Lecture 23: The Compactness Theorem

1 Compactness Theorem

Definition 1 (Finite Satisfiability)**.** *A set* Φ *of formulas is* finitely satisfiable *iff every finite subset* Φ' *of* Φ *is satisfiable.*

Compactness Theorem: *A set of formulas* Φ *is satisfiable iff it is finitely satisfiable.*

The compactness theorem is often used in its contrapositive form: *A set of formulas* Φ *is unsatisfiable iff there is some finite subset of* Φ *that is unsatisfiable.*

The theorem is true for both first order logic and propositional logic. The proof for first order logic is outside the scope of this course, but we will give the proof for propositional logic. We will only consider the case of a countable number of propositions. The proof for larger cardinalities is essentially the same, but requires some form of the axiom of choice.

1.1 Proof of Compactness Theorem for Propositional Logic

We first prove a useful result about finite satisfiability, which allows us to enlarge a set of formulas while preserving their finite satisfiability.

Lemma 1 (Finite Satisfiability Lemma)**.** *Let* Σ *be a finitely satisfiable set of formulas and* φ *be a formula. Then* $\Sigma \cup \{\varphi\}$ *is finitely satisfiable or* $\Sigma \cup \{\neg\varphi\}$ *is finitely satisfiable.*

Proof. We prove the contrapositive statement. Suppose $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are both not finitely satisfiable. Then there exist finite subsets Σ_1 and Σ_2 of Σ, such that $\Sigma_1 \cup \{\varphi\}$ is not satisfiable and $\Sigma_2 \cup \{\neg\varphi\}$ is not satisfiable. So $models(\Sigma_1) \cap models(\varphi) = \emptyset$, which means $models(\Sigma_1) \subseteq models(\neg\varphi)$. Similarly, we get $models(\Sigma_2) \subseteq models(\varphi)$. Then $models(\Sigma_1 \cup \Sigma_2) = models(\Sigma_1) \cap models(\Sigma_2) \subseteq models(\neg\varphi) \cap models(\varphi) = \emptyset$. So $\Sigma_1 \cup \Sigma_2$ is not satisfiable. Since $\Sigma_1 \cup \Sigma_2$ is a finite subset of Σ, therefore Σ is not finitely satisfiable. □

Proof of the compactness theorem: Let Σ be a finitely satisfiable set of propositional formulas and let $AP(\Sigma)$ be countable. Then, by the Relevance Lemma, we can assume $Prop = AP(\Sigma) = \{p_i : i \in N\}$ to be a countable set of propositions. We define an increasing sequence of sets of formulas as follows: $\Sigma_0 = \Sigma$ and for all $i \in N$,

$$\Sigma_{i+1} = \begin{cases} \Sigma_i \cup \{p_i\}, & \text{if } \Sigma_i \cup \{p_i\} \text{ is finitely satisfiable} \\ \Sigma_i \cup \{\neg p_i\}, & \text{otherwise} \end{cases}$$

And we define Σ' to be their union: $\Sigma' = \bigcup_{i=0}^{\infty} \Sigma_i$.

Claim: *For all $i \in N$, Σ_i is finitely satisfiable.*

Proof: By induction. The base case is $\Sigma_0 = \Sigma$, which is finitely satisfiable by assumption. Assume that Σ_k is finitely satisfiable for $k \in N$. Then, by Lemma 1, one of $\Sigma_k \cup \{p_k\}$ and $\Sigma_k \cup \{\neg p_k\}$ must be finitely satisfiable, so Σ_{k+1} is also finitely satisfiable. $\square$

Claim: *Σ' is finitely satisfiable.*

Proof: Consider a finite subset of X of Σ'. Since X is finite, and $\Sigma_k \subseteq \Sigma_{k+1}$ for all $k \in N$, there exists a finite $j \in N$, such that $X \subseteq \bigcup_{i=0}^{j} \Sigma_i = \Sigma_j$. Since Σ_j is finitely satisfiable, therefore X is satisfiable. $\square$

Now, by the construction of Σ', at least one of p_i and $\neg p_i$ is in Σ' for all $i \in N$. However, Σ' is finitely satisfiable, so it cannot contain both p_i and $\neg p_i$. Therefore, Σ' contains exactly one of p_i and $\neg p_i$ for all $i \in N$. Then we can define a truth assignment $\tau : Prop \to \{0,1\}$ and a function $v : Prop \to Literals$, where $Literals = \{p_i, \neg p_i : i \in N\}$, as follows:

$$\tau(p) = \begin{cases} 1, & \text{if } p \in \Sigma' \\ 0, & \text{if } \neg p \in \Sigma' \end{cases} \qquad v(p) = \begin{cases} p, & \text{if } p \in \Sigma' \\ \neg p, & \text{if } \neg p \in \Sigma' \end{cases}$$

Claim: $\tau \models \Sigma$.

Proof: Let $\varphi \in \Sigma$. Define $X = \{v(p) : p \in AP(\varphi)\}$. Then, by the definition of τ and v, we have $\tau \models X$, $X \subseteq \Sigma'$, and $AP(X) = AP(\varphi)$. Since $AP(\varphi)$ is finite, $X \cup \{\varphi\}$ is a finite subset of Σ'. Since Σ' is finitely satisfiable, therefore $X \cup \{\varphi\}$ is satisfiable. Then, by the Relevance Lemma, there exists $\tau' \in 2^{Prop}$, $\tau'|_{AP(\varphi)} \models \varphi$ and $\tau'|_{AP(\varphi)} \models X$. Also by the Relevance Lemma, we have $\tau|_{AP(\varphi)} \models X$. Since X contains a literal for each proposition in $AP(\varphi)$, therefore τ and τ' must assign the same value to propositions in $AP(\varphi)$, that is, $\tau|_{AP(\varphi)} = \tau'_{AP(\varphi)}$. Therefore $\tau|_{AP(\varphi)} \models \varphi$, and by the Relevance Lemma, $\tau \models \varphi$. Since φ was an arbitrary formula in Σ, therefore $\tau \models \Sigma$. $\square$

We have shown that Σ is satisfiable. This completes the proof of the theorem.

2 Applications of Compactness

2.1 Definability Results in First Order Logic

Compactness can be used to prove results about the definability of class of structures. For example, in the previous lecture, we showed that $2C \notin EC$ and $EC_\Delta \cap co - EC_\Delta = EC$.

2.2 Proving Theorems Outside Logic

Compactness can also be used to prove results in mathematical fields other than logic. For example, in Assignment 6 you are asked to prove the 3-color version of the following theorem using compactness:

Theorem 1. *A graph is k-colorable iff every finite subgraph is k-colorable.*

This theorem can then be combined with the famous four color theorem to prove an infinite version of the four color theorem.

Theorem 2 (Four color theorem)**.** *Every finite planar graph is* 4*-colorable.*

Theorem 3. *Every infinite planar graph is* 4*-colorable.*

Proof. Let G be an infinite planar graph. Since every subgraph of a planar graph is also planar, by Theorem 2, every finite subgraph of G is 4-colorable. Then, by Theorem 1, G is also 4-colorable. □

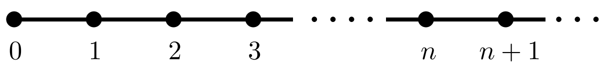
Some other examples of theorems that can be proved using compactness:

- König's lemma: Every finitely-branching infinite tree has an infinite path.
- Every partial order can be extended to a total order.

2.3 Constructing New Models

In first order logic, compactness is frequently used to construct new and interesting models of familiar structures. Here we give a simple example, where starting from a connected graph G, we construct a non-connected graph G' such that G and G' satisfy the same first order properties.

Consider the infinite graph G whose set of nodes is N (the natural numbers) such that for each $n \in N$, there is an undirected edge from n to $n+1$. Note that G is a connected graph, since there is a finite path from i to j for $i, j \in N$.



$$G = (V, E) = (N, \{(n, n+1), (n+1, n) : n \in N\})$$

Let $\Sigma = \{\varphi : G \models \varphi\}$ be the set of all sentences that are true in G. Note that, by definition, $G \models \Sigma$. We define two new constants (0-ary function symbols) a and b, and define $\psi_n = \neg path_n(a, b)$ to be a formula that says there is no path of length n from a to b. Then $(V, E, 0, k)$ satisfies $\Sigma \cup \{\psi_n : n < k\}$, since there is no path of length less than k between node 0 and node k in G (here a is interpreted as 0 and b is interpreted as k). Therefore, $\Sigma' = \Sigma \cup \{\psi_n : n \in N\}$ is finitely satisfiable, and hence, by compactness, Σ' is satisfiable.

Let (V', E', a^0, b^0) be a model of Σ'. Then there is no finite path between a^0 and b^0 in $G' = (V', E')$. So G' is disconnected. However, because $\Sigma \subseteq \Sigma'$, any model of Σ' is also a model of Σ. Since the symbols a and b do not occur in Σ, we get that G' is a model of Σ. But Σ' was defined to be all sentences satisfied by G. Thus G and G' satisfy the same sentences, that is, $G \equiv G'$. We cannot distinguish between G and G' using first order logic. G' is called a *non-standard* model of G. Note that G is connected, but G' is not. As a consequence, we immediately get the following results:

Theorem 4. *The class of connected graphs is not EC_Δ.*

Proof. Suppose a set of sentences Φ defines the class of connected graphs. Then G is a model of Φ (because it is connected), and therefore $\Phi \subseteq \Sigma$. But then G' is also a model of Φ (because G' is a model of Σ). This contradicts the fact that G' is not connected. Therefore no such set Φ exists and the class of connected graphs is not EC_Δ. $\square$

Theorem 5. *Reachability is not definable in first order logic.*

Proof. Suppose we could define a formula $reach(x, y)$ to mean that y is reachable from x. Then $(\forall x)(\forall y)reach(x, y)$ defines the class of connected graphs. But this contradicts the fact that this class is not EC_Δ. Thus, reachability is not definable. $\square$

Similarly, non-standard models can be defined for all the familiar mathematical structures such as the natural numbers and the real numbers. For example, one can construct non-standard models of the reals which contain infinitismal numbers that are greater than zero but smaller than all other positive reals.