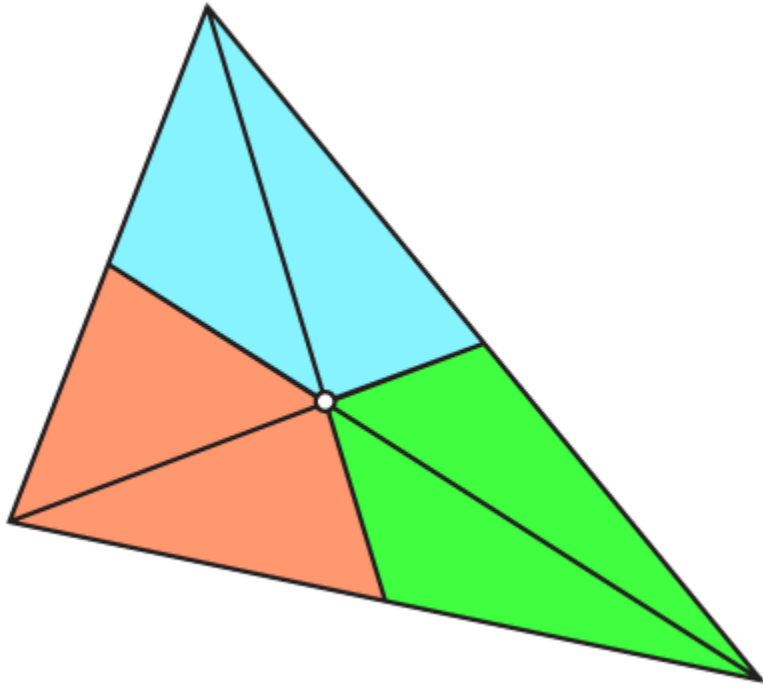
$$\tilde{a}_{ii} := \text{Area}(\text{cell } i)$$

Won't make big difference for smooth functions

<http://users.led-inc.eu/~phk/mesh-dualmesh.html>

Approximate with diagonal matrix

Simplest: Barycentric Lumped Mass



<http://www.alecjacobson.com/weblog/?p=1146>

Area/3 to each vertex

Ingredients

- ***Cotangent Laplacian L***

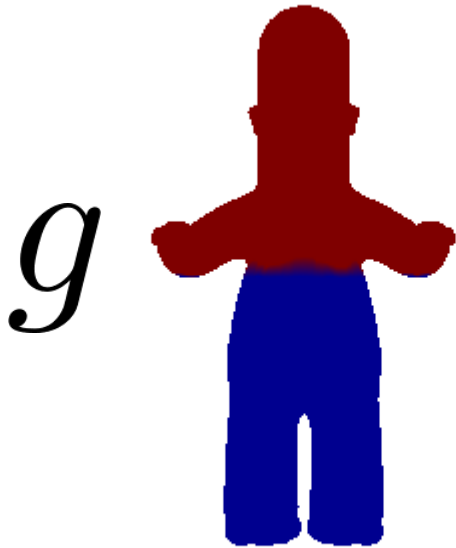
*Per-vertex function to integral of its
Laplacian against each hat*

- ***Area weights A***

*Integrals of pairwise products of hats
(or approximation thereof)*

Solving the Poisson Equation

$$\Delta f = g \rightarrow L \vec{f} = A \vec{g}$$



g

**Must integrate
to zero**



f

**Determined up
to constant**

Helmholtz Equation

$$\Delta f = \lambda f$$

$$\int \Delta f \cdot \psi = -\int \nabla f \cdot \nabla \psi = \int \lambda f \cdot \psi$$

$$\Rightarrow La = \lambda Ma$$

Generalized
Eigenvalue Problem

Important Detail: Boundary Conditions

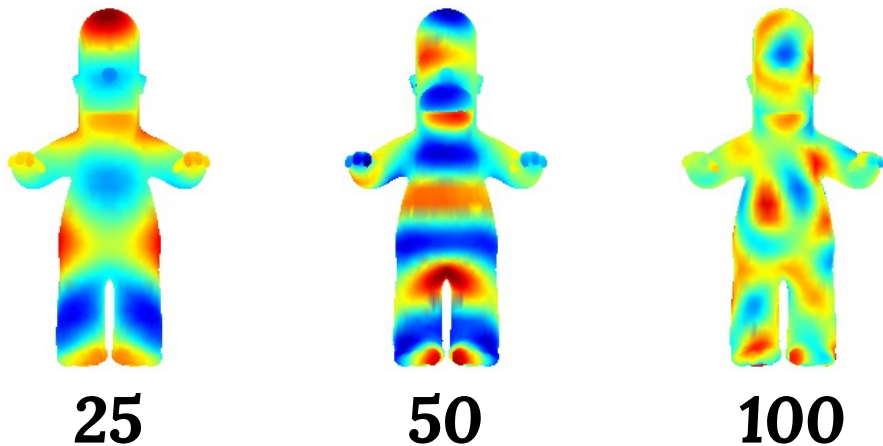
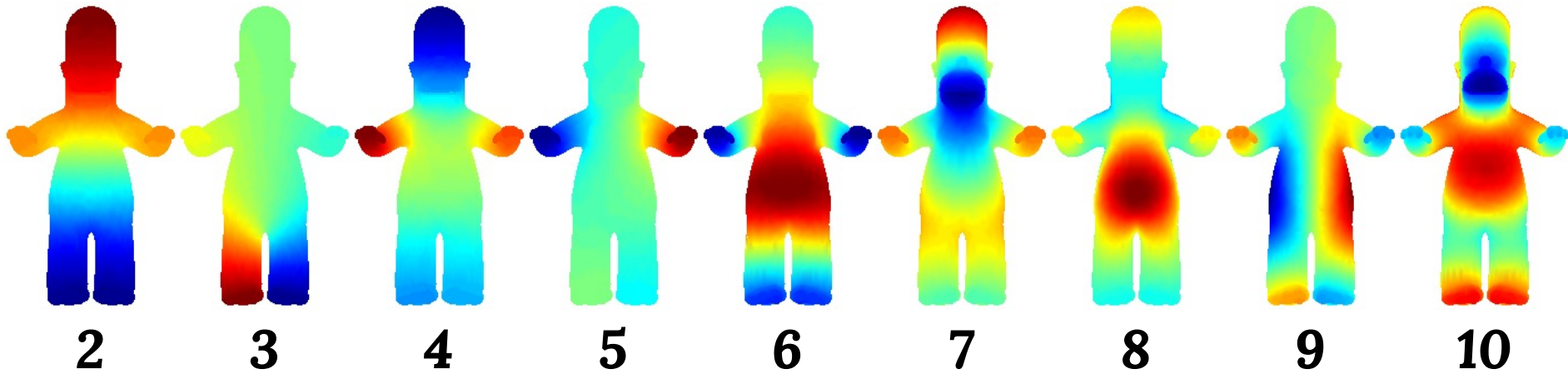
$$\begin{aligned}\Delta f(x) &= g(x) \quad \forall x \in \Omega \\ f(x) &= u(x) \quad \forall x \in \Gamma_D \\ \nabla f \cdot n &= v(x) \quad \forall x \in \Gamma_N\end{aligned}$$

Strong
form

$$\begin{aligned}\int_{\Omega} \nabla f \cdot \nabla \phi &= \int_{\Gamma_N} v(x) \phi(x) d\Gamma - \int_{\Omega} f(x) \phi(x) d\Omega \\ f(x) &= u(x) \quad \forall x \in \Gamma_D\end{aligned}$$

Weak form

Eigenhomers



What is
smallest
eigenvalue?

**On board:
FEM approach?**

Higher-Order Elements

https://www.femtable.org

finite element method

Periodic Table of the Finite Elements

	The $\mathcal{P}_r^- \Lambda^k$ family				The $\mathcal{P}_r \Lambda^k$ family				The $\mathcal{Q}_r^- \Lambda^k$ family				The $\mathcal{S}_r \Lambda^k$ family			
	$k=0$	$k=1$	$k=2$	$k=3$	$k=0$	$k=1$	$k=2$	$k=3$	$k=0$	$k=1$	$k=2$	$k=3$	$k=0$	$k=1$	$k=2$	$k=3$
$n=1$																
$n=2$																
$n=3$																

<https://www.femtable.org/>

Point Cloud Laplace: Easiest Option

$$W_{ij} = \exp \left(-\frac{\|x_i - x_j\|^2}{t} \right)$$

$$D_{ii} = \sum_j W_{ji}$$

$$L = D - W$$

$$Lf = \lambda Df$$

