A Simple Proof of Menger’s Theorem

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ABSTRACT

A proof of Menger’s theorem is presented.

We use the notation and terminology of Bondy and Murty [1].

Let \( D \) be a directed graph. If \( \{u\}, \{v\}, \) and \( S \) are disjoint subsets of \( V(D) \) and \( u \) and \( v \) are nonadjacent, then \( S \) separates \( u \) and \( v \) if every \((u, v)\)-path has a vertex in \( S \).

Proofs of Menger’s theorem are given in [2–14].

Menger’s Theorem. If no set of fewer than \( n \) vertices separates nonadjacent vertices \( u \) and \( v \) in a directed graph \( D \), then there are \( n \) internally disjoint \((u, v)\)-paths.

Proof. The proof uses induction on \( n \). The theorem is trivial for \( n = 1 \). Suppose \( u \) and \( v \) are separated by no set of less than \( n + 1 \) vertices \((n \geq 1)\). By the induction hypothesis there are \( n \) internally disjoint \((u, v)\)-paths \( P_1, \ldots, P_n \). Since the set of second vertices of \( P_1, \ldots, P_n \) does not separate \( u \) and \( v \), there is a \((u, v)\)-path \( P \) whose initial arc is not on \( P_i \), \( i = 1, \ldots, n \). Let \( x \) be the first vertex on \( P \) after \( u \) which is also on some \( P_i \), \( 1 \leq i \leq n \). Let \( P_{n+1} \) be the \((u, x)\)-section of \( P \). Assume \( P_1, \ldots, P_n \) have been chosen so that the distance in \( D - \{u\} \) from \( x \) to \( u \) is a minimum. If \( x = u \) we are done, so assume not.

In \( D - \{x\} \) there are \( n \) internally disjoint \((u, v)\)-paths \( Q_1, \ldots, Q_n \), again by the induction hypothesis. Assume \( Q_1, \ldots, Q_n \) have been chosen so that a minimum number of arcs in \( B = A(D) - \bigcup_{i=1}^{n+1} A(P_i) \) are used. Let \( H \) be the directed graph consisting of the vertices and arcs of \( Q_1 \),
..., \( Q_n \), together with the vertex \( x \). Choose some \( P_k \), \( 1 \leq k \leq n + 1 \), whose initial arc is not in \( A(H) \). Let \( y \) be the first vertex on \( P_k \) after \( u \) which is in \( V(H) \). If \( y = v \) we are done, so assume not.

If \( y = x \) then let \( R \) be the shortest \((x, v)\)-path in \( D - \{u\} \). Let \( z \) be the first vertex of \( R \) on some \( Q_j \), \( 1 \leq j \leq n \). Then the distance in \( D - \{u\} \) from \( z \) to \( v \) is less than the distance from \( x \) to \( v \). This contradicts the choice of \( P_1, \ldots, P_n, P_{n+1} \).

If \( y \) is on some \( Q_i \), \( 1 \leq i \leq n \), then the \((u, y)\)-section of \( Q \) has an arc in \( B \). Otherwise, two paths in \( \{P_1, \ldots, P_n, P_{n+1}\} \) intersect at a vertex other than \( u, v, \) or \( x \). Now if we replace the \((u, y)\)-section of \( Q \) by the \((u, y)\)-section of \( P_k \) we get \( n \) internally disjoint \((u, v)\)-paths in \( D - \{x\} \) using less arcs in \( B \) than \( Q_1, \ldots, Q_n \). This is a contradiction.

A similar proof can be used for the undirected version of Menger's theorem.

References
