

THE LEXICOGRAPHIC PRODUCT OF GRAPHS

BY GERT SABIDUSSI

1. **The automorphism group of the lexicographic product.** The purpose of this section is to improve an earlier result [3] giving a necessary and sufficient condition under which the automorphism group of the lexicographic product of two graphs X, Y is equal to the wreath product ([1; 81]; [3, Definition 1]) of the groups of X and Y . Using the terminology and notation of [3] we have the following:

THEOREM 1. *Let X be any graph with $E(X) \neq \square$, and suppose that Y is such that $|V(Y; y) \cap V(Y; y')| < |Y|$ for any two distinct vertices y, y' of Y . Then a necessary and sufficient condition that $G(X) \circ G(Y) = G(X \circ Y)$ is that Y be connected if $R \neq \Delta$, and that Y' be connected if $S \neq \Delta$.*

In [3], X and Y were assumed to be almost locally finite, and finite, respectively. Finite graphs trivially satisfy the condition that $|V(Y; y) \cap V(Y; y')| \leq d(Y; y) < |Y|$.

The proof of Theorem 1 will be broken up into a sequence of lemmas. Note first that if X and Y satisfy the hypotheses of the theorem, then $X \circ Y$ is not isomorphic to Y . For if it were, then $|V(X \circ Y; (x, y)) \cap V(X \circ Y; (x, y'))| < |Y|$ for any $x \in X$, and distinct vertices $y, y' \in Y$. But if $d(X; x) \geq 1$, then

$$|V(X \circ Y; (x, y)) \cap V(X \circ Y; (x, y'))| \geq |V(X; x) \times V(Y)| \geq |Y|.$$

It follows from this that if $|K_x| = 1$, then $\phi Y_x = Y_x$, with $|K_{x'}| = 1$.

It therefore suffices to consider the case $|Y| \geq 2, |K_x| \geq 2$, and to prove that C_x is complete. Let $c_1, c_2 \in C_x, y_i \in B_{c_i}, i = 1, 2$, and suppose that $\rho_X(c_1, c_2) = 2$. This assumption is made throughout the following sequence of lemmas.

LEMMA 1. *Let A, B, C, M, N be sets such that $|N| < |B|$, and $A \times B = (C \times B) \cup M - N$, then $A \supset C$. If, moreover, $|M| < |B|$, then $A = C$.*

This is obvious.

LEMMA 2. $V(X; t_y) \subset V(X; c_1) \cap V(X; c_2)$ for each $y \in Y$ with $d(Y; y) < |Y|$.

Proof. Consider $W = V(X \circ Y; (x, y_1)) \cap V(X \circ Y; (x, y_2))$. Since $\rho_X(c_1, c_2) = 2$,

$$(1) \quad \phi W = (V(X; c_1) \cap V(X; c_2)) \times V(Y) \neq \square.$$

For W itself we obtain $W = (V(X; x) \times V(Y)) \cup D$, where

$$D = \{x\} \times (V(Y; y_1) \cap V(Y; y_2)).$$

Received December 20, 1960. Written with the support of the National Science Foundation, Grant No. NSF-G14084.

Hence for any $y \in Y$,

$$W = (V(X \circ Y; (x, y)) - A_y) \cup D,$$

where $A_y = \{x\} \times V(Y; y) \subset V(X \circ Y; (x, y))$. Therefore if we put $U_y = \{t_y\} \times V(Y; z_y)$, we obtain

$$(2) \quad \phi W = (V(X; t_y) \times V(Y) \cup U_y) - \phi A_y \cup \phi D.$$

By hypothesis the cardinal of A_y is $< |Y|$. Hence by (1), (2), and Lemma 1, $V(X; t_y) \subset V(X; c_1) \cap V(X; c_2)$.

We shall henceforth denote $V(X; c_1) \cap V(X; c_2)$ by Q .

LEMMA 3. Let w_1, w_2 be two distinct vertices of Y . Then

$$V(X; t_{w_1}) \cap V(X; t_{w_2}) \subset Q$$

with equality if $[t_{w_1}, t_{w_2}] \not\subseteq X$.

Proof. Let y_1, y_2, W, D , etc. be as in the proof of Lemma 2, and let $w_1, w_2 \in Y, w_1 \neq w_2$. Then

$$V(X \circ Y; (x, w_1)) \cap V(X \circ Y; (x, w_2)) = V(X; x) \times V(Y) \cup D',$$

where $D' = \{x\} \times (V(Y; w_1) \cap V(Y; w_2))$. Hence

$$W = V(X \circ Y; (x, w_1)) \cap V(X \circ Y; (x, w_2)) - D' \cup D,$$

and therefore

$$(3) \quad \phi W = V(X \circ Y; (t_{w_1}, z_{w_1})) \cap V(X \circ Y; (t_{w_2}, z_{w_2})) - \phi D' \cup \phi D.$$

We now compare (1) and (3) distinguishing three cases:

Case (i): $t_{w_1} = t_{w_2}$. Here the right-hand side of (3) equals $(V(X; t_{w_1}) \times V(Y)) \cup D'' - \phi D' \cup \phi D$, where $D'' = \{t_{w_1}\} \times (V(Y; z_{w_1}) \cap V(Y; z_{w_2}))$. Hence by Lemma 1, $V(X; t_{w_1}) \cap V(X; t_{w_2}) = V(X; t_{w_1}) = Q$.

Case (ii): $[t_{w_1}, t_{w_2}] \in X$. Here

$$(4) \quad \phi W = (V(X; t_{w_1}) \cap V(X; t_{w_2})) \times V(Y) \cup A_1 \cup A_2 - \phi D' \cup \phi D,$$

where $A_i = \{t_{w_i}\} \times V(Y; z_{w_i}), i = 1, 2$. The conclusion again follows from Lemma 1. Similarly in case (iii): $\rho_X(t_{w_1}, t_{w_2}) = 2$.

COROLLARY 1. Let $t \in C_x$ be such that $|B_t| \geq 2$. Then $V(X; t) = Q$.

COROLLARY 2. If $t \in V(C_x) \cap Q$, then $|B_t| = 1$.

Proof. If $|B_t| \geq 2$, then by Corollary 1, $t \in Q = V(X; t)$, a contradiction.

COROLLARY 3. Let $t \in C_x$. (i) If $t \notin Q$, then $d(C_x; t) \leq 1$. (ii) If $t \in Q$, then $\rho_X(t, t') \leq 1$ for each $t' \in C_x$.

Proof. (i): If $d(C_x; t) \geq 2$, let $t_1, t_2 \in V(C_x; t), t_1 \neq t_2$. Then by Lemma 3, $t \in V(X; t_1) \cap V(X; t_2) \subset Q$, a contradiction. (ii): If $\rho_X(t, t') \geq 2$, then by Lemma 3, $t \in Q = V(X; t) \cap V(X; t')$.

COROLLARY 4. *If $V(C_x) \cap Q \neq \square$, then C_x is a star whose center is the only vertex of C_x belonging to Q . If $V(C_x) \cap Q = \square$, then each component of C_x has at most two vertices.*

Proof. Suppose two distinct vertices t_1, t_2 of C_x belong to Q . By Corollary 3 (ii), $[t_i, c_1] \in C_x, i = 1, 2$. Hence $d(C_x; c_1) \geq 2$ contrary to Corollary 3 (i). Thus $|V(C_x) \cap Q| \leq 1$. If $V(C_x) \cap Q = \square$, then by Corollary 3 (ii) each vertex of C_x is of degree 0 or 1, and hence each component of C_x is either a single vertex or a single edge. If $V(C_x) \cap Q = \{c_0\}$, then by Corollary 3 (ii) every vertex of $C_x - c_0$ is adjacent to c_0 , and no two such vertices are adjacent to each other (Corollary 3 (i)). Hence C_x is a star with center c_0 .

LEMMA 4. *If Y contains two distinct vertices w_1, w_2 such that $V(X; t_{w_1}) \cap V(X; t_{w_2}) \neq Q$, then $V(X; t_{w_1}) \cap V(X; t_{w_2}) \cup \{t_{w_i}\} = Q, i = 1$ or 2 , and $|V(Y; y_1) \cap V(Y; y_2)| \geq 1 + \min \{d(Y; y) : y \in Y\}$.*

Proof. We use the same notation as in the proof of Lemma 3. By Lemma 3, $[t_{w_1}, t_{w_2}] \in X$. Hence

$$(5) \quad \phi W = A \cup A_1 \cup A_2 - \phi D' \cup \phi D,$$

where $A = (V(X; t_{w_1}) \cap V(X; t_{w_2})) \times V(Y)$. Note that A, A_1, A_2 are disjoint.

By hypothesis and Lemma 3 there exists a $t \in Q - (V(X; t_{w_1}) \cap V(X; t_{w_2}))$. $T = \{t\} \times V(Y)$ is disjoint from A , and hence $T \subset A_1 \cup A_2 \cup \phi D$. Since $|\phi D| < |Y| = |T|$, T meets $A_1 \cup A_2$, say $T \cap A_1 \neq \square$. Then $t = t_{w_1} \in Q$, and hence $T \cap A_2 = \square$. Therefore $T \subset A_1 \cup \phi D$ so that

$$(6) \quad \phi D \supset T - A_1 = \{t_{w_1}\} \times (V(Y) - V(Y; z_{w_1})) \neq \square.$$

By Corollary 4, $t_{w_1} \in Q$ implies $t_{w_2} \notin Q$, whence $\phi D' \supset A_2$. Thus $|D'| \geq |A_2| \geq \min \{d(Y; y) : y \in Y\}$. Since $V(X; t_{w_1}) \cap V(X; t_{w_2}) \subset Q, \phi D' \subset \phi D$. Hence $\phi D \supset A_2$, and since A_2 is disjoint from $\{t_{w_1}\} \times (V(Y) - V(Y; z_{w_1}))$, we have $|V(Y; y_1) \cap V(Y; y_2)| = |\phi D| \geq 1 + \min \{d(Y; y) : y \in Y\}$.

Proof of the completeness of C_x . We assume as in the preceding lemmas that there exist $c_1, c_2 \in C_x$ such that $\rho_X(c_1, c_2) = 2$.

Case (i): $V(C_x) \cap Q \neq \square$. By Corollary 4, C_x is a star whose center c_0 is the only vertex of C_x in Q . By Corollary 2, $|B_{c_0}| = 1$, say $B_{c_0} = \{y_0\}$. Note first that if $y \in Y, y \neq y_0$, then $[y, y_0] \in Y$. For if y and y_0 are not adjacent, then by Lemma 3, $c_0 \in Q = V(X; t_w) \cap V(X; c_0)$, a contradiction. Next let $P = (w_0, \dots, w_r)$ be a path of Y joining $w_0 \in B_{c_1}$ with $w_r \in B_{c_2}$, and suppose that $y_0 \notin P$. Then $(t_{w_0}, \dots, t_{w_r})$ is a path of C_x joining c_1 and c_2 and not containing c_0 . But this is impossible since c_0 is the center of C_x . It follows that B_{c_1} and B_{c_2} belong to different components of $Y - y_0$, and hence $V(Y; y_1) \cap V(Y; y_2) = \{y_0\}$. Note finally that $V(X; c_0) \cap V(X; c_1) \neq Q$, hence by Lemma 4, and the connectedness of $Y, |V(Y; y_1) \cap V(Y; y_2)| \geq 2$, a contradiction.

Case (ii): $V(C_x) \cap Q = \square$. In this case C_x is disconnected. For if $|C_x| = 2$,

then $V(C_x) = \{c_1, c_2\}$, and since $\rho_x(c_1, c_2) = 2$, C_x is disconnected. If $|C_x| \geq 3$ the disconnectedness of C_x is a consequence of no component of C_x having more than two vertices. C_x being disconnected, Y is likewise disconnected. It follows from this that there is at most one $t_0 \in C_x$ with $|B_{t_0}| \geq 2$. Otherwise by Corollary 1, $R \neq \Delta$, and then by hypothesis Y is connected, a contradiction. We shall call a component of Y *small* if its order is ≤ 2 . Note that if K is a component of Y with $V(K) \cap B_{t_0} = \square$, then K is small.

Let w_1, w_2 be vertices of two distinct small components of Y . Then $\rho_x(t_{w_1}, t_{w_2}) = 2$, and by Lemma 3, $Q = V(X; t_{w_1}) \cap V(X; t_{w_2})$. Moreover, $d(Y; w_i) \leq 1 < |Y|, i = 1, 2$, hence by Lemma 2, $V(X; t_{w_i}) \subset Q = V(X; t_{w_1}) \cap V(X; t_{w_2}), i = 1, 2$, i.e. $t_{w_1} R t_{w_2}$, a contradiction. Therefore Y has at most one small component Z .

Suppose Z has two vertices v_1, v_2 . If $V(X; t_{v_1}) \cap V(X; t_{v_2}) = Q$, then by the same argument as in the preceding paragraph $t_{v_1} R t_{v_2}$. Hence $V(X; t_{v_1}) \cap V(X; t_{v_2}) \neq Q$. Therefore by Lemma 4, $t_{v_i} \notin Q, i = 1$ or 2 , contrary to the assumption that $V(C_x) \cap Q = \square$. Thus Z consists of a single vertex. We claim that $Z = y_1$ or y_2 . If $c_1 = t_0$, then $[c_2, t_0] \notin C_x$. Hence $B_{c_2} = \{y_2\}$, and y_2 is not adjacent to any vertex in B_{t_0} . Thus the component of Y containing y_2 is small, i.e. $y_2 = Z$. Suppose then that $c_1 \neq t_0$. If $d(C_x; c_1) = 0$, then the component of Y containing y_1 is y_1 . If $d(C_x; c_1) = 1$, let $t \in V(C_x; c_1)$. If $t \neq t_0$, then $B_t = \{y\}$, and $[y, y_1]$ is a small component of Y distinct from Z . Therefore $t = t_0$, and then $[c_2, t_0] \notin C_x$; otherwise $d(C_x; t_0) \geq 2$, contrary to Corollary 3 (i). As before, $[c_2, t_0] \notin C_x$ implies $Z = y_2$.

We may therefore assume without loss of generality that $Z = y_1$. Then $d(Y; y_1) = 0$, and by Lemma 2,

$$(7) \quad V(X; c_1) \subset V(X; c_2).$$

It also follows that $V(C_x) = \{c_1, c_2\}$, and $t_0 = c_2$. Next, $d(Y; y_1) = 0$ implies $V(X \circ Y; (x, y_1)) = V(X; x) \times V(Y)$. Hence

$$(8) \quad V(X \circ Y; (x, y_2)) - \{x\} \times V(Y; y_2) = V(X \circ Y; (x, y_1)).$$

Also

$$(9) \quad \phi V(X \circ Y; (x, y_1)) = V(X; c_1) \times V(Y).$$

By (8) and (9),

$$V(X; c_2) \times V(Y) \cup \{c_2\} \times V(Y; z_{v_2}) - \phi(\{x\} \times V(Y; y_2)) = V(X; c_1) \times V(Y).$$

If the inclusion (7) is proper, $\phi(\{x\} \times V(Y; y_2))$ must contain a set of the form $\{u\} \times V(Y), u \in V(X; c_2)$. But $\phi(\{x\} \times V(Y; y_2)) \subset \{c_2\} \times V(Y)$. Hence $V(X; c_1) = V(X; c_2)$, contrary to the disconnectedness of Y . This completes the proof of the completeness of C_x .

The remaining part of the proof of Theorem 1 is identical with that given in [3; 695, 696].

2. Lexicographically idempotent graphs. We wish to make two remarks

concerning the maximal complete subgraphs of such graphs. The first is that if X is idempotent and has no isolated vertices, then its maximal complete subgraphs are infinite and uniformly distributed over X in the sense described by Theorem 2. The second is that the order of the maximal subgraphs is not in any way determined by the order of X . In order to show this we define the lexicographic product of infinitely many graphs and point out some of its properties.

LEMMA 5. *If X is an idempotent graph without isolated vertices, then every maximal complete subgraph of X is infinite.*

Proof. The union of a tower of complete subgraphs of X being complete, it follows from Zorn's lemma that X has a maximal complete subgraph. Let \mathfrak{M} be the set of all maximal complete subgraphs of X , and put $m = \min \{|A| : A \in \mathfrak{M}\}$. Suppose that m is finite. Let $A_0 \in \mathfrak{M}$ be of order m , and let ϕ be an isomorphism of X onto $X \circ X$. Then ϕA_0 is a maximal complete subgraph of $X \circ X$. If $\phi A_0 \subset X_a$ for some $a \in X$, let $b \in X$ be adjacent to a , then the maximal subgraph Y_0 of $X \circ X$ with $V(Y_0) = \{(b, b)\} \cup V(\phi A_0)$ is a complete subgraph of $X \circ X$, contrary to the maximality of ϕA_0 . (X_a denotes the maximal subgraph of $X \circ X$ with $V(X_a) = \{a\} \times V(X)$.) Hence

$$(10) \quad \phi A_0 \subsetneq X_a$$

for each $a \in X$. For given $a \in X$ let $B_a = X_a \cap \phi A_0$. Either $B_a = \square$ or else B_a is a maximal complete subgraph of X_a . For if B is a complete subgraph of X_a properly containing B_a , then the maximal subgraph Z_0 of $X \circ X$ with $V(Z_0) = V(B) \cup V(\phi A_0)$ is a complete subgraph of $X \circ X$ properly containing ϕA_0 , a contradiction.

By (10) there exist $a_1, a_2 \in X, a_1 \neq a_2$, such that $B_{a_i} \neq \square, i = 1, 2$. Since m is finite, $|B_{a_i}| < m, i = 1, 2$. Let A_i be the maximal subgraph of X with $V(A_i) = \{x \in X : (a_i, x) \in B_{a_i}\}, i = 1, 2$. Since B_{a_i} is a maximal complete subgraph of $X_{a_i}, A_i \in \mathfrak{M}$. But $|A_i| = |B_{a_i}| < m$, a contradiction. It follows that m is infinite.

THEOREM 2. *Let X be an idempotent graph without isolated vertices, A a finite complete subgraph of X, K an infinite complete subgraph of X . Then there exists a maximal complete subgraph B of X such that $A \subset B$ and $|B| \geq |K|$.*

Proof. We use induction on $|A|$. If $|A| = 1$, i.e. if A is a single vertex x , let $\phi x = (a, b)$, where $\phi : X \approx X \circ X$, and let $a' \in X$ be adjacent to a . Let $K_{a'}$ be a subgraph of $X_{a'}$ isomorphic to K , then the maximal subgraph C of X with $V(C) = \phi^{-1}(V(K_{a'}) \cup \{\phi x\})$ is complete, $x \in C$, and $|C| = |K|$. Any maximal complete subgraph B of X containing C will then satisfy the conditions of the theorem.

Now assume the theorem true for all complete subgraphs of X of order $< n$. Let $A \subset X$ be complete, $|A| = n$. Let $x_1, \dots, x_s, s \leq n$, be the vertices of X for which $B_{x_i} = X_{x_i} \cap \phi A \neq \square$. Each B_{x_i} is complete. Case (i): $s = 1$.

In this case repeat the construction in the preceding paragraph, replacing x by A . Case (ii): $s \geq 2$. Then $|B_{x_i}| < n, i = 1, \dots, s$, hence by induction hypothesis there exists a complete subgraph B_i of X_{x_i} with $B_{x_i} \subset B_i$ and $|B_i| \geq |K|$. Moreover the maximal subgraph C of X with $V(C) = \phi^{-1}(\bigcup_{i=1}^s V(B_i))$ is complete, has order $\geq |K|$, and contains A .

We will now give an example of a connected idempotent graph X showing that the order of the maximal complete subgraphs of X is independent of $|X|$. We shall make use of the following general construction.

Let $\{X_\alpha : \alpha \in A\}$ be a family of graphs indexed by a totally ordered set A , and let $x_\alpha^{(0)} \in X_\alpha, \alpha \in A$. Put $R_\alpha = (X_\alpha, x_\alpha^{(0)}), \alpha \in A$, and define the lexicographic product $X = \prod_{\alpha \in A} R_\alpha$ as follows. Let W be the Cartesian product of the sets $V(X_\alpha), \alpha \in A$, and denote by $p_\alpha : W \rightarrow V(X_\alpha)$ the projection of W onto its α -th coordinate. Then

- (i) $V(X) = \{x \in W : p_\alpha x = x_\alpha^{(0)} \text{ for almost all } \alpha \in A\}$;
- (ii) $[x, y] \in E(X)$ if and only if $[p_{\alpha_0} x, p_{\alpha_0} y] \in E(X_{\alpha_0})$, where α_0 is the smallest $\alpha \in A$ such that $p_\alpha x \neq p_\alpha y$.

The product so defined has the following properties:

- (a) If each X_α is connected, then X is likewise connected.
- (b) A n.a.s.c. that X be complete is that each R_α be complete.
- (c) If B is a non-empty proper subset of A such that $B < A - B$, then

$$\prod_{\alpha \in A} R_\alpha \approx \prod_{\alpha \in B} R_\alpha \circ \prod_{\alpha \in A - B} R_\alpha .$$

It follows from (c) that if $B < A - B$ and if A, B , and $A - B$ are order-isomorphic, then any graph of the form $\prod_{\alpha \in A} R_\alpha$ is idempotent provided $R_\alpha \approx R$ for each $\alpha \in A$. If A is the set of rationals with the usual order we denote the graph $\prod_{\alpha \in A} R_\alpha, R_\alpha \approx R$, by R^* .

We can now give the example mentioned above. Let n be an infinite cardinal, N a set of cardinal $n, x^{(0)} \notin N$. Define $S_n = (X, x^{(0)})$ by $V(X) = \{x^{(0)}\} \cup N, E(X) = \{[x, x^{(0)}] : x \in N\}$. Then S_n^* is idempotent, connected, $|S_n^*| = n$, but S_n^* has no uncountable complete subgraph.

REFERENCES

1. M. HALL, JR., *The Theory of Groups*, New York, 1959.
2. F. HARARY, *On the group of the composition of two graphs*, this Journal, vol. 26(1959), pp. 29-34.
3. G. SABIDUSSI, *The composition of graphs*, this Journal, vol. 26(1959), pp. 693-696.

TULANE UNIVERSITY
AND
McMASTER UNIVERSITY