Exercices sem. 15

1. $$K = \begin{array}{c|cc} & X & Y \\ \hline X & \sigma^2 & \rho\sigma^2 \\ Y & \rho\sigma^2 & \sigma^2 \end{array} \longrightarrow \begin{array}{l} \mathbb{E}[X^2] = \sigma^2 \quad \mathbb{E}[XY] = \rho\sigma^2. \\ \mathbb{E}[Y^2] = \sigma^2 \end{array}$$

Trouvez $I(X;Y)$.

<u>Sol</u>:
$$\begin{aligned} I(X;Y) &= h(X) + h(Y) - h(XY) \\ &= \tfrac{1}{2}\log 2\pi e\sigma^2 + \tfrac{1}{2}\log 2\pi e\sigma^2 - \tfrac{1}{2}\log (2\pi e)^2 (\det K) \\ &= \log 2\pi e\sigma^2 - \log 2\pi e - \tfrac{1}{2}\log(\det K) \\ &= \log\sigma^2 - \tfrac{1}{2}\log(\det K) \end{aligned}$$

$$\det K = \sigma^4 - \rho^2\sigma^4 = \sigma^4(1-\rho^2).$$

$$\Rightarrow \log(\det K) = 2\log\sigma^2 + \log(1-\rho^2).$$

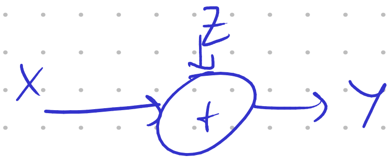
$$= -\tfrac{1}{2}\log(1-\rho^2).$$

(Comparer avec Capacité du canal gaussien $= I(X;Y)$, $Y = X+Z$)

$\Rightarrow \rho^2 = \frac{P}{N}$.

2. Bruit uniformément distribué :

$X \sim \mathrm{Unif}\left[\frac{-1}{2}, \frac{1}{2}\right] \qquad Z \sim \mathrm{Unif}\left[\frac{-a}{2}, \frac{a}{2}\right]$



$Y = X + Z.$

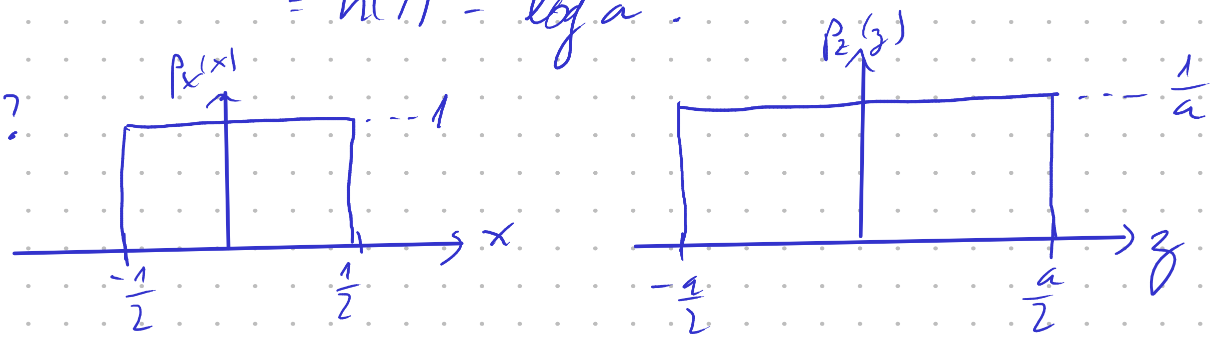
(a) Trouvez $I(X;Y)$ en fonction de a :

Soln :

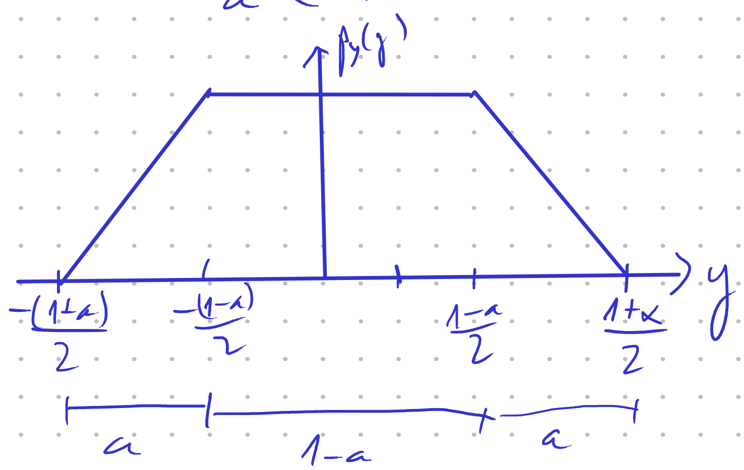
$$
\begin{aligned}
I(X;Y) &= h(Y) - h(Y|X) \\
&= h(Y) - h(X+Z|X) \\
&= h(Y) - h(Z|X) \\
&= h(Y) - h(Z) \\
&= h(Y) - \log a .
\end{aligned}
$$

car X et Z sont indépendants

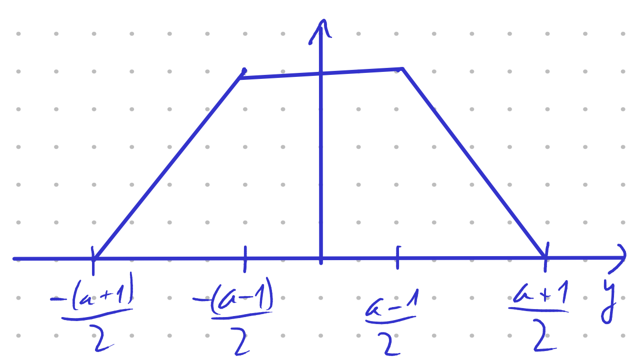
$p_Y(y) = ?$



$a < 1$

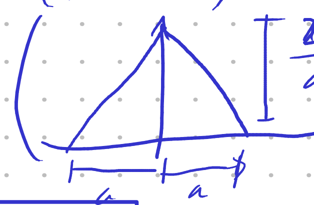


$a > 1$



$$h(Y) = -\int_{\mathbb{R}} p_Y(y) \log p_Y(y)\, dy . \qquad A = \begin{cases} 0 & \text{si } y \in \left[-\frac{(1-a)}{2}, \frac{1-a}{2} \right] \\ 1 & \text{sinon} \end{cases} \quad (a<1)$$

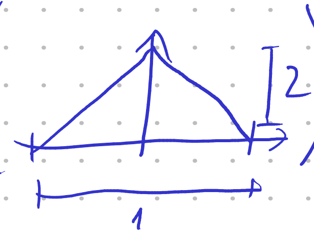
$$\begin{aligned} &= h(Y A) = H(A) + h(Y|A) \\ &= \underbrace{h(a)}_{\hookrightarrow -a \log a - (1-a)\log(1-a)} + h(Y|A) \\ &= h(a) + (1-a)\, h(Y|A=0) + a\, h(Y|A=1) \\ &= h(a) + (1-a) \log(1-a) + a\, h\Big(\end{aligned}$$



$\Big)$

Faits : ① $h(\lambda X) = h(X) + \log |\lambda|$
$(\lambda \in \mathbb{R})$

② $h\Big($



$\Big) = \dfrac{1}{2\ln 2} - 1 .$

$$= h(a) + (1-a)\log(1-a) + a \left(\frac{1}{2\ln 2} - \cancel{1} + \overset{\cancel{1} + \log a}{\overbrace{\log 2a}} \right)$$

$$\begin{aligned} &= \cancel{-a\log a} \cancel{- (1-a)\log(1-a)} + \cancel{(1-a)\log(1-a)} \\ &\quad + a \left(\frac{1}{2\ln 2} + \cancel{\log a} \right) \\ &= \frac{a}{2\ln 2} . \end{aligned}$$

(Si $a > 1$, on devrait avoir $\log a + \dfrac{1}{2a\ln 2}$.)

$$\Rightarrow I(X;Y) = h(Y) - \log a$$

$$= \begin{cases} \dfrac{a}{2\ln 2} - \log a & \text{si} \quad a \le 1 \\[2ex] \dfrac{1}{2a\ln 2} & \text{si} \quad a > 1 . \end{cases}$$

(b) Pour $a=1$, trouvez la cap. du canal si X est limité à l'intervalle $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

$$C(W) = \max_{p_X} I(X;Y) = \max_{p_X} \left[h(Y) - h(Z) \right]$$

$$= \max_{p_X} h(Y) - \cancelto{0}{\log 1}$$

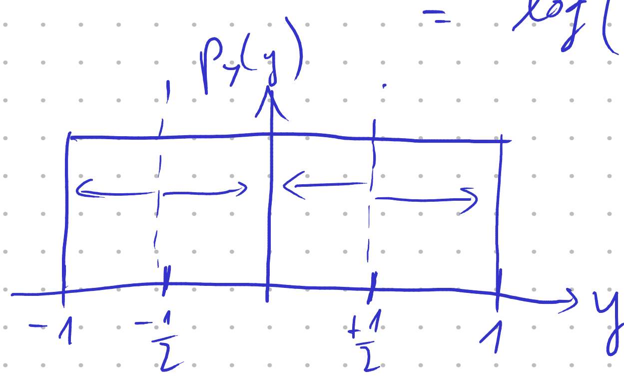
$$\rightarrow h(Y) \le h(\mathrm{Unif}[-1,1])$$

$$= \log(2) = 1.$$

$|X| \le \frac{1}{2}$

$|Z| \le \frac{1}{2}$

$\Rightarrow |Y| \le 1$



$\Rightarrow$ Cette borne est atteinte si $X \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$ avec

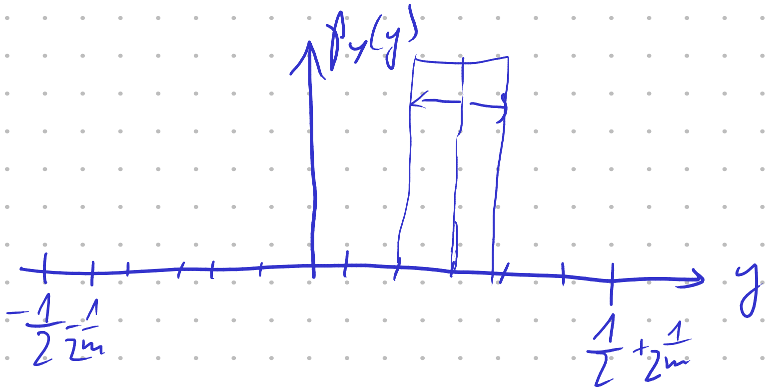
$p_X\left(-\frac{1}{2}\right) = \frac{1}{2}$

$p_X\left(\frac{1}{2}\right) = \frac{1}{2}$.

$\Rightarrow$ Une distribution discrète fait le travail.

(c) Généralisez au cas où a est arbitraire :

Si $a = \frac{1}{m}$ où $m = 2, 3, 4 \ldots$



$$I(X;Y) = h(Y) - h(Z)$$
$$= h(Y) - \log a$$
$$= \log\left(1+\frac{1}{m}\right) - \log\frac{1}{m}$$
$$= \log\frac{1+\frac{1}{m}}{\frac{1}{m}}$$
$$= \log(m+1).$$