

Definition 1 Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of the vertex set of G into non-empty classes. The quotient G/\mathcal{P} of G by \mathcal{P} is the graph whose vertices are the sets V_1, \dots, V_k and whose edges are the pairs $[V_i, V_j]$, $i \neq j$, such that there are $u_i \in V_i$, $u_j \in V_j$ with $[u_i, u_j] \in E(G)$. The mapping $\pi_{\mathcal{P}} : V(G) \rightarrow V(G/\mathcal{P})$ defined by $\pi_{\mathcal{P}}(u) = V_i$ such that $u \in V_i$, is the natural map for \mathcal{P} .

Quotients often provide a way of deriving the structure of an object from the structure of a larger one; factor groups are a good example. In the present context it is worth our while to investigate when the natural map $\pi_{\mathcal{P}} : V(G) \rightarrow V(G/\mathcal{P})$ is a homomorphism. Observe that if $\pi_{\mathcal{P}}$ is a homomorphism, then it is automatically faithful.

Lemma 1 A map $\phi : V(G) \rightarrow V(H)$ is a graph homomorphism if and only if the pre-image $\phi^{-1}(I)$ of every independent subset I of $V(H)$ is an independent set.

Corollary 1 Let G be a graph and $\mathcal{P} = \{V_1, \dots, V_k\}$ a partition of the vertex set of G . Then $\pi_{\mathcal{P}}$ is a homomorphism if and only if V_i is an independent set for each i .

Proposition 1 For every homomorphism $\phi : G \rightarrow H$ there is a partition \mathcal{P} of $V(G)$ into independent sets and a monomorphism $\psi : G/\mathcal{P} \rightarrow H$ such that $\phi = \psi \circ \pi_{\mathcal{P}}$.

A homomorphism ϕ of a graph G into H gives rise to an equivalence relation \equiv_{ϕ} , the kernel of ϕ , defined on V by $u \equiv_{\phi} v$ if and only if $\phi(u) = \phi(v)$. This, in turn, induces a partition \mathcal{P}_{ϕ} on the vertex set of G ; it is this partition that works in the Corollary 1 and Proposition 1. We can then speak of the *quotient of G by ϕ* and, abusing the notation slightly, we write G/ϕ for G/\mathcal{P}_{ϕ} . The language of quotients also gives us a more natural definition of a complete homomorphism.

Proposition 2 A homomorphism $\phi : G \rightarrow H$ is complete if and only if $\psi : G/\phi \rightarrow H$ (defined in Proposition 1) is an isomorphism.

Definition 2 A complete homomorphism $\phi : G \rightarrow H$ is elementary if there is unique pair of (nonadjacent) vertices $u, v \in V(G)$ which are identified by ϕ . We call H an elementary quotient of G .

One simple but useful consequence is that any homomorphism can be viewed as a sequence of identifications of pairs of vertices by complete homomorphisms.

Lemma 2 Let G and H be graphs and $\phi : G \rightarrow H$ a homomorphism. Then there is a natural number k and graphs $G = G_0, G_1, \dots, G_k$ such that G_{i+1} is an elementary quotient of G_i when $i < k$ and $G_k \cong G/\phi$.

The proof follows by repeated applications of the following simple lemma.

Lemma 3 *Let $\phi : G \rightarrow H$ be a homomorphism. Suppose there are vertices $u \neq v \in V(G)$ such that $h(u) = h(v)$. Then there is a graph $G' = (V', E')$, an elementary homomorphism $h' : G \rightarrow G'$ identifying u and v and a homomorphism $h'' : G' \rightarrow H$ such that $G/h' \cong G'$ and $h = h'' \circ h'$.*

Proof: Let $G = (V, E)$. Define h' by $h'(u) = v$ and $h'(z) = z$ if $z \neq u$. Define $V' = V \setminus \{u\}$ and $E' = E \cup \{vz : uz \in E\}$. It is easy to verify that $h' : G \rightarrow V'$ is a homomorphism remembering that $uv \notin E$. Define $h'' : G' \rightarrow H$ by setting $h''(z) = h(z)$. Clearly $h''h'(z) = h(z)$ if $z \neq u$ and $h''h'(u) = h''(v) = v = h(v) = h(u)$. It is equally easy to see that $G/h' \cong G'$ as there is only one class, say V_0 , of the partition induced by h' that has more than one element. This means that $H' : G/h' \rightarrow G'$ defined by $H'(\{z\}) = h'(z)$ and $H'(\{u, v\}) = h'(u) = h'(v)$ is an isomorphism. \square