

INFINITE GENERALIZED FRIENDSHIP GRAPHS

Charles DELORME and Geňa HAHN

Université de Paris-Sud, L.R.I., Bât. 490, 91405 Orsay Cedex, France

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We give necessary and sufficient conditions for the existence of infinite generalized friendship graphs and show that there are 2^c non-isomorphic ones of each admissible order c and chromatic number. Further we prove that such graphs and their complements are almost always regular of degree equal to the order and that various generalizations of the Friendship Theorem do not hold for infinite generalized friendship graphs.

Introduction

A *friendship graph* (ours are simple and either finite or infinite) is one in which every pair of vertices has exactly one common neighbour. It is well known (see, for example, [9, 13, 14, 19]) that friendship graphs either consist of triangles all joined at one vertex or are regular of infinite degree. The finite case of this result is known as the Friendship Theorem and is often expressed in the following way:

Every finite friendship graph contains a vertex adjacent to all other vertices. Finite friendship graphs are rare. In contrast, infinite ones abound, there are 2^c of them of order c , for each infinite cardinal c , by a result of [4]. Several generalizations of friendship graphs have been studied. Skala [17] considers graphs in which any two vertices either are adjacent or have exactly one common neighbour. Bose and Shrikhande [2] and Le Conte de Poly [5] require that each pair of vertices have exactly $\lambda \geq 1$ common neighbours. Carstens and Kruse [3], Plesník [16] and Sudolský [18] ask that any set of t vertices have exactly $\lambda \geq 1$ common neighbours. Yet another, described by Doyen [7], uses three parameters: for any two disjoint sets of t and u vertices, respectively, there are λ vertices which are adjacent to the t -set and non-adjacent to the u -set.

We shall call *generalized friendship graphs* those which have at least $t + \lambda$ vertices and in which every set of t vertices has exactly λ (common) neighbours. The class of generalized friendship graphs will be denoted by \mathcal{G}_t^λ and its subclass consisting of infinite graphs by \mathcal{F}_t^λ . The parameters t and λ are positive integers and, in order to avoid trivialities, we assume that $t \geq 2$ and $\lambda \geq 1$. Note that the notation \mathcal{G}_t^λ remains meaningful for all positive t and λ .

The finite graphs in \mathcal{G}_t^λ , $t \geq 3$, have been completely characterized. They are just

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the complete graphs $K_{t+\lambda}$ ([3], [18]). The finite graphs in \mathcal{G}_2^λ , $\lambda \geq 2$, are regular by a result of [5] and Erdős (cited in [2]); Mulder [15] also gives a proof. Erdős does more by showing that all graphs in \mathcal{G}_2^λ are regular for $\lambda \geq 2$. Some necessary conditions for the existence of finite generalized friendship graphs with $t=2$ are found in [2] and some examples constructed by Doyen [7]. Our purpose is to look at \mathcal{F}_t^λ , $\lambda \geq 2$. We answer Doyen's question as to the existence of infinite generalized friendship graphs and show that not only are they regular (when they exist) but that there are many of them.

Results

We shall denote the cardinality of a set S by $|S|$ and the common neighbourhood of a set S of vertices, defined as $\bigcap_{x \in S} N(x)$, by $N(S)$. Notation not explained is that of [1]. Let us repeat the often crucial assumption:

$$t \geq 2 \quad \text{and} \quad \lambda \geq 1.$$

The following lemma is fundamental.

Lemma 1. *Let $G \in \mathcal{F}_t^\lambda$ be of order c and let x be a vertex of G . Then either x has degree c or $N(x)$ contains λ -sets L_i such that $\sup |N(L_i)| = c$.*

Proof. Let $G \in \mathcal{F}_t^\lambda$ be of order c and let x be any of its vertices. If $d(x) = c$ there is nothing to prove. Hence only the case $d(x) < c$ need be considered. Note that $t < c$ and observe that each of the c t -element subsets of $V(G)$ containing x has its λ neighbours in $N(x)$. Since there are strictly fewer than c λ -sets in $N(x)$ there is one with c neighbours or infinitely many ones, say L_i , with $\sup |N(L_i)| = c$. \square

There is a useful corollary.

Corollary 1. *If $G \in \mathcal{F}_t^\lambda$ is not regular of degree $|V(G)|$, then $t > \lambda$.*

Proof. If G has a vertex of degree less than $|V(G)|$ and if $t \leq \lambda$ then any t -subset of a λ -set L with $d(x) < |N(L)| \leq |V(G)|$ contradicts G being in \mathcal{F}_t^λ . \square

Proposition 1. *Let $G \in \mathcal{F}_t^\lambda$ be of order c . There is a clique S in G with $t-1$ vertices and with $N(S)$ infinite.*

Proof. We induct on t . For $t=2$ the claim is just Lemma 1. For $t > 2$, let x be a vertex of infinite degree in G and let H be the subgraph of G induced by $N(x)$. Now $H \in \mathcal{F}_{t-1}^\lambda$ and contains a clique T on $t-2$ vertices with $N(T)$ infinite, by hypothesis. Put $S = T \cup \{x\}$. \square

Several interesting consequences are at hand.

Proposition 2. *If $\mathcal{F}_t^\lambda \neq \emptyset$, then $t \leq \lambda + 1$.*

Proof. Any t -set in the neighbourhood of the clique found in Proposition 1 already has $t - 1$ neighbours. \square

In what follows we assume $\lambda \geq 2$. Note, however, that the results remain true if $\lambda = 1$ for graphs other than the graph consisting of c triangles all joined at one vertex (c infinite).

Proposition 3. *If $G \in \mathcal{F}_t^\lambda$ is of order c then it is regular of degree c .*

Proof. In view of Corollary 1 and Proposition 2 only the case $t = \lambda + 1$ requires proof. We assume $t \geq 3$ since $\lambda \geq 2$. If x is a vertex of G whose degree is less than c , its neighbourhood contains vertices y_i with $\sup d(y_i) = c$, by Lemma 1. The infinite graphs induced by $N(y_i)$ contain x , are in $\mathcal{F}_{t-1}^\lambda$ and are, therefore, regular of degree $d(y_i)$. Hence $d(x) = c$ in any case.

Corollary 2. *Each vertex of a graph $G \in \mathcal{F}_t^\lambda$ of order c lies in a $(t - 1)$ -clique S with $|N(S)| = c$.*

Proof. The inductive construction of Proposition 1 can begin at any vertex, by Proposition 3. \square

We also note that each maximal clique in G has order k , $t + 1 \leq k \leq t + \lambda$.

Proposition 4. *The complement of a graph of order c in \mathcal{F}_t^λ is regular of degree c .*

Proof. Let $G \in \mathcal{F}_t^\lambda$ be of order c . Let x be an arbitrary vertex of G and let $y \in N(x)$. Consider the graph H induced by $N(y)$. Since $G \in \mathcal{F}_t^\lambda$, $H \in \mathcal{F}_{t-1}^\lambda$. If $t = 2$, H is λ -regular. Hence x already has c non-neighbours in H , a fortiori in G . If $t > 2$ the complement of H is regular of degree c by hypothesis. \square

Proposition 5. *Let $G \in \mathcal{F}_t^\lambda$ be of order c , and let $S \cup T$ be a proper partition of $V(G)$. Then $N(S) \neq T$.*

Proof. If $|S| < c$, Proposition 4 says that each $x \in S$ has c non-neighbours in T . If $|S| = |T| = c$ then any t -set in, say, S has too many neighbours whenever $N(S) = T$. \square

So far we have only been concerned with conditions necessary for the non-emptiness of \mathcal{F}_t^λ . The remainder of the paper deals with the sufficiency of the basic condition, that is, $t \leq \lambda + 1$. Although there are straightforward ways of constructing graphs in \mathcal{F}_t^λ for $t \leq \lambda + 1$, we want to do more. Inspired by the

elegant technique of [4], we show that for each d , $t+1 \leq d \leq \aleph_0$, and each infinite cardinal c , the class \mathcal{F}_t^λ , $t \leq \lambda + 1$, contains 2^c non-isomorphic graphs of order c and chromatic number d . Our basic tool is a modified m -cube. An m -cube is a graph on the vertex set $\{0, 1\}^m$, m a positive integer, with two vertices adjacent if and only if the m -tuples differ in exactly one coordinate. For $m > 2$ we obtain the graph C_m by deleting one vertex and specifying two vertices u and v of even distance, with $d(u) = m$ and $d(v) = m - 1$. Since the m -cube is bipartite and regular of degree m , C_m is bipartite with minimum degree $m - 1$. Note that no automorphism of C_m interchanges u and v . Let now X be a well ordered set of infinite cardinality c , let $m > n > 2$ be integers and let F be a function from the set of unordered pairs of elements of X into $\{m, n\}$. With X , m , n , F given, we construct the graph G_F as follows. Begin with X as the vertex set. For each pair $\alpha < \beta$ in X insert a new copy of $C_{F(\alpha, \beta)}$, identifying u with α and v with β . We claim that if $F \neq F'$ then G_F and $G_{F'}$ are not isomorphic. To see this, observe that from G_F we can recover X with its order, as well as F . Indeed, the elements of X are the vertices of G_F of infinite degree and m , n and F can be obtained by considering certain subgraphs of G_F . Let α, β be distinct vertices of infinite degree in G_F and let $H_{\alpha, \beta}$ be obtained from G_F by deleting all the vertices of infinite degree other than α and β . The 2-connected component of $H_{\alpha, \beta}$ containing α and β is $C_{F(\alpha, \beta)}$ and $\alpha < \beta$ if and only if $d(\alpha) > d(\beta)$ in this component.

Let now t and λ be given and let G be any graph not containing the complete bipartite graph $K_{t, \lambda + 1}$. Let $G_0 = G$ and, given G_i , obtain G_{i+1} by adding $\lambda - r$ new distinct vertices for every set of t vertices in G_i having r neighbours in G_i , together with the edges joining the t -set to these $\lambda - r$ new neighbours. Note that the chromatic number of G_i is preserved in G_{i+1} provided it is at least $t + 1$, as is the absence of $K_{t, \lambda + 1}$. Let $G^* = \bigcup_{i \in \mathbb{N}} G_i$, \mathbb{N} denoting the set of non-negative integers.

It is easy to see that $G^* \in \mathcal{F}_t^\lambda$: for any t -set S there is a least i such $S \subset V(G_i)$ and, hence, $|N(S)| = \lambda$ in G_j , $j > i$. Further, if H is a finite subgraph, there is a least i such that $H \subset G_i$. If $i > 0$ then the vertices of H in $G_i \setminus G_{i-1}$ have degree at most t .

This proves a lemma:

Lemma 2. *Let G be an infinite graph of order c not containing $K_{t, \lambda + 1}$ and having chromatic number d , $t + 1 \leq d \leq \aleph_0$. Then $G^* \in \mathcal{F}_t^\lambda$ is of order c and chromatic number d . If H is a finite subgraph of G^* of minimum degree at least $t + 1$ then H is in fact a subgraph of G .*

We can now prove a proposition:

Proposition 6. *There are 2^c d -chromatic graphs of order c in \mathcal{F}_t^λ whenever $t \leq \lambda + 1$, $\lambda \geq 2$ and $t + 1 \leq d \leq \aleph_0$.*

Proof. We will construct 2^c graphs of order c which are non-isomorphic, have chromatic number d and contain no $K_{t,\lambda+1}$. They will further have the property that non-isomorphic G and H yield non-isomorphic G^* and H^* . Let m and n be integers, $m > n > t + 1$. Let X be a well-ordered set of infinite order c and F any function from the set of unordered pairs of elements of X into $\{m, n\}$. The graph G_F contains no $K_{t,\lambda+1}$ since $d(u, v) \geq 2$ and C_k contains no $K_{2,3}$. Also, since C_k is bipartite and the distance between the special vertices u and v is even, G_F is bipartite, hence 2-chromatic. Finally, there are 2^c functions F that give non-isomorphic G_F . For each d , $t + 1 \leq d \leq \aleph_0$, let G_d be a finite graph of girth at least 5 and chromatic number d . These exist, see [6], [8] or [12]. Let $G_{\aleph_0} = \bigcup_{d \in \mathbb{N}} G_d$ be the disjoint union of the G_d . For each F and each d , $t + 1 \leq d \leq \aleph_0$, let $G_{F,d}$ be the disjoint union of G_F with G_d . The graphs $G_{F,d}$ and $G_{F',d'}$ are clearly non-isomorphic if $(F, d) \neq (F', d')$. The graphs $G_{F,d}^*$ are in \mathcal{F}_t^λ and have chromatic number d and order c . It remains to show that the $G_{F,d}^*$ are also non-isomorphic. Observe that G_F can be recovered as the infinite connected subgraph induced by the set of vertices which lie in finite subgraphs of degree at least $t + 1$ in $G_{F,d}^*$, by Lemma 2. This completes the proof. \square

Remark. We have proved the following theorems.

Theorem 1. *The class \mathcal{F}_t^λ either is empty or contains 2^c non-isomorphic graphs of order c and chromatic number d , for each c and each d , $t + 1 \leq d \leq \aleph_0$. It is non-empty if and only if $t \leq \lambda + 1$.*

Theorem 2. *Let $\lambda \geq 2$ and let $G \in \mathcal{F}_t^\lambda$ be of order c . Then both G and its complement are regular of degree c .*

As in [4], we observe that the bounds for the chromatic number are best possible. The lower bound follows from the remark following Corollary 2, the upper from Corollary 5.6 of [4] which says that any graph of chromatic number $d > \aleph_0$ contains arbitrarily large complete bipartite graphs.

Our construction being slightly different from that of [4], the answer given in that paper to its referee's suggestion is re-inforced: there are 2^c graphs, of order c in \mathcal{F}_t^λ of each admissible chromatic number, which are not obtained by the construction of [4]. It suffices to use another 'arrow' graph to join the vertices $\alpha < \beta < \mu$ in addition to the graph Γ given in [4]. One possibility is to take two copies Γ_1 and Γ_2 of Γ and identify the vertices a of Γ_1 and b of Γ_2 . The two special vertices required will be a of Γ_2 and b of Γ_1 . See [4].

Finally, a word about the Friendship Theorem. There are three obvious generalizations of it which we sum up as: for any $G \in \mathcal{G}_t^\lambda$ there is a partition of $V(G)$ into $X \cup Y$ such that $N(X) = Y$ and $|X| \in \{1, t - 1, \lambda\}$. We have shown this to be invalid for infinite graphs other than the graphs consisting of triangles all joined at one vertex.

For finite graphs these generalizations hold trivially if $t \geq 3$ by the results of [3] and [18]. If $t = 2$, $\lambda \geq 2$, it is easy to see that they fail for graphs which are not complete.

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