

A note on k -cop, l -robber games on graphs[☆]

Geňa Hahn^{a,1}, Gary MacGillivray^b

^a*Département d'Informatique et de Recherche Opérationnelle, Université de Montréal, C.P. 6128, succursale Centre-ville, Montréal, Que., Canada H3C 3J7*

^b*Department of Mathematics and Statistics, University of Victoria, Victoria, B.C., Canada V8W 3P4*

Received 4 March 2003; received in revised form 18 November 2004; accepted 12 December 2005

Available online 18 July 2006

To Claude Berge, the inveterate player, who never really left us

Abstract

We give an algorithmic characterisation of finite cop-win digraphs. The case of $k > 1$ cops and $k \geq l \geq 1$ robbers is then reduced to the one cop case. Similar characterisations are also possible in many situations where the movements of the cops and/or the robbers are somehow restricted.

© 2006 Elsevier B.V. All rights reserved.

Keywords: Pursuit; Cop-win; Copnumber; k -Cop-win; Graph; Digraph

1. Introduction

We use semi-standard conventions for graphs and digraphs. In particular, an undirected graph will have a set of edges, $E(G)$, while a digraph will have a set of arcs, $A(G)$. All our graphs are finite.

Cops and Robber is a perfect information pursuit game played on a directed graph $G = (V(G), A(G))$, with loops at some, but not necessarily all, vertices. All players must move and this implies that the digraphs under consideration must have outdegree at least one if we wish to avoid trivialities (which we do). Fix an integer $k > 0$; this will be the number of cops chasing the unique robber. A *round* consists of a pair of *moves*, a cops' move, followed by a robber's move. At round 0, each of the $k \geq 1$ cops chooses a vertex of G (they can share vertices), and the robber chooses a vertex of G . At round i , $i > 0$, each cop moves from the vertex currently occupied to one of its out-neighbours, then the robber does the same. The cops win the game if one of them occupies the same vertex as the robber (the robber is *caught*). If this happens on a cops' move, the game ends and the robber does not move in the last round; note that it can also happen on a robber's move. The robber wins if he can move never to be caught. A digraph is *k -cop-win* if k cops have a winning strategy on it, and *robber-win* otherwise. Historically, 1-cop-win (undirected, reflexive) graphs have been called *cop-win*.

[☆] Research partially supported by grants from NSERC.

¹ Research partially supported by a grant from MITACS.

E-mail addresses: hahn@iro.umontreal.ca (G. Hahn), gmacgill@math.uvic.ca (G. MacGillivray).

This game was first described for reflexive undirected graphs with one cop and one robber by Nowakowski and Winkler [11], and Quilliot [12], independently. Both papers characterise the cop-win graphs. A good discussion of previous work on Cops and Robber, and related pursuit games, can be found in [3,6].

Goldstein and Reingold (mentioned in [6]) have proved that, for fixed k , there is a polynomial time backtracking algorithm that determines whether a finite, reflexive, undirected graph G is k -cop-win. On the other hand, they also proved that when k is a parameter, it is EXPTIME-complete to decide whether k cops can win from a given initial position on such a graph.

Cops and robber games on digraphs have not received as much attention. Cayley digraphs are considered in [8] and, implicitly, in [5] (Alspach et al. [1] recently looked at the cop number of Cayley graphs on dihedral groups). In [6] it is proved that when k is a parameter, it is EXPTIME-complete to decide whether a strongly connected digraph is k -cop-win.

Beyond the one-cop case [11,12], no characterisations are known. It is a long standing open problem to characterise the k -cop-win reflexive undirected graphs for $k > 1$. The reflexive digraphs on which a single cop can always win have also not been characterised.

This paper is an attempt at a unified approach to both directed and undirected graphs, reflexive or not. While the techniques are simple, they do allow for some progress and for a first consideration of digraphs. The algorithm, in addition to being general, is easier and faster than algorithms previously used based on Theorem 1. A first version (not giving strategy information) of the algorithm described in Section 2 of this paper has been implemented and used in an honours thesis by Hole [9]. The full algorithm has been implemented in various ways by Benoit Thériault [13] as a part of his M.Sc. thesis. He found that the order in which the vertex labels are updated influences the performance—updating the cop vertices c_{xy} before the robber vertices r_{xy} is many times faster in his tests than the other order. He also found that representing the positions of the cops (robbers) by vectors of length n (the order of the digraph) whose i th coordinates give the number of cops (at the i th vertex allows representing $k!$ positions by one vector and so speeds up processing. See [13].

In the next section, we give an algorithmic characterisation of cop-win finite digraphs. By an *algorithmic characterisation* we mean that a digraph D is cop-win if and only if an auxiliary digraph M_D , constructible in polynomial time, has a certain labelling, also constructible in polynomial time. Any k -cop game is reduced to a 1-cop game in the subsequent section. We conclude the paper by describing how our methods can be applied to some variants of the game.

We insist on the adjective “algorithmic”—while it is nice to know whether or not a digraph is k -cop-win, that knowledge can only be obtained by running an algorithm on the digraph. It would certainly be nicer to have some closed-form characterisation of such digraphs, so that we could say that certain classes of graphs are k -cop-win, as is the case in the characterisation of cop-win undirected reflexive graphs in [11,12].

Theorem 1 (Nowakowski and Winkler, Quilliot). *An undirected reflexive graph G is cop-win if and only if its vertices can be ordered v_1, v_2, \dots, v_n so that for each $i, 1 \leq i < n$, there is a $j, i < j \leq n$ such that $\{v_s : v_i v_s \in E(G), s \geq i\} \subseteq \{v_s : v_j v_s \in E(G), s \geq i\}$.*

One can deduce from the theorem that chordal graphs and bridged graphs are cop-win, see [2].

2. One cop, one robber

Consider a digraph D on which a game of cop and robber is played. After each move, the situation of the game can be described simply by saying where each player is and whose turn it is to move. This can be accomplished easily by considering $V(D) \times V(D) \times \{c, r\}$. For ease of reading, we will use c_{xy} and r_{xy} instead of (x, y, c) and (x, y, r) , respectively. Thus, we can read c_{xy} as *on cop's move, the cop is at x and the robber at y* , and read r_{xy} analogously. We will refer to both c_{xy} and r_{xy} as *configurations*.

We can, and will, associate with a digraph D its *move digraph* $M = M_D$ on vertex set $V_M = C_M \cup R_M$, where $C_M = \{c_{xy} : x, y \in V(D)\}$ and $R_M = \{r_{xy} : x, y \in V(D)\}$. There is an arc from c_{xy} to r_{wy} if $x \neq y$ and $xw \in E(D)$, and there is an arc from r_{xy} to c_{xz} if $x \neq y$ and $yz \in E(D)$. Observe that M has $2|V(D)|^2$ vertices and at most $2|A(D)|$ arcs, and can be constructed in time $O(|V(D)|^2)$. The arcs of M from a vertex c_{xy} represent the possible moves for the cop, those from r_{xy} for the robber.

The following lemma is obvious.

Lemma 2. *There is a path from p_{xy} to $p'_{x'y'}$ in $M(D)$ if and only if there is a sequence of moves, starting with p 's move, which takes the cop from x to x' and the robber from y to y' .*

Corollary 3. *If the digraph D is 1-cop-win then there is a vertex $c_{xy} \in V_M$ and a path from c_{xy} to p_{zz} for some $z \in V(D)$ and $p \in \{c, r\}$.*

Our purpose now is to prove a qualified converse of the corollary (the plain converse is clearly false, as a cycle of length at least four shows).

One characterisation of cop-win reflexive graphs given in [11] is in terms of a relation \leq on the vertices of the graph. The relation is defined recursively by setting $x \leq_0 x$ for each vertex x of the graph. For each ordinal α , define $x \leq_\alpha y$ if and only if for each $u \in N(x)$ there exists $v \in N(y)$ such that $u \leq_\beta v$ for some $\beta < \alpha$. Let ρ be the least ordinal such that $\leq_\rho = \leq_{\rho+1}$ and define $\leq = \leq_\rho$ (note that $\rho \leq |V|$). The graph is cop-win if and only if $x \leq y$ for all vertices x and y . This characterisation also applies to infinite graphs.

The algorithm below bears a strong resemblance to the recursive definition of \leq (but the similarity was discovered after the fact). It labels each vertex v of M_D with a non-negative integer $\ell(v)$ which indicates the number of rounds, that is, of **cop's** moves, to a cop's win. If it is the robber's move and the vertex label is k , the cop will need k moves *after* the robber has moved.

```

set  $\ell(v) = \infty$  for all  $v \in V_M$ 
set  $\ell(c_{xx}) = \ell(r_{xx}) = 0$  for each  $x \in V(D)$ 
repeat until no change results
  for each pair  $(x, y)$ ,  $x \neq y$ 
    if  $\ell(c_{xy}) = \infty$ , set  $\ell(c_{xy}) := 1 + \min\{\ell(r_{x'y'}) : x' \in N^+(x)\}$ 
  for each pair  $(x, y)$ ,  $x \neq y$ 
    if  $\ell(r_{xy}) = \infty$ , set  $\ell(r_{xy}) := \max\{\ell(c_{xy'}) : y' \in N^+(y)\}$ 
    
```

Since for each pair of vertices of D (x, y) only two vertices of M_D are considered at each iteration, and since the label of a vertex is never changed once it is not ∞ (an easy induction on the number of times the main loop of the algorithm is executed), the algorithm terminates in at most $O(|V(D)|^2)$ iterations, each of which takes at most $O(|V(D)|^3)$ steps.

We prove that the algorithm does what it is meant to in a lemma. Let us call a strategy from a configuration c_{xy} *optimal for the cop* if no other strategy gives a win in fewer moves. A strategy is *optimal for the robber* from a position r_{xy} if no other strategy forces a longer game (this includes an infinite game, i.e. a robber's winning strategy).

Lemma 4. *Let D be a directed graph and M_D its move digraph. Then*

- (1) *the cop has an optimal winning strategy in t rounds from the configuration c_{xy} if and only if $\ell(c_{xy}) = t$ when the algorithm terminates;*
- (2) *the robber has an optimal strategy in t rounds from the configuration r_{xy} if and only if $\ell(r_{xy}) = t$ when the algorithm terminates.*

Proof. The case $t = \infty$ is clear for both c_{xy} and r_{xy} . For t finite, we proceed by induction on t . We have that $\ell(c_{xy}) = 0$ if and only if $x = y$ and in this case there is nothing to prove. If $\ell(r_{xy}) = 0$, it is because $x = y$ or $\ell(c_{xy'}) = 0$ for all $y' \in N^+(y)$, and so the claim is true here as well.

- (1) Now $\ell(c_{xy}) = t$ if and only if we have $\ell(c_{xy}) = 1 + \min\{\ell(r_{x'y'}) : x' \in N^+(x)\}$, otherwise the algorithm could not have terminated. Thus, in this case, the cop's move will be to an x' that realises the minimum and from which, by the induction hypothesis, the cop can win in $t - 1$ rounds. Conversely, if the cop can win from c_{xy} in t rounds, there is an $x' \in N^+(x)$ such that the cop wins in $t - 1$ rounds from $c_{x'y'}$ for any $y' \in N^+(y)$. This means that $\ell(c_{x'y'}) \leq t - 1$ for all $y' \in N^+(y)$, that is, $\ell(r_{xy}) = t - 1$, by the induction hypothesis, and so $\ell(c_{xy}) = t$.
- (2) If $\ell(r_{xy}) = t$, then the robber has a move to a $y' \in V(D)$ such that $\ell(c_{xy'}) = t$ and to no z with $\ell(c_{xz}) > t$, lest the algorithm be contradicted. By the first part, the cop's optimal strategy is in t rounds, that is, the robber survives for t moves. Conversely, suppose that the robber has an optimal strategy of t moves from r_{xy} . Then the cop cannot win in fewer than t rounds, that is, $\max\{\ell(r_{xy'}) : y' \in N^+(y)\} = t$ and so $\ell(r_{xy}) = t$. \square

Theorem 5. Let D be a directed graph and let M_D be its move digraph. Then D is 1-cop-win if and only if there is a vertex $x \in V(D)$ such that for all $y \in V(D)$, $\ell(c_{xy}) < \infty$ in M_D .

Proof. Follows from Lemma 4. \square

Corollary 6. A finite strongly connected directed graph is cop win if and only if, in M_D , $\ell(v) < \infty$ for all $v \in V_M$.

Corollary 7. A finite undirected graph D is cop-win if and only if, in M_D , $\ell(v) < \infty$ for all $v \in V_M$.

The algorithm gives an easy way for each player to optimise his move. If the cop is at x and the robber at y , the cop will move to x' such that $\ell(c_{xy}) := 1 + \min\{\ell(r_{x'y}) : x' \in N^+(x)\}$, while the robber will move to y' such that $\ell(r_{xy}) := \max\{\ell(r_{xy'}) : y' \in N^+(y)\}$. Indeed, the name of the vertex x' (or y' , respectively) can simply be included in the labels of c_{xy} and r_{xy} by replacing $\ell(p_{xy})$ by $\ell'(p_{xy}) = (\ell(p_{xy}), z')$, with $z' = x'$ or y' depending on whether $p = c$ or r . The algorithm is easy to modify to update as needed. The length of an optimal game (that is, one where both players use an optimal strategy, including in their choice of starting vertices) is also given by the algorithm—it is $L(D) = \min_{x \in V(D)} \max_{y \in V(D)} \{\ell(c_{xy})\}$. Indeed, this is a simple way to test if the (di)graph D is cop-win; it is so if and only if $L(D) < \infty$. Further, $L(D)$ can be computed (updated) throughout the execution of the algorithm and so can be a part of the output at almost no additional cost.

3. More cops, more robbers

The algorithm of the preceding section can be used to test if $k \geq 1$ cops can catch $1 \leq l \leq k$ robbers, provided that “catch” means that each robber shares a vertex with at least one cop. For k cops (and, analogously, for l robbers) to be on at most k vertices means that they are at a point in V^k . For a point $u \in V^k$ write $[u] = \{u_1, u_2, \dots, u_k\}$. It is now natural to introduce the following.

Fix integers $0 < l \leq k$ and a digraph $D = (V, A)$ with minimum outdegree at least 1. Define $C_k(D) = (V^k, A_k)$ by setting $A_k = \{(u, v) : u, v \in V^k, v_i \in N_D^+(u_i), i = 1, 2, \dots, k\}$. Define $R_l(D) = (V^l, A_l)$ similarly. Thus, a move by either the cops or the robbers on D corresponds to moves by one cop on $C_k(D)$ or by one robber on $R_l(D)$. It remains to connect the two. The (k, l, D) game graph \mathcal{D} is the disjoint union of copies of $C_k(D)$ and $R_l(D)$. Formally, $\mathcal{D} = ((V^k \times \{c\}) \cup (V^l \times \{r\}), A(c) \cup A(r))$ where $A(c) = \{(u, c), (v, c) : u, v \in V^k, (u, v) \in A_k\}$ and $A(r) = \{(u, r), (v, r) : u, v \in V^l, (u, v) \in A_l\}$. The game can now be played on \mathcal{D} : the cops move from vertex to vertex of $V^k \times \{c\}$ while the robbers move on $V^l \times \{r\}$ (the cops choose their position first, the robbers second, then they alternate). The cops win if at some stage they are at some (u, c) with the robbers at (v, r) such that $[v] \subseteq [u]$.

The algorithm can be applied to \mathcal{D} directly.

If we insist that the game be played on a whole connected graph, we can augment the model somewhat. The k, l, D augmented game digraph \mathcal{D}^* is constructed from \mathcal{D} by first adding two new control vertices α from which arcs go to all the vertices of $C_k(D)$, and β , from which arcs go to all the vertices of $R_l(D)$, and by adding the arcs indicating how the cops control the space around them on D . Thus, $V(\mathcal{D}^*) = V(\mathcal{D}) \cup \{\alpha, \beta\}$, $\alpha, \beta \notin V(\mathcal{D})$, and $A(\mathcal{D}^*) = A(\mathcal{D}) \cup \{(\alpha, x) : x \in V^k \times \{c\}\} \cup \{(\beta, y) : y \in V^l \times \{r\}\} \cup \{(u, v) : u = ((x_1, x_2, \dots, x_k), c), v = (y, r), y \in \prod_{i=1}^k N^+(x_i)\}$.

Note that the augmented game digraph is never reflexive (α and β do not have loops) and that \mathcal{D}^* may be reflexive even though \mathcal{D} is not (cops can trade places).

Theorem 8. A digraph D is k -cop-win if and only if the augmented game digraph \mathcal{D}^* is 1-cop-win.

Proof. Suppose the cops have a winning strategy on D . We describe a winning strategy for the cop in the one-cop game on \mathcal{D}^* . The cop begins by choosing α . This forces the robber to start at β or in $R_l(D)$, otherwise he is immediately caught. The cop then moves to the vertex of $C_k(D)$ corresponding to the cops' initial move on D . The robber's move puts him in $R_l(D)$ whatever his starting point was. The cop then moves among the vertices of $C_k(D)$ according to the cops' winning strategy on D . By the construction of \mathcal{D}^* , the robber is eventually caught. Hence, \mathcal{D}^* is cop-win.

Suppose that the robbers have a winning strategy on D . On \mathcal{D}^* , the cop must start by choosing a vertex in $C_k(D)$ or α , otherwise the robber plays on $C_k(D)$ and can never be caught. The robber starts by choosing β if the cop is at α or the vertex of $C_k(D)$ corresponding to the robbers' winning initial position in D if the cop is in $C_k(D)$. In the first

case, the cop's next move gets him to a vertex in $C_k(D)$ while the robber moves to the vertex of $R_l(D)$ corresponding to the robbers' initial winning position D . The robber then mirrors, on $R_k(D)$, their winning strategy on D and so wins on \mathcal{D}^* . \square

4. Variants of the game

The move digraph, the game graph and the augmented game graph can all be defined for infinite digraphs with no vertices of out-degree zero. By changing our labelling algorithm into a transfinite recursion as in [11], the infinite k -cop-win digraphs of this type can also be characterised.

Variations of the game in which the movements of the cops and/or robber are somehow constrained have been considered in [3,10]. In [10] the cops and robber must use disjoint sets of edges, except that loops may be used by either player. The restriction where the cops must always move so as to stay within distance one of each other is considered in [3]. Another variation, in which the subgraphs induced by the vertices occupied by the cops must always be connected, is also mentioned in [3].

The move digraph, defined in Section 2, can be used to algorithmically characterise the 1-cop-win finite digraphs on which the cop and robber must use disjoint sets of arcs between distinct vertices, and both can use loops. All that is needed is to define the arcs of M so that they correspond only to moves along the appropriate arcs.

The game digraph, defined in Section 3, can be used for any situation in which any arrangement of cops that can occur during the game is a legal starting position, and both the cops and the robber may use any arc of D . This is accomplished by redefining the graph $C_k(D)$ to have as vertices the arrangements of cops that are allowed according to the rules of the game. As an example, consider the game in which the subgraphs induced by the vertices occupied by the cops must always be connected. In this case, the vertices of C_D are the k -tuples $(a_1, a_2, \dots, a_k) \in V(D)^k$ such that $\{a_1, a_2, \dots, a_k\}$ induce connected subgraphs of D (on at most k vertices).

Nothing stops us now from generalising further: as long as the rule for the cops' and the robber's moves are expressible by adjacencies in \mathcal{D} , the approach remains valid.

The requirement that *all* the robbers be caught at the same time seems not really to be a restriction in the sense that if a robber is caught, the cop that got her can always mimic her moves and stay with her, until all the others are caught, provided each gets caught on a cops' move (or all are caught robber's moves) so that some do not become free while others get caught. This, however is not the case. Consider the graph D consisting of two cycles, say x_0, x_1, \dots, x_s and y_0, y_1, \dots, y_t , $s, t \geq 6$, with edges $x_i x_{i+1}$, $y_i y_{i+1}$, addition modulo s, t , respectively, as well as $x_0 y_0, x_2 y_2$. If a robber is willing to meet a cop face to face (and then escape), two cops cannot catch two robbers. If each robber must avoid occupying the same vertex as a cop unless forced to, then two cops can catch the two robbers on this graph.

A more interesting and realistic version seems to be one in which the robbers get caught one by one and sent off to jail, freeing the cops to pursue the others (modelled by the addition of a vertex J which receives arcs from all the vertices to which robbers have access and by the addition of the rule that the robbers next move must be to J if she shares a vertex with a cop). But this is just a k cops, 1 robber version as all the robbers but one can be ignored while the single one is caught. The length of the game could be influenced, however, but we do not understand very much the function which gives, for naturals k, l and a (k, l) -cop-win graph, the length of the shortest game of k cops and l robbers.

5. Open questions

It is clear that, for any finite digraph D , putting a cop at every vertex guarantees that the cops win. Therefore, there is a least number of cops needed to win on D ; this is called the *cop number* of D . The cop number of graphs belonging to various classes has often been considered in the literature. See [2–6,10], for example. Nothing is known about the version with l robbers.

We still have no structural characterisation of cop-win digraphs that would allow us to show that some classes of digraphs contain only cop-win graphs. A theorem like that of [11,12] seems elusive.

As with a characterisation, it would be nice to have a closed-form bound on the length of the game, perhaps in terms of some other parameters. This is elusive as most "obvious parameters" do not work. For example, for each t there is a chordal, diameter two, finite graph on which the game takes t rounds. See [7].

Acknowledgement

Thanks are due to Benoit Thériault for his helpful remarks.

References

- [1] B. Alspach, D. Hanson X. Li, Pursuit-Evasion in Cayley graphs on dihedral groups, preprint, 2003.
- [2] R. Anstee, M. Farber, On bridged graphs and cop-win graphs, *J. Combin. Theory B* 44 (1988) 22–28.
- [3] N. Clarke, Constrained cops and robbers, Ph.D. Thesis, Dalhousie University, 2002.
- [4] S.L. Fitzpatrick, R.J. Nowakowski, Copnumber of graphs with strong isometric dimension two, *Ars Combin.* 59 (2001) 65–73.
- [5] P. Frankl, On a pursuit game on Cayley graphs, *Combinatorica* 7 (1987) 67–70.
- [6] A.S. Goldstein, E.M. Reingold, The complexity of pursuit on a graph, *Theoret. Comput. Sci.* 143 (1995) 93–112.
- [7] G. Hahn, F. Laviolette, N. Sauer, R.E. Woodrow, On cop-win graphs, *Discrete Math.* 258 (2002) 27–41.
- [8] Y.O. Hamidoune, On a pursuit game on Cayley digraphs, *European J. Combin.* 8 (1987) 285–289.
- [9] I. Hole, Discrete pursuit-evasion algorithms, Honors thesis, Dalhousie University, 2004.
- [10] S. Neufeld, R. Nowakowski, A vertex-to-vertex pursuit game played with disjoint sets of edges, *Finite and infinite combinatorics in sets and logic*, (Banff, AB, 1991), NATO Advanced Science Institutes Series C Mathematical and Physical Sciences, vol. 411, Kluwer Academic Publishers, Dordrecht, 1993, 299–312.
- [11] R. Nowakowski, P. Winkler, Vertex-to-vertex pursuit in a graph, *Discrete Math.* 43 (1983) 235–239.
- [12] A. Quilliot, Thèse de 3ème cycle, Université de Paris VI, 1978, pp. 131–145.
- [13] B. Thériault, Reconnaissance des graphes policiers-gagnants, M.Sc. Thesis, Université de Montréal, 2006.