

THE COMPOSITION OF GRAPHS

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1. **Introduction.** In a recent paper [1] Harary introduces a new binary operation on graphs called *composition* (or *lexicographic multiplication*) (see [1; 31], or Definition 2, below), and discusses the automorphism group of the composition in terms of the automorphism groups of the components. It is stated in [1; 32] that $G(X \circ Y) = G(X) \circ G(Y)$ (see [1; 30], or Definition 1, below) if and only if at most one of X and Y is complete. This is however incorrect, the condition being only necessary but not sufficient. It is the purpose of this note to give the correct statement of Harary's theorem, and to enlarge slightly the class of graphs under consideration.

By a *graph* X we mean a set $V(X)$ (the set of *vertices* of X) together with a set $E(X)$ (the set of *edges* of X) of unordered pairs of *distinct* elements of $V(X)$. Unordered pairs will be denoted by brackets. If A is a set, $|A|$ denotes the cardinal of A . By $|X|$ we mean $|V(X)|$. For $x \in V(X)$ we put $V(X; x) = \{y \in V(X) \mid [x, y] \in E(X)\}$. $d(X; x) = |V(X; x)|$ is the *degree* of x in X . A graph X is *almost locally finite* if for any two distinct vertices x, y , $V(X; x) \cap V(X; y)$ is finite. By the *complement* of a graph X we mean the graph X' given by $V(X') = V(X)$, $E(X') = \{[x, y] \mid x, y \in V(X'), x \neq y, [x, y] \notin E(X)\}$. An *automorphism* of X is a one-one function ϕ of $V(X)$ onto $V(X)$ such that $[\phi x, \phi y] \in E(X)$ if and only if $[x, y] \in E(X)$. By $G(X)$ we denote the automorphism group of X . Note that $G(X) = G(X')$.

DEFINITION 1. Let A and B be sets, G and H groups of one-one functions of A onto itself, and B onto itself, respectively. Define $G \circ H$ (the *composition* of G and H) to be the group of all one-one functions f of $A \times B$ onto itself for which there exist $g \in G$ and $h_a \in H$, $a \in A$, such that

$$f(a, b) = (ga, h_a b)$$

for all $(a, b) \in A \times B$.

DEFINITION 2. Let X and Y be graphs. By the *lexicographic product* (or *composition*) $X \circ Y$ we mean the graph given by

$$V(X \circ Y) = V(X) \times V(Y),$$

$$E(X \circ Y) = \{[(x, y), (x', y')] \mid [x, x'] \in E(X),$$

$$\text{or } x = x' \text{ and } [y, y'] \in E(Y)\}.$$

It is easily verified that $X \circ (Y \circ Z) \cong (X \circ Y) \circ Z$, and that $(X \circ Y)' = X' \circ Y'$. Idempotency is possible, e.g. if C_n is the complete n -graph, and n is infinite, then $C_n \circ C_n \cong C_n$.

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It is clear that $G(X) \circ G(Y) \subset G(X \circ Y)$. We wish to give a necessary and sufficient condition that $G(X \circ Y) = G(X) \circ G(Y)$.

DEFINITION 3. Let X be a graph. Define equivalence relations R and S on $V(X)$ as follows:

$$\begin{aligned} xRy & \text{ if and only if } V(X; x) = V(X; y), \\ xSy & \text{ if and only if } V(X; x) \cup \{x\} = V(X; y) \cup \{y\}. \end{aligned}$$

Note that xRy implies $[x, y] \notin E(X)$, xSy and $x \neq y$ imply $[x, y] \in E(X)$. Hence if xRy and xSy , then $x = y$. Both xRy and xSy each imply the existence of a $\phi \in G(X)$ such that $\phi x = y$, $\phi y = x$, and $\phi z = z$ for each $z \in V(X)$ which is adjacent to both x and y .

Let R', S' be the corresponding relations on $V(X')$. Then xRy if and only if $xS'y$, and xSy if and only if $xR'y$.

By Δ we denote $\Delta = \{(x, x) \mid x \in V(X)\}$.

In order to avoid difficulties arising from idempotency we have to impose certain conditions concerning the finiteness of X and Y .

THEOREM. Let X be an almost locally finite graph, Y a finite graph, and let R, S be the above relations defined on $V(X)$. Then a necessary and sufficient condition that $G(X \circ Y) = G(X) \circ G(Y)$ is that Y be connected if $R \neq \Delta$, and that Y' be connected if $S \neq \Delta$.

2. Proof of theorem. Necessity. Let $x_0Rx_1, x_0 \neq x_1$, and suppose that Y is disconnected. Let U be a component of Y . Define $\psi: V(X \circ Y) \rightarrow V(X \circ Y)$ by

$$\psi(x, y) = \begin{cases} (x, y), & \text{if } y \notin V(U), \text{ or if } y \in V(U) \text{ and } x \neq x_0, x_1 \\ (x_{1-i}, y), & \text{if } y \in V(U) \text{ and } x = x_i, i = 0 \text{ or } 1. \end{cases}$$

It is easily verified that $\psi \in G(X \circ Y)$. Obviously $\psi \notin G(X) \circ G(Y)$.

Now let $x_0 \neq x_1$ and x_0Sx_1 . Then $x_0R'x_1$. Hence if Y' is disconnected, there is a $\psi' \in G(X' \circ Y') - G(X') \circ G(Y')$. But $G(X' \circ Y') - G(X') \circ G(Y') = G(X \circ Y) - G(X) \circ G(Y)$ because $X' \circ Y' = (X \circ Y)'$.

Sufficiency. For $x \in V(X)$ we shall denote by Y_x the maximal subgraph of $X \circ Y$ with

$$V(Y_x) = \{x\} \times V(Y).$$

Clearly $Y \cong Y_x$. Let $\phi \in G(X \circ Y)$. We have to show that given $x \in V(X)$ there is an $x' \in V(X)$ with $\phi Y_x = Y_{x'}$. For fixed x and any $y \in V(Y)$ we will denote $\phi(x, y)$ by (t_y, z_y) .

If $|Y| = 1$, then $X \circ Y \cong X$, hence there is nothing to prove. We will therefore assume that $|Y| \geq 2$. Let K_x be the component of X which contains the vertex x . If K_x consists only of x , and if x is the only isolated vertex of X , then

clearly $\phi Y_x = Y_x$. If X contains more than one isolated vertex, then $R \neq \Delta$, hence by hypothesis Y is connected, and therefore $\phi Y_x = Y_x$ for some $x' \in V(X)$.

We will henceforth assume that $|K_x| \geq 2$. By ρ_X, ρ_Y, ρ we denote distance in $X, Y, X \circ Y$, respectively, whenever distance is defined. For fixed $x \in V(X)$ let C_x be the maximal subgraph of X with

$$V(C_x) = \{c \in V(X) \mid V(Y_c) \cap V(\phi Y_x) \neq \square\}$$

(\square denotes the vacuous set) and for $c \in V(C_x)$ let

$$B_c = \{y \in V(Y) \mid t_y = c\}.$$

We shall show that C_x is a complete graph. Let $c_1, c_2 \in V(C_x)$. It is immediate from $|K_x| \geq 2$ and the definition of $X \circ Y$ that

$$\rho((x, y_1), (x, y_2)) \leq 2,$$

where $y_i \in B_{c_i}, i = 1, 2$. Hence $\rho((c_1, z_{y_1}), (c_2, z_{y_2})) \leq 2$, and therefore $\rho_X(c_1, c_2) \leq 2$. Now assume $\rho_X(c_1, c_2) = 2$. It follows that $\rho((c_1, z_{y_1}), (c_2, z_{y_2})) = 2$, and hence $\rho((x, y_1), (x, y_2)) = 2$. But this implies either (1) that y_1 and y_2 belong to different components of Y , or (2) that $\rho_Y(y_1, y_2) = 2$. Let $W = V(X \circ Y; (x, y_1)) \cap V(X \circ Y; (x, y_2))$. Then $\phi W = V(X \circ Y; (c_1, z_{y_1})) \cap V(X \circ Y; (c_2, z_{y_2}))$.

In case (1),

$$W = V(X; x) \times V(Y), \quad \phi W = (V(X; c_1) \cap V(X; c_2)) \times V(Y).$$

By the finiteness hypotheses on X and Y it follows immediately that $d(X; x)$ is finite. Now note that

$$\begin{aligned} d(X \circ Y; (x, y_i)) &= d(X; x) \mid Y \mid + d(Y; y_i), \\ d(X \circ Y; (c_i, z_{y_i})) &= d(X; c_i) \mid Y \mid + d(Y; z_{y_i}), \quad i = 1, 2. \end{aligned}$$

It follows that $|Y|$ divides $d(Y; y_i) - d(Y; z_{y_i}), i = 1, 2$. Hence either $d(Y; y_i) = d(Y; z_{y_i})$, and then $d(X; x) = d(X; c_i), i = 1, 2$, or (without loss of generality) $d(Y; y_i) - d(Y; z_{y_i}) = k \mid Y \mid$, where $k \geq 1$. But then $d(Y; y_i) \geq |Y|$, a contradiction. $d(X; x) = d(X; c_i)$ together with $|W| = |\phi W|$ implies $V(X; c_1) = V(X; c_2)$, i.e. $c_1 R c_2$, a contradiction.

In case (2), $W' = V(Y; y_1) \cap V(Y; y_2) \neq \square$, and

$$W = (\{x\} \times W') \cup (V(X; x) \times V(Y)),$$

ϕW as in case (1). As in case (1), $V(X; x)$ is finite, and we note that $|Y|$ divides $|\phi W|$, while it does not divide $|W|$ (because $1 \leq |W'| < |Y|$). Hence in both cases we arrive at a contradiction. It follows that $\rho_X(c_1, c_2) \leq 1$. But since c_1 and c_2 were arbitrary vertices of C_x , this means that C_x is complete.

If $|C_x| = 1$ for all $x \in V(X)$, the proof is complete. If $|C_x| \geq 2$ for some $x \in V(X)$, we show that Y' is disconnected. It suffices to show that if $y_1 \in B_{c_1}, y_2 \in B_{c_2}, c_1 \neq c_2$, then $[y_1, y_2] \in E(Y)$. By the completeness of $C_x, c_1 \neq c_2$

implies $[c_1, c_2] \in E(X)$. Hence $e = [(c_1, z_{y_1}), (c_2, z_{y_2})] \in E(X \circ Y)$, and therefore $\phi^{-1}e = [(x, y_1), (x, y_2)] \in E(X \circ Y)$, i.e. $[y_1, y_2] \in E(Y)$.

The proof of the theorem will be complete if we show that $S \neq \Delta$. In order to do this let $c_1, c_2 \in V(C_x)$, $c_1 \neq c_2$, and let $a \in V(X)$ be such that $[a, c_1] \in E(X)$. If $a \in V(C_x)$, then by the completeness of C_x , $[a, c_2] \in E(X)$. If $a \notin V(C_x)$, then $e = [(a, b), (c_1, z_{y_1})] \in E(X \circ Y)$, where $b \in V(Y)$, $y_1 \in B_{c_1}$. Hence $\phi^{-1}e = [(a', b'), (x, y_1)] \in E(X \circ Y)$, where $(a', b') = \phi^{-1}(a, b)$. $a \notin V(C_x)$ implies $a' \neq x$, hence $[a', x] \in E(X)$. Therefore $e' = [(a', b'), (x, y_2)] \in E(X \circ Y)$, where $y_2 \in B_{c_2}$, and this in turn implies $\phi e' = [(a, b), (c_2, z_{y_2})] \in E(X \circ Y)$. $a \neq c_2$ (because $a \notin V(C_x)$), hence $[a, c_2] \in E(X)$. Thus $c_1 S c_2$.

REFERENCE

1. F. HARARY, *On the group of the composition of two graphs*, this Journal, vol. 26(1959), pp. 29-34.

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