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ON THE INTERCHANGE  
OF DERIVATIVE AND EXPECTATION  
FOR LIKELIHOOD RATIO DERIVATIVE ESTIMATORS

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**ABSTRACT**

Sufficient conditions for the validity of interchange between derivative and expectation, in the context of likelihood ratio gradient estimation, were given in L'Ecuyer (1990). The aim of this paper is to shed additional light on these conditions and introduce specific variants of them which are often easier to check. Sufficient conditions for the derivative estimator to have finite moments up to a given order are also given and illustrated by examples. In particular, we give an example of an unbiased derivative estimator which satisfies the interchange conditions, but which has infinite variance.

( DERIVATIVE ESTIMATION, LIKELIHOOD RATIO )

# 1. Introduction

This note is a follow-up to L'Ecuyer (1990). It clarifies the conditions given there for the unbiasedness of the likelihood ratio (LR) derivative estimators. It also introduces specific variants of these conditions, which are often easier to check. Finally, it gives a minor correction to the first two examples of L'Ecuyer (1990). For an overview of derivative estimation, see Glynn (1990) and L'Ecuyer (1991).

Let  $\{P_\theta, \theta \in \Theta\}$  be a family of probability measures, defined over the same measurable space  $(\Omega, \mathcal{B})$ , where  $\Theta$  is some open interval in  $\mathbb{R}$ . Let  $h(\theta, \omega)$  denote the sample value (cost), where the sample point  $\omega \in \Omega$  obeys the probability law  $P_\theta$ . We assume that  $h(\theta, \cdot)$  is measurable. If  $G$  is a given probability measure on  $(\Omega, \mathcal{B})$  that dominates all the  $P_\theta$ 's, then the expected value (cost) can be written as a function of  $\theta$  as

$$\alpha(\theta) = \int_{\Omega} h(\theta, \omega) dP_\theta(\omega) = \int_{\Omega} h(\theta, \omega) L(G, \theta, \omega) dG(\omega), \quad (1)$$

where  $L(G, \theta, \omega) = (dP_\theta/dG)(\omega)$ . Under appropriate regularity conditions, one can differentiate  $\alpha$  by differentiating inside the integral:

$$\alpha'(\theta) = \int_{\Omega} \psi(\theta, \omega) dG(\omega), \quad (2)$$

where

$$\psi(\theta, \omega) = L(G, \theta, \omega)h'(\theta, \omega) + h(\theta, \omega)L'(G, \theta, \omega) \quad (3)$$

when this derivative exists. Here and throughout the paper, the prime denotes the derivative with respect to  $\theta$ . In practice, one can often use simulation to compute realizations of the random variable (3), called the *likelihood ratio* (LR) derivative estimator, to estimate the derivative of the performance measure  $\alpha$ . Such a derivative estimator can be very useful in different contexts, including stochastic optimization, sensitivity analysis, and interpolation (Glynn 1990, L'Ecuyer 1991).

In this note, we are interested in *sufficient* conditions that are not too hard to check in practice and under which (2) is valid, which would imply that (3) is an *unbiased* estimator

of the derivative. Such conditions are given for instance in Glynn (1986), L'Ecuyer (1990), Reiman and Weiss (1989), and Rubinstein (1989), in most cases for more specific setups than the one examined here. Furthermore, unbiasedness is not the only question of interest. One would also like to know, for instance, whether the estimator has finite variance, or finite moments of higher orders. Later on, we will give an example of an unbiased LR derivative estimator which has infinite variance.

As a special case, if  $P_\theta \equiv P$ , independent of  $\theta$ , one can take  $G = P$  and (3) becomes  $\psi(\theta, \omega) = h'(\theta, \omega)$ , which is often called the IPA derivative estimator (Glasserman 1991, L'Ecuyer 1990). As another special case, if, for some fixed  $\theta_0 \in \Theta$ ,  $P_{\theta_0}$  dominates  $\{P_\theta, \theta \in \Theta\}$ , then one can take  $G = P_{\theta_0}$  and, at  $\theta = \theta_0$ , (3) becomes

$$\psi(\theta, \omega) = h'(\theta, \omega) + h(\theta, \omega) \frac{\partial}{\partial \theta} L(P_{\theta_0}, \theta, \omega) \Big|_{\theta=\theta_0}. \quad (4)$$

Assumption A1 below is parameterized by an integer  $k \geq 1$ . For example, A1(2) denotes that assumption with  $k = 2$ . It is proven in L'Ecuyer (1990) that under A1(1), Equation (2) is valid at  $\theta = \theta_0$  for any  $\theta_0 \in \Upsilon$ . A1( $k$ ) for  $k > 1$  also implies that the  $k$ -th moment of  $\psi(\theta, \omega)$  is finite for  $\theta \in \Upsilon$ .

**ASSUMPTION A1( $k$ ).** Let  $H(\theta, \omega) = h(\theta, \omega)L(G, \theta, \omega)$ , let  $\Upsilon \subseteq \Theta$  be an open interval, and let  $\Xi \subseteq \Omega$  be a measurable set such that  $G(\Xi) = 1$ .

- (a) For all  $\theta \in \Upsilon$ , equation (1) holds and  $\psi(\theta, \omega)$  exists for  $G$ -almost all  $\omega$ .
- (b) For every  $\omega \in \Xi$ , there is a  $D(\omega) \subseteq \Upsilon$ , where  $\Upsilon \setminus D(\omega)$  is at most a denumerable set, such that  $H(\cdot, \omega)$  exists and is continuous everywhere in  $\Upsilon$ , and is also differentiable everywhere in  $D(\omega)$ .
- (c) There exists a  $G$ -integrable function  $\Gamma : \Omega \rightarrow [0, \infty)$  such that

$$\sup_{\theta \in D(\omega)} |\psi^k(\theta, \omega)| \leq \Gamma(\omega) \quad (5)$$

for every  $\omega$  in  $\Xi$ . ■

Note that the  $\psi(\theta, \omega)$  which is referred to in (5) is the one defined by equation (3), not (4). Therefore, A1 must be verified for the general expression (3), not for (4). When  $G = P_{\theta_0}$ , (4) is equal to (3) only at  $\theta = \theta_0$ . In Example 4.1 of L'Ecuyer (1990), the fourth displayed equation of page 1372 represents (4), and part (c) of A1(1) was in fact verified with (4) instead of (3), which is not correct. The same error was also made in Example 4.2. The initial motivation of the present note was to correct that. Its first version had less than a page. But while trying to verify A1 in different application contexts, it appeared to the author that other variants of A1, sometimes tailored for specific contexts, could prove useful in practice. Questions like random horizons and finite moments were also given some attention. As a result, the note grew up to its present state.

In the next section, we give an alternative formulation for part (c) of A1, which is often easier to verify. We also extend Theorem 1 of L'Ecuyer (1990). In §3, we examine the case where  $\omega$  can be *viewed* as a sequence of independent random variables with known  $\theta$ -dependent densities. The number of such variables can be fixed (deterministic) or random (e.g., a stopping time). For both of these cases, we give more specific conditions that imply A1. In §4, we verify (correctly, this time) the validity of the interchange for Examples 4.1 and 4.2 of L'Ecuyer (1990), and illustrate the results of §3 by an additional example. We also show that even when the estimator is always unbiased, the variance may become infinite for some values of  $\theta$  and  $\theta_0$ .

## 2. An Alternative Formulation of the Assumption

When verifying A1 in practice, complications sometimes arise in trying to find the  $G$ -integrable function  $\Gamma$ . We now give a slightly different formulation of that assumption, which is often more “user-friendly”. That formulation is more handy when the family  $\{P_\theta, \theta \in \Theta\}$  is dominated by some “worst-case” probability measure  $Q$  which is easy to find, but not necessarily equal to  $G$ , and such that it is easier to bound  $L(Q, \theta, \omega)$  than  $L(G, \theta, \omega)$ . For example (see Example 4.1), for the exponential distribution with mean  $\theta$ ,  $L(P_{\theta_0}, \theta, \cdot)$  is unbounded for  $\theta > \theta_0$ , but  $L(P_b, \theta, \cdot)$  is bounded for  $\theta < b$ . One can then choose  $Q = P_b$  for  $b$

larger than any  $\theta$  of interest. This technique can be applied in general to distributions for which the likelihood ratio is *monotone* in  $\theta$  (e.g., distributions from the exponential). The measure  $Q$  can then be  $P_\theta$ , where  $\theta$  is taken at the worst-case extremity of the interval of interest.

The Proposition that follows extends Theorem 1 of L'Ecuyer (1990). It shows that  $A1'(k)$  is equivalent to  $A1(k)$ , gives a continuous differentiability condition for  $\alpha$ , and finite-moment conditions for the derivative estimator.

**ASSUMPTION  $A1'(k)$ .** Let parts (a–b) of  $A1(k)$  hold. Suppose that there exists a probability measure  $Q$  over  $(\Omega, \mathcal{B})$  and a  $Q$ -integrable function  $\tilde{\Gamma} : \Omega \rightarrow [0, \infty)$  such that  $dG/dQ$  exists and

$$\sup_{\theta \in D(\omega)} \left| \psi^k(\theta, \omega) \frac{dG}{dQ}(\omega) \right| \leq \tilde{\Gamma}(\omega) \quad (6)$$

for all  $\omega \in \Xi$ . ■

**PROPOSITION 1.** (a)  $A1(k)$  is equivalent to  $A1'(k)$ .

(b) Under  $A1(1)$ , equation (2) is valid and  $\alpha(\theta)$  is differentiable at every  $\theta \in \Upsilon$ .

(c) Under  $A1(1)$ , if  $\psi(\cdot, \omega)$  is continuous throughout  $\Upsilon$  for each  $\omega \in \Xi$ , then  $\alpha$  is continuously differentiable in  $\Upsilon$ .

(d) If  $A1(k)$  holds for  $k \geq 1$ , then the derivative estimator has a finite moment of order  $k$ , uniformly bounded over  $\Upsilon$ .

PROOF.  $A1(k)$  clearly implies  $A1'(k)$  (take  $Q = G$ ). So, it remains to prove that (6) implies (5). Let

$$\Gamma(\omega) = \begin{cases} \sup_{\theta \in D(\omega)} |\psi^k(\theta, \omega)| & \text{if } \omega \in \Xi, \\ 0 & \text{otherwise.} \end{cases}$$

Then, (5) holds and, since  $dG/dQ$  is always non-negative, one has

$$\int_{\Omega} \Gamma(\omega) dG(\omega) = \int_{\Xi} \Gamma(\omega) dG(\omega) = \int_{\Xi} \Gamma(\omega) \frac{dG}{dQ}(\omega) dQ(\omega) \leq \int_{\Xi} \tilde{\Gamma}(\omega) dQ(\omega) < \infty,$$

which proves (a). For  $\theta_0 \in \Upsilon$ , the proof of Theorem 1 in L'Ecuyer (1990) implies differentiability of  $\alpha(\theta)$  at  $\theta = \theta_0$ . Since  $\theta_0$  can be chosen anywhere in  $\Upsilon$ , this proves (b). If  $\psi(\cdot, \omega)$  is continuous in  $\Upsilon$ , then, for any  $\theta_0 \in \Upsilon$ ,

$$\lim_{\theta \rightarrow \theta_0} \alpha'(\theta) = \lim_{\theta \rightarrow \theta_0} \int_{\Omega} \psi(\theta, \omega) dG(\omega) = \int_{\Omega} \lim_{\theta \rightarrow \theta_0} \psi(\theta, \omega) dG(\omega) = \int_{\Omega} \psi(\theta_0, \omega) dG(\omega) = \alpha'(\theta_0).$$

Here, the first equality holds because of (b) and since  $\theta \in \Upsilon$  when  $|\theta - \theta_0|$  is small enough. The second equality follows from Lebesgue's dominated convergence Theorem. This completes the proof of (c). Finally (d) follows directly from (5). ■

An interesting application of Proposition 1(b) is to estimate the derivative *everywhere* in  $\Upsilon$  by a *single* simulation under  $G$ . This is useful for some optimization approaches; see Rubinstein (1991). Proposition 1(b), when it applies, provides an assurance that such a derivative estimator is everywhere unbiased. Proposition 1(d) with  $k = 2$  gives a sufficient condition for the estimator to have a finite variance.

Now, assume that the likelihood ratio  $L(G, \theta, \omega)$  can be expressed as a ratio of (perhaps multivariate) densities, or probability mass functions, as follows:

$$L(G, \theta, \omega) = f_{\theta}(\omega)/g(\omega). \quad (7)$$

This covers Equation (6) in L'Ecuyer (1990) and is general enough for most discrete-event simulations. One has

$$L'(G, \theta, \omega) = \frac{\partial}{\partial \theta} f_{\theta}(\omega)/g(\omega) = L(G, \theta, \omega) S(\theta, \omega), \quad (8)$$

where

$$S(\theta, \omega) = \frac{\partial}{\partial \theta} \ln f_{\theta}(\omega) \quad (9)$$

is called the *score function*. Using (8), (3) can be rewritten as

$$\psi(\theta, \omega) = L(G, \theta, \omega)[h'(\theta, \omega) + h(\theta, \omega)S(\theta, \omega)]. \quad (10)$$

Assuming that  $(dG/dQ)(\omega)$  exists, one has  $(dG/dQ)(\omega) = L(Q, \theta, \omega)/L(G, \theta, \omega)$  if  $L(G, \theta, \omega) > 0$ , and  $\psi(\theta, \omega) = L(Q, \theta, \omega) = 0$  otherwise. In both cases,

$$\psi^k(\theta, \omega)(dG/dQ)(\omega) = L(Q, \theta, \omega)L^{k-1}(G, \theta, \omega)[h'(\theta, \omega) + h(\theta, \omega)S(\theta, \omega)]^k. \quad (11)$$

In A1', we need to bound the right-hand-side expression in (11) by a  $Q$ -integrable function. Observe that for  $k = 1$ ,  $G$  is no longer involved in that. Therefore, if A1'(1) holds for some  $Q$ , it follows that A1(1) holds for all  $G$  such that  $dG/dQ$  exists. In other words, the interchange will be valid whatever be the distribution  $G$  from which we simulate, provided  $dG/dQ$  exists and the other conditions in A1(1) hold for  $Q$ . This is the main strength of A1'. For higher order moments, things do not simplify so nicely, though, because (11) depends on both  $G$  and  $Q$ . However, introducing some  $Q \neq G$  can still be helpful, as will be illustrated in the next section.

### 3. Simulations Driven by a Sequence of Independent Variates

#### 3.1. Fixed Horizon

Let  $\zeta_1, \zeta_2, \dots$  be a sequence of independent random variables and for each  $n > 0$ , let  $\mathfrak{F}_n$  be the sigma field generated by  $\{\zeta_1, \dots, \zeta_n\}$ . Suppose that for some fixed (deterministic) integer  $t > 0$ , the cost function  $h(\theta, \omega)$  is  $\mathfrak{F}_t$ -measurable for every  $\theta \in \Theta$ . Here, one can define the underlying measurable space  $(\Omega, \mathcal{B})$  in such a way that  $\omega$  can be viewed as the sequence  $\zeta_1, \dots, \zeta_t$ . Assume that for each  $i$ ,  $\zeta_i$  has a density  $f_{i,\theta}$ . (Our development is also valid for the case where some or all of these densities are replaced by probability mass functions.) For each  $i$ ,  $1 \leq i \leq t$ , let  $g_i$  and  $q_i$  be densities such that  $\{\zeta \mid f_{i,\theta}(\zeta) > 0 \text{ for some } \theta \in \Theta\} \subseteq \{\zeta \mid g_i(\zeta) > 0\} \subseteq \{\zeta \mid q_i(\zeta) > 0\}$ . These sets of densities define probability measures  $P_\theta$ ,  $G$ , and  $Q$ , respectively, over  $(\Omega, \mathcal{B})$ . Let  $E_\theta$ ,  $E_G$ , and  $E_Q$  denote the corresponding mathematical expectations. Here, the  $g_i$ 's are the densities we use to generate sample paths (in the simulation), while the  $q_i$ 's are used to verify A1'. For each  $\theta \in \Theta$ , one has

$$L(G, \theta, \omega) = \prod_{i=1}^t \frac{f_{i,\theta}(\zeta_i)}{g_i(\zeta_i)}; \quad (12)$$

$$L(Q, \theta, \omega) = \prod_{i=1}^t \frac{f_{i,\theta}(\zeta_i)}{q_i(\zeta_i)}; \quad (13)$$

$$S(\theta, \omega) = \sum_{i=1}^t d_i, \quad (14)$$

where

$$d_i = \frac{\partial}{\partial \theta} \ln(f_{i,\theta}(\zeta_i)) = \frac{f'_{i,\theta}(\zeta_i)}{f_{i,\theta}(\zeta_i)}. \quad (15)$$

Under this setup, we show that the following assumption implies A1( $k$ ).

**ASSUMPTION A2( $k$ ).** Let  $\Upsilon \subseteq \Theta$  be an open interval, and let  $\Xi \subseteq \Omega$  be a measurable set such that  $Q(\Xi) = 1$ . Interpret  $\omega$  as  $\omega = (\zeta_1, \dots, \zeta_t)$ .

(a) For any  $\theta \in \Upsilon$ ,  $h'(\theta, \omega)$  and each  $d_i$  exist for  $Q$ -almost all  $\omega$ . For every  $\omega \in \Xi$ , there is a  $D(\omega) \subseteq \Upsilon$ , where  $\Upsilon \setminus D(\omega)$  is at most a denumerable set, such that  $h(\theta, \omega)$  and each  $f_{i,\theta}(\zeta_i)$  are continuous in  $\theta$  everywhere in  $\Upsilon$ , and also differentiable in  $\theta$  everywhere in  $D(\omega)$ .

(b) All the following functions are  $Q$ -integrable (for each  $i$ ):

$$\sup_{\theta \in D(\omega)} [h(\theta, \omega)]^{2k}, \quad \sup_{\theta \in D(\omega)} [h'(\theta, \omega)]^{2k}, \quad \sup_{\theta \in D(\omega)} \left[ \frac{(\frac{\partial}{\partial \theta} f_{i,\theta}(\zeta_i))^k}{q_i(\zeta_i) g_i^{k-1}(\zeta_i)} \right]^2, \quad \sup_{\theta \in D(\omega)} \left[ \frac{f_{i,\theta}^k(\zeta_i)}{q_i(\zeta_i) g_i^{k-1}(\zeta_i)} \right]^2. \quad \blacksquare$$

**PROPOSITION 2.** *Under A2( $k$ ), A1'( $k$ ) holds (and Proposition 1 applies).*

PROOF. First note that from our assumptions, every set that has  $Q$ -measure zero also has  $G$ -measure zero. Therefore,  $G(\Xi) = 1$ . Since each  $f_{i,\theta}(\zeta_i)$  is differentiable with respect to  $\theta$  in  $D(\omega)$ ,

$$H(\theta, \omega) = h(\theta, \omega) \prod_{i=1}^t \frac{f_{i,\theta}(\zeta_i)}{g_i(\zeta_i)},$$

and from (10), it is easily seen that A2 implies parts (a) and (b) of A1. It remains to verify part (c). In what follows, we make use of the fact that for any real numbers  $x$  and  $y$ ,  $|xy| \leq x^2 + y^2$  and  $|x + y|^k \leq 2^k(|x|^k + |y|^k)$ . For  $\omega \in \Xi$  and  $\theta \in D(\omega)$ , one has

$$\begin{aligned} & \left| \psi^k(\theta, \omega)(dG/dQ)(\omega) \right| \\ &= L(Q, \theta, \omega) L^{k-1}(G, \theta, \omega) |h'(\theta, \omega) + h(\theta, \omega)S(\theta, \omega)|^k \\ &\leq 2^k L(Q, \theta, \omega) L^{k-1}(G, \theta, \omega) \left[ |h'(\theta, \omega)|^k + \left| h(\theta, \omega) \sum_{j=1}^t d_j \right|^k \right] \end{aligned}$$



$$\begin{aligned}
&\leq 2^k L(Q, \theta, \omega) L^{k-1}(G, \theta, \omega) \left[ |h'(\theta, \omega)|^k + |h(\theta, \omega)|^k t^{k-1} \sum_{j=1}^t |d_j|^k \right] \\
&\leq 2^k \left[ [L(Q, \theta, \omega) L^{k-1}(G, \theta, \omega)]^2 + |h'(\theta, \omega)|^{2k} \right. \\
&\quad \left. + \sum_{j=1}^t \left( t^{2(k-1)} |h(\theta, \omega)|^{2k} + [|d_j|^k L(Q, \theta, \omega) L^{k-1}(G, \theta, \omega)]^2 \right) \right]. \tag{16}
\end{aligned}$$

Let  $\tilde{\Gamma}(\omega)$  be the supremum over  $\theta$ , for  $\theta \in D(\omega)$ , of (16). One has

$$\begin{aligned}
&E_Q \left[ \sup_{\theta \in D(\omega)} [|d_j|^k L(Q, \theta, \omega) L^{k-1}(G, \theta, \omega)]^2 \right] \\
&= E_Q \left[ \sup_{\theta \in D(\omega)} \left( \frac{|d_j|^k f_{j,\theta}^k(\zeta_j)}{q_j(\zeta_j) g_j^{k-1}(\zeta_j)} \prod_{i=1, i \neq j}^t \frac{f_{i,\theta}^k(\zeta_i)}{q_i(\zeta_i) g_i^{k-1}(\zeta_i)} \right)^2 \right] \\
&\leq E_Q \left[ \sup_{\theta \in D(\omega)} \left( \frac{|d_j|^k f_{j,\theta}^k(\zeta_j)}{q_j(\zeta_j) g_j^{k-1}(\zeta_j)} \right)^2 \prod_{i=1, i \neq j}^t \sup_{\theta \in D(\omega)} \left( \frac{f_{i,\theta}^k(\zeta_i)}{q_i(\zeta_i) g_i^{k-1}(\zeta_i)} \right)^2 \right] \\
&= E_Q \left[ \sup_{\theta \in D(\omega)} \left( \frac{|\frac{\partial}{\partial \theta} f_{j,\theta}(\zeta_j)|^k}{q_j(\zeta_j) g_j^{k-1}(\zeta_j)} \right)^2 \right] \prod_{i=1, i \neq j}^t E_Q \left[ \sup_{\theta \in D(\omega)} \left( \frac{f_{i,\theta}^k(\zeta_i)}{q_i(\zeta_i) g_i^{k-1}(\zeta_i)} \right)^2 \right] \\
&< \infty. \tag{17}
\end{aligned}$$

The last equality follows from the independence of the  $\zeta_i$ 's. Also,

$$\begin{aligned}
E_Q \left[ \sup_{\theta \in D(\omega)} [L(Q, \theta, \omega) L^{k-1}(G, \theta, \omega)]^2 \right] &\leq E_Q \left[ \prod_{i=1}^t \sup_{\theta \in D(\omega)} \left( \frac{f_{i,\theta}^k(\zeta_i)}{q_i(\zeta_i) g_i^{k-1}(\zeta_i)} \right)^2 \right] \\
&= \prod_{i=1}^t E_Q \left[ \sup_{\theta \in D(\omega)} \left( \frac{f_{i,\theta}^k(\zeta_i)}{q_i(\zeta_i) g_i^{k-1}(\zeta_i)} \right)^2 \right] \\
&< \infty. \tag{18}
\end{aligned}$$

From (16–18) and part (b) of A2, A1' follows. ■

### 3.2. Random Horizon

Now, suppose that  $h(\theta, \omega)$  is  $\mathfrak{F}_\tau$ -measurable for some (random) stopping time  $\tau$ . This means that for each integer  $t > 0$ , the event  $\{\tau \leq t\}$  is  $\mathfrak{F}_t$ -measurable. We suppose that  $\tau$  depends on  $\theta$  only through the distribution  $P_\theta$ , so that for  $\omega$  fixed,  $\tau$  does not depend on  $\theta$  (i.e.,  $\tau$  is just a function of  $\omega$ ). The constant  $t$  in the previous setup must be replaced by  $\tau$ .

We can now view  $\omega$  as representing  $(\zeta_1, \dots, \zeta_\tau)$ . Verifying A1' in that context is a little more complicated, because the likelihood ratio is now the product of a random (generally unbounded) number of terms. We will give an assumption A3, similar to A2, under which A1 holds in this context. Of course, one can try to verify A1 directly. Alternatively, A3 is a *sufficient* condition that is often easier to check, because in A3, the left-hand-side expression in (6) is already broken up into smaller pieces.

Glynn (1986), in his Theorem 4.9, proves the validity of (2) when  $h(\theta, \omega)$  represents the total cost for the first  $\tau$  steps in a *finite-state* Markov chain with state-dependent one step rewards (and under some additional assumptions). His proof exploits the fact that his state space is finite and everything is bounded. Reiman and Weiss (1989) give assumptions based on a concept called *amiability*, under which (2) is valid when the cost is  $\mathfrak{S}_\tau$ -measurable for some stopping time  $\tau$ ,  $f_{i,\theta}$  does not depend on  $i$ , and  $g_i = f_{i,\theta_0}$ . By looking at the proofs of their theorems, one can see that their assumptions imply A1(1). Glynn (1991) and Glynn and Iglehart (1989) also give finite variance conditions for importance sampling cost (rather than derivative) estimators, over random horizons.

**ASSUMPTION A3(k).** Let part (a) of A2 hold with  $t$  replaced by  $\tau$ . Suppose that there exist measurable functions  $\Gamma_{1i} : \Omega \rightarrow [1, \infty)$  and  $\Gamma_{2i} : \Omega \rightarrow [1, \infty)$ ,  $i = 1, 2, \dots$ , such that for each  $i$ ,  $\zeta$ , and  $\omega$ , one has

$$\sup_{\theta \in \Upsilon} \left( \frac{f_{i,\theta}^k(\zeta)}{q_i(\zeta)g_i^{k-1}(\zeta)} \right)^8 \leq \Gamma_{1i}(\zeta), \quad (19)$$

$$\sup_{\theta \in \Upsilon} \left( \frac{(\frac{\partial}{\partial \theta} f_{i,\theta}(\zeta))^k}{q_i(\zeta)g_i^{k-1}(\zeta)} \right)^4 \leq \Gamma_{2i}(\zeta), \quad (20)$$

and such that all the following functions are  $Q$ -integrable:

$$\sup_{\theta \in D(\omega)} [h(\theta, \omega)]^{4k}, \quad \sup_{\theta \in D(\omega)} [h'(\theta, \omega)]^{2k}, \quad \sup_{\theta \in D(\omega)} [\tau]^{4k-2}, \quad \sum_{j=1}^{\tau} \Gamma_{2j}(\zeta_j), \quad \prod_{j=1}^{\tau} \Gamma_{1j}(\zeta_j). \blacksquare$$

**PROPOSITION 3.** *Assumption A3(k) implies A1(k).*

PROOF. Part (a) holds because  $L(G, \theta, \omega)$  and  $L(Q, \theta, \omega)$  are assumed to exist (which implies (1)), and because of the differentiability of  $f_{i,\theta}$  with respect to  $\theta$  for  $G$ -almost all  $\omega$ .

Part (b) also follows directly from part (a) of A2, with  $t$  replaced by  $\tau$  in the definition of  $h(\theta, \omega)$ . For part (c) of A1', one has, using the analog of (16–17),

$$\begin{aligned}
& \sup_{\theta \in D(\omega)} \left| \psi^k(\theta, \omega)(dG/dQ)(\omega) \right| \\
& \leq 2^k \sup_{\theta \in D(\omega)} \left[ [L(Q, \theta, \omega)L^{k-1}(G, \theta, \omega)]^2 + |h'(\theta, \omega)|^{2k} \right. \\
& \quad \left. + \sum_{j=1}^{\tau} \left( \tau^{2(k-1)} |h(\theta, \omega)|^{2k} + [|d_j|^k L(Q, \theta, \omega)L^{k-1}(G, \theta, \omega)]^2 \right) \right]. \tag{21} \\
& \leq 2^k \left[ \prod_{i=1}^{\tau} \sup_{\theta \in D(\omega)} \left( \frac{f_{i,\theta}^k(\zeta_i)}{q_i(\zeta_i)g_i^{k-1}(\zeta_i)} \right)^2 + \sup_{\theta \in D(\omega)} |h'(\theta, \omega)|^{2k} + \tau^{2k-1} \sup_{\theta \in D(\omega)} |h(\theta, \omega)|^{2k} \right. \\
& \quad \left. + \sum_{j=1}^{\tau} \sup_{\theta \in D(\omega)} \left( \frac{|\frac{\partial}{\partial \theta} f_{j,\theta}(\zeta_j)|^k}{q_j(\zeta_j)g_j^{k-1}(\zeta_j)} \right)^2 \prod_{i=1, i \neq j}^{\tau} \sup_{\theta \in D(\omega)} \left( \frac{f_{i,\theta}^k(\zeta_i)}{q_i(\zeta_i)g_i^{k-1}(\zeta_i)} \right)^2 \right] \\
& \leq \prod_{i=1}^{\tau} (\Gamma_{1i}(\zeta_i))^{1/4} + \sup_{\theta \in D(\omega)} |h'(\theta, \omega)|^{2k} + \tau^{2k-1} \sup_{\theta \in D(\omega)} |h(\theta, \omega)|^{2k} + \sum_{j=1}^{\tau} (\Gamma_{2j}(\zeta_j))^{1/2} \prod_{i=1, i \neq j}^{\tau} (\Gamma_{1i}(\zeta_i))^{1/4} \\
& \leq \prod_{i=1}^{\tau} \Gamma_{1i}(\zeta_i) + \sup_{\theta \in D(\omega)} |h'(\theta, \omega)|^{2k} + \tau^{4k-2} + \sup_{\theta \in D(\omega)} |h(\theta, \omega)|^{4k} + \sum_{j=1}^{\tau} \left( \Gamma_{2j}(\zeta_j) + \prod_{i=1}^{\tau} (\Gamma_{1i}(\zeta_i))^{1/2} \right) \\
& \leq \prod_{i=1}^{\tau} \Gamma_{1i}(\zeta_i) + \sup_{\theta \in D(\omega)} |h'(\theta, \omega)|^{2k} + \tau^{4k-2} + \sup_{\theta \in D(\omega)} |h(\theta, \omega)|^{4k} + \sum_{j=1}^{\tau} \Gamma_{2j}(\zeta_j) + \tau \prod_{i=1}^{\tau} (\Gamma_{1i}(\zeta_i))^{1/2} \\
& \leq \sup_{\theta \in D(\omega)} |h'(\theta, \omega)|^{2k} + \sup_{\theta \in D(\omega)} |h(\theta, \omega)|^{4k} + \tau^{4k-2} + \tau^2 + 2 \prod_{i=1}^{\tau} \Gamma_{1i}(\zeta_i) + \sum_{j=1}^{\tau} \Gamma_{2j}(\zeta_j). \tag{22}
\end{aligned}$$

From Assumption A3, part (c) of A1' holds with  $\tilde{\Gamma}$  defined by the expression (22). ■

#### 4. Verifying Interchange Conditions and Finite Moments for Some Examples

We will use formulation (11) to verify A1' for Example 4.1 of L'Ecuyer (1990), and formulation (10) to verify A1 for Example 4.2 of L'Ecuyer (1990). Parts (a) and (b) of A1 have been verified there, but part (c) was not verified correctly. We then give another example to illustrate A3.

#### 4.1. A GI/M/1 Queue with Mean Service Time $\theta$

This is a slight generalization of Example 4.1 in L'Ecuyer (1990). Let  $h(\theta, \omega)$  be the average sojourn time for the first  $t$  customers in a GI/M/1 queue, initially empty, with mean service time  $\theta$ . Let  $\omega = (S_1, A_1, \dots, A_{t-1}, S_t)$ , where for each  $i$ ,  $W_i$  and  $S_i$  are the waiting time and service time of customer  $i$ , respectively, and  $A_i$  is the interarrival time between customers  $i$  and  $i + 1$ , whose distribution is assumed independent of  $\theta$ . Let  $G = P_{\theta_0}$ , where  $\theta_0$  is the parameter value at which we perform the simulation. We will now try to find the largest interval  $\Upsilon = (\theta_{\min}, \theta_{\max})$  for which A1' holds. Assume that

$$0 < \theta_{\min} < \theta_0 < \theta_{\max} < \frac{k}{k-1}\theta_0 \quad (23)$$

(where  $1/0 = \infty$ ). Let us take  $b > \theta_0\theta_{\max}/(k\theta_0 - (k-1)\theta_{\max})$  and let  $Q = P_b$  (a “worst-case” service time distribution). Then,

$$\begin{aligned} L(Q, \theta, \omega) &= \left(\frac{b}{\theta}\right)^t \exp\left[\left(\frac{1}{b} - \frac{1}{\theta}\right) \sum_{i=1}^t S_i\right], \\ L(G, \theta, \omega) &= \left(\frac{\theta_0}{\theta}\right)^t \exp\left[\left(\frac{1}{\theta_0} - \frac{1}{\theta}\right) \sum_{i=1}^t S_i\right], \\ S(\theta, \omega) &= \frac{1}{\theta^2} \sum_{i=1}^t (S_i - \theta), \end{aligned}$$

$(dG/dQ)(\omega) = L(Q, \theta_0, \omega)$  exists, and

$$\begin{aligned} &\psi^k(\theta, \omega)(dG/dQ)(\omega) \\ &= L(Q, \theta, \omega)L^{k-1}(G, \theta, \omega)h^k(\theta, \omega)S^k(\theta, \omega) \\ &= \left[\frac{1}{t} \sum_{i=1}^t (W_i + S_i)\right]^k \left[\frac{1}{\theta^2} \sum_{i=1}^t (S_i - \theta)\right]^k \left(\frac{b\theta_0^{k-1}}{\theta^k}\right)^t \exp\left[\left(\frac{1}{b} + \frac{k-1}{\theta_0} - \frac{k}{\theta}\right) \sum_{i=1}^t S_i\right]. \end{aligned}$$

For each  $i$ ,  $W_i \leq \sum_{j=1}^{i-1} S_j$ , so that

$$\left[\frac{1}{t} \sum_{i=1}^t (W_i + S_i)\right]^k \leq \left[\frac{1}{t} \sum_{i=1}^t \sum_{j=1}^i S_j\right]^k \leq \left[\sum_{i=1}^t S_i\right]^k.$$

Also, from our choice of  $b$ , we have  $b > \theta_{\max} > \theta_0$  and  $1/b + (k-1)/\theta_0 - k/\theta < 0$ . So,

$$\exp\left[\left(\frac{1}{b} + \frac{k-1}{\theta_0} - \frac{k}{\theta}\right) \sum_{i=1}^t S_i\right] \leq 1.$$

Therefore,

$$|\psi^k(\theta, \omega)|(dG/dQ)(\omega) \leq \left[ \sum_{i=1}^t S_i \right]^k \left[ \frac{1}{\theta_{\min}^2} \sum_{i=1}^t (S_i + b) \right]^k \left( \frac{b}{\theta_{\min}} \right)^{kt}. \quad (24)$$

The latter is  $Q$ -integrable, because the exponential distribution has finite moments of all orders. This verifies part (c) of A1'. The other conditions of A1 were already verified in L'Ecuyer (1990) and therefore, Proposition 1 applies.

Observe that for  $k = 1$ , condition (23) becomes

$$0 < \theta_{\min} < \theta_0 < \theta_{\max} < \infty. \quad (25)$$

Therefore, with  $G = P_{\theta_0}$  for some fixed  $\theta_0 > 0$ , for any  $\Upsilon = (\theta_{\min}, \theta_{\max})$  satisfying (25), one obtains through (2) an unbiased estimator of the derivative *everywhere* in  $\Upsilon$ . In other words, one can simulate by generating service times with any parameter value  $\theta > 0$ , and compute

$$h(\theta, \omega)L(P_{\theta_0}, \theta, \omega)S(\theta, \omega) = \left[ \frac{1}{t} \sum_{i=1}^t (W_i + S_i) \right] \left[ \frac{1}{\theta^2} \sum_{i=1}^t (S_i - \theta) \right] \left( \frac{\theta_0}{\theta} \right)^t \exp \left[ \left( \frac{1}{\theta_0} - \frac{1}{\theta} \right) \sum_{i=1}^t S_i \right].$$

The latter gives an unbiased estimator of  $\alpha'(\theta)$  for all  $\theta > 0$ . For a finite variance, the condition (23) becomes (take  $k = 2$ ):  $\theta_{\max} < 2\theta_0$ .

We have just shown that condition (23) is *sufficient* for a finite  $k$ -th moment. We now show that it is also *necessary*. Indeed, if  $(k - 1)\theta \geq k\theta_0$ , one has

$$\begin{aligned} & \int_{\Omega} \psi^k(\theta, \omega) dG(\omega) \\ &= \int L^k(G, \theta, \omega) h^k(\theta, \omega) S^k(\theta, \omega) dG(\omega) \\ &\geq \int_0^{\infty} \cdots \int_0^{\infty} \left[ \frac{1}{t} \sum_{i=1}^t S_i \right]^k \left[ \frac{1}{\theta^2} \sum_{i=1}^t (S_i - \theta) \right]^k \left( \frac{\theta_0}{\theta} \right)^{kt} \\ &\quad \exp \left[ \left( \frac{k}{\theta_0} - \frac{k}{\theta} \right) \sum_{i=1}^t S_i \right] \left( \frac{1}{\theta_0} \right)^t \exp \left[ -\frac{1}{\theta_0} \sum_{i=1}^t S_i \right] dS_1 \cdots dS_t \\ &\geq \int_0^{\infty} \cdots \int_0^{\infty} I_S \frac{t^k}{\theta^{2k}} \left( \frac{\theta_0}{\theta} \right)^{kt} \left( \frac{1}{\theta_0} \right)^t \exp \left[ \left( \frac{k-1}{\theta_0} - \frac{k}{\theta} \right) \sum_{i=1}^t S_i \right] dS_1 \cdots dS_t \\ &\quad - \int_0^{\infty} \cdots \int_0^{\infty} (1 - I_S) (\theta + 1)^k \left( \frac{t}{\theta} \right)^k \left( \frac{\theta_0}{\theta} \right)^{kt} \left( \frac{1}{\theta_0} \right)^t \exp \left[ \left( \frac{k-1}{\theta_0} - \frac{k}{\theta} \right) (\theta + 1)t \right] dS_1 \cdots dS_t \\ &= \infty, \end{aligned}$$

where  $I_S$  denotes the indicator function:

$$I_S = \begin{cases} 1 & \text{if } \sum_{i=1}^t S_i \geq (\theta + 1)t, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, (23) is a necessary and sufficient condition for having a finite moment of order  $k$ . For example, for any  $\theta_0 > 0$ , the derivative estimator  $\psi(\theta, \omega)$  is unbiased for all  $\theta > 0$ , but has finite variance only for  $\theta < 2\theta_0$ , and infinite variance otherwise.

#### 4.2. A GI/G/1 Queue with Service Time Determined by a Bernoulli ( $\theta$ )

This is essentially Example 4.2 of L'Ecuyer (1990). We look at a GI/G/1 queue as in Example 4.1, but with a different service time distribution. Let  $0 < a < b < \infty$ ,  $0 < c < \theta < d < 1$ , and suppose that the service time is  $b$  with probability  $\theta$ , and  $a$  with probability  $1 - \theta$ . Let  $C_i = 1$  if  $S_i = b$ , and  $C_i = 0$  if  $S_i = a$ . Here,  $C_i$  is Bernoulli ( $\theta$ ) and determines the service time of customer  $i$ . Take  $Q = G = P_{\theta_0}$ , for  $c < \theta_0 < d$ , which means that we will verify A1( $k$ ) directly. One has

$$L(G, \theta, \omega) = \prod_{i=1}^t \frac{\theta^{C_i} (1 - \theta)^{1 - C_i}}{\theta_0^{C_i} (1 - \theta_0)^{1 - C_i}}$$

and

$$\begin{aligned} S(\theta, \omega) &= \frac{\partial}{\partial \theta} \ln L(G, \theta, \omega) = \frac{\partial}{\partial \theta} \ln \prod_{i=1}^t [\theta^{C_i} (1 - \theta)^{1 - C_i}] \\ &= \frac{1}{\theta} \sum_{i=1}^t C_i - \frac{1}{1 - \theta} \sum_{i=1}^t (1 - C_i) = \sum_{i=1}^t \frac{C_i - \theta}{\theta(1 - \theta)}. \end{aligned}$$

Recall that  $S_i \leq b$ ,  $0 < c < \theta_0 < d < 1$ , and  $0 < c < \theta < d < 1$ . Therefore,  $\theta^{C_i} (1 - \theta)^{1 - C_i} \leq 1$ ,  $|C_i - \theta| \leq 1$ ,  $\theta_0^{C_i} \geq c$ ,  $(1 - \theta_0)^{1 - C_i} \geq 1 - d$ , and  $\theta(1 - \theta) \geq c(1 - d)$ . Then,

$$\begin{aligned} |\psi(\theta, \omega)| &= \left| \frac{1}{t} \sum_{i=1}^t (W_i + S_i) \right| \left| \prod_{i=1}^t \frac{\theta^{C_i} (1 - \theta)^{1 - C_i}}{\theta_0^{C_i} (1 - \theta_0)^{1 - C_i}} \right| \left| \sum_{i=1}^t \frac{C_i - \theta}{\theta(1 - \theta)} \right| \\ &\leq \left[ \sum_{i=1}^t S_i \right] \left( \frac{1}{c(1 - d)} \right)^t \sum_{i=1}^t \frac{1}{\theta(1 - \theta)} \leq \frac{bt^2}{(c(1 - d))^{t+1}}. \end{aligned}$$

Take  $\Gamma(\omega) = bt^2 / (c(1 - d))^{t+1}$ . This is a constant and is therefore trivially  $P_{\theta_0}$ -integrable. This completes the verification of A1(1), so that Proposition 1 applies for  $k = 1$ . It is also easily seen that the moments of  $\psi(\theta, \omega)$  of all orders are finite.

### 4.3. Derivative of the Expected Number of Customers per Busy Cycle

For the GI/M/1 queue examined in Example 4.1, redefine  $h(\theta, \omega)$  as the number  $\tau$  of customers in the first busy cycle, that is  $\tau = \min\{i \geq 1 \mid \text{customer } i + 1 \text{ does not wait}\}$ . We assume that the system is started empty. Here,  $\zeta_i = (S_i, A_i)$  and  $\omega = (S_1, A_1, \dots, S_\tau, A_\tau)$ . Take  $G = P_{\theta_0}$ , where  $\theta_0 < 1/\lambda$  (the latter is the mean interarrival time). We want to simulate the system at parameter value  $\theta_0$  to estimate the derivative at  $\theta$ . We will now seek an interval  $\Upsilon$ , containing  $\theta_0$ , for which A3(k) holds.

Since the interarrival time distribution does not depend on  $\theta$ , only the service times and their density function affect the likelihood ratio, the score function, and the left-hand-side of (19–20). Therefore, to simplify the notation, we can view  $\zeta_i$  as just  $S_i$  and  $f_{i,\theta}$  as the service time density. One then has

$$\begin{aligned} f_{i,\theta}(\zeta) &= \frac{1}{\theta} \exp[-\zeta/\theta]; \\ \frac{\partial}{\partial \theta} f_{i,\theta}(\zeta) &= \frac{\zeta - \theta}{\theta^3} \exp[-\zeta/\theta]; \\ h'(\theta, \omega) &= 0; \\ L(G, \theta, \omega) &= \left(\frac{\theta_0}{\theta}\right)^\tau \exp\left[\left(\frac{1}{\theta_0} - \frac{1}{\theta}\right) \sum_{i=1}^{\tau} S_i\right]. \end{aligned}$$

The LR estimator of the derivative of  $E_\theta[\tau]$  is then

$$\psi(\theta, \omega) = \tau S(\theta, \omega) L(G, \theta, \omega) = \tau \left(\sum_{i=1}^{\tau} \frac{S_i - \theta}{\theta^2}\right) \left(\frac{\theta_0}{\theta}\right)^\tau \exp\left[\left(\frac{1}{\theta_0} - \frac{1}{\theta}\right) \sum_{i=1}^{\tau} S_i\right].$$

For fixed  $\omega = (\zeta_1, \dots, \zeta_\tau)$ ,  $S_i$  and  $\tau$  are fixed, so that  $f_{i,\theta}(\zeta_i)$  and  $h(\theta, \omega) = \tau$  are everywhere differentiable in  $\theta$ . Therefore, part (a) of A2 holds.

For each given pair  $0 < \theta_0 < 1/\lambda$  and  $0 < \theta < 1/\lambda$ , one can choose a different value of  $b$  and take  $q_i = f_{i,b}$  for each  $i$ . Let

$$b = \frac{\theta\theta_0}{k\theta_0 - (k-1)\theta},$$

i.e.,

$$\frac{1}{b} = \frac{k}{\theta} - \frac{k-1}{\theta_0}. \quad (26)$$

Suppose that  $\theta_0$  and  $\theta$  are such that  $0 < b < 1/\lambda$ . From Theorem III.3.1 in Gut (1988), page 78, one has  $E_b[\tau^{4k}] < \infty$ . Since the exponential distribution has a finite moment generating function in some neighborhood of zero, it also follows from Theorem III.3.2 in Gut (1988), page 81, that there exists a  $z_0(b) > 1$  such that  $E_b[z^\tau] < \infty$  for  $0 \leq z \leq z_0(b)$ . Suppose now that

$$\left(\frac{b\theta_0^{k-1}}{\theta^k}\right)^8 < z_0(b). \quad (27)$$

Then,  $E_b[(b\theta_0^{k-1}/\theta^k)^{8\tau}] < \infty$ . Note that for  $\theta = \theta_0$ , one has  $b = \theta_0$  and (27) holds. Further, for fixed  $\theta_0$  and  $b$ , there exists a neighborhood of  $\theta_0$ , say  $\Upsilon = (\theta_{\min}, \theta_{\max})$ , in which (27) is satisfied.

To complete the verification of A3 for that fixed  $b$ , under these assumptions, define  $\Gamma_{1i}(\zeta) = (b\theta_0^{k-1}/\theta_{\min}^k)^8$  and  $\Gamma_{2i}(\zeta) = ((\zeta + \theta_{\max})^k b^k / \theta_{\min}^{3k})^4$  for each  $\zeta$  and  $i$ . One has

$$\begin{aligned} \sup_{\theta \in \Upsilon} \left( \frac{f_{i,\theta}^k(\zeta)}{q_i(\zeta)g_i^{k-1}(\zeta)} \right)^8 &= \sup_{\theta \in \Upsilon} \left( \frac{b\theta_0^{k-1}}{\theta^k} \exp \left[ \left( \frac{1}{b} + \frac{k-1}{\theta_0} - \frac{k}{\theta} \right) \zeta \right] \right)^8 \\ &\leq \left( \frac{b\theta_0^{k-1}}{\theta_{\min}^k} \right)^8 = \Gamma_{1i}(\zeta); \\ \sup_{\theta \in \Upsilon} \left( \frac{(\frac{\partial}{\partial \theta} f_{i,\theta}(\zeta))^k}{q_i(\zeta)g_i^{k-1}(\zeta)} \right)^4 &= \sup_{\theta \in \Upsilon} \left( \frac{(\zeta - \theta)^k b \theta_0^{k-1}}{\theta^{3k}} \exp \left[ \left( \frac{1}{b} + \frac{k-1}{\theta_0} - \frac{k}{\theta} \right) \zeta \right] \right)^4 \\ &\leq \left( \frac{(\zeta + \theta_{\max})^k b^k}{\theta_{\min}^{3k}} \right)^4 = \Gamma_{2i}(\zeta); \\ E_b \left[ \sum_{j=1}^{\tau} \Gamma_{2j}(S_j) \right] &= E_b[\tau] E_b \left[ \left( \frac{(S_j + \theta_{\max})^k b^k}{\theta_{\min}^{3k}} \right)^4 \right] < \infty; \\ E_b \left[ \prod_{i=1}^{\tau} \Gamma_{1i}(S_i) \right] &= E_b[(b\theta_0^{k-1}/\theta_{\min}^k)^{8\tau}] < \infty. \end{aligned}$$

The next to last equality is the well known Wald's equation. This completes the verification of A3(k) for that given  $\Upsilon$ .

From that reasoning, sufficient conditions for A3(k) are that

$$k/\theta - (k-1)/\theta_0 > \lambda \quad (28)$$

and that (27) holds with  $b$  defined by (26). For  $k = 1$ , one has  $b = \theta$ , (28) becomes  $\theta < 1/\lambda$ , and (27) becomes  $z_0(b) > 1$ , which always holds.



For a more specific example, consider an M/M/1 queue with arrival rate  $\lambda = 1$  and mean service time  $\theta < 1$ . In this case, one has  $E_b[z^\tau] < \infty$  if and only if  $z \leq z_0(b) = (1+b)^2/(4b)$  (see Kleinrock 1975, §5.9). The condition (27) then becomes:

$$\frac{4b^9}{(1+b)^2} \leq \left( \frac{\theta^k}{\theta_0^{k-1}} \right)^8. \quad (29)$$

For  $k = 2$ , for each  $\theta_0 \in (0, 1)$ , there is an interval  $I(\theta_0)$  containing  $\theta_0$  in which (29) holds. For example, one has:

$\theta_0$	$I(\theta_0)$
0.2	(.1261, .2407)
0.5	(.4090, .5450)
0.8	(.7514, .8233)

Further, for  $k = 2$ , (28) becomes  $\theta \leq 2\theta_0/(\theta_0 + 1)$ , which also holds when  $\theta \in I(\theta_0)$ . We have now proved that the variance of the LR derivative estimator is finite for  $\theta \in I(\theta_0)$ . Outside that interval, we have not proven anything, which means that the variance could also be finite in a larger region.

## 5. Conclusion

We have studied sufficient conditions for unbiasedness and finite moments, for LR derivative estimators. Clearly, the conditions given are not necessary, but they could prove useful in practice. Necessary and sufficient conditions, namely uniform integrability, are often difficult to check directly (see, e.g., Glasserman 1991).

We have illustrated the practical use of our different sets of assumptions by a few simple examples. More complicated applications, like queueing network driven by a sequence of independent variates, would also fit into the framework of Section 3, and their analysis will be similar to that of our examples. In our examples, the probability distributions are from the exponential family. In fact, our treatment is not really specific to the exponential family, but likelihood ratios simplify nicely for the exponential family, and most commonly used distributions belong to that family.

The interchange between derivative and expectation is not the only major issue in derivative estimation. Finite vs infinite variance (or higher moments) is also crucial. One must be very careful in practice, because as illustrated by Example 4.1, an unbiased LR derivative estimator can have an infinite variance. Further, when the variance is infinite, the variance estimators are usually very misleading: they typically take a very small value with large probability, and an extremely large value with a tiny (almost zero) probability. What happens is essentially the same as with “importance sampling” for function (as opposed to derivative) estimation (see Glynn 1991 and Glynn and Iglehart 1989).

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