

## THE REPAIR vs. REPLACEMENT PROBLEM: A STOCHASTIC CONTROL APPROACH

PIERRE L'ECUYER

*Département d'informatique, Université Laval, Ste-Foy, Qué., Canada, G1K-7P4*

AND

ALAIN HAURIE

*GERAD, Ecole des Hautes Etudes Commerciales, 5255 Decelles, Montréal, Canada, H3T-1V6*

### SUMMARY

This paper is concerned with a *repair vs. replacement* problem for a system observed in continuous time and subject to random failure. When the system fails, a decision is taken, based on the observed values of the (random) repair cost and the system's age, to either perform a minimal repair or to replace the system by a new one. At any time, the system may also be replaced preventively, and the aim of the decision maker is to minimize the total expected discounted cost. A dynamic programming approach is proposed to solve this problem. Sufficient conditions under which a *repair-limit* rule is optimal are given. An efficient algorithm is devised for the computation of such an optimal policy, and a numerical illustration is worked out.

KEY WORDS Dynamic programming Stochastic control Optimal replacement Repair limit

### 1. INTRODUCTION

The aim of this paper is to propose a very general formulation of the so-called *repair vs. replacement* problem for a system observed in continuous time, to develop a dynamic programming approach for its solution, to find conditions under which a *repair-limit* rule is optimal and, finally, to devise an efficient algorithm for the computation of such an optimal policy.

The *repair vs. replacement* problem is a variant of the classical preventive replacement problem for an age-dependent failure-prone system<sup>1-4</sup> where, at failure time, there is a possibility either to perform a minimal repair or to replace the system by a new one. The model presented in Sections 2 and 3 generalizes earlier studies<sup>5-7</sup> along several lines: (i) a very general cost structure is assumed; (ii) no *a priori* assumptions on the form of the optimal policy are made; (iii) the problem is formulated in the general framework of discrete-event dynamic programming with continuous state variables. In Section 4, we explore the conditions under which a *repair-limit* rule is an optimal policy. In a *repair-limit* rule, a decision to replace the system is taken either at a predetermined age limit, or at failure time, provided that the cost of a minimal repair is higher than a predetermined limit which is generally dependent on the age of the system. In Section 5, we propose an efficient algorithm to compute an  $\epsilon$ -optimal policy, for the case where a *repair-limit* rule is optimal. In Section 6, the algorithm is illustrated

on a numerical example. The Appendix contains the mathematical developments involved in establishing the results of the paper.

This application is based on recent theoretical developments in dynamic programming for discrete-event systems with state variables in Borel spaces,<sup>8-11</sup> and on an advanced implementation of a successive approximations algorithm in such a general framework.<sup>12</sup> This approach has already permitted a complete solution of a continuous-time group preventive replacement problem,<sup>12</sup> and the present paper gives another example of its usefulness for the treatment of complex maintenance problems. In this general modern framework, discrete event stochastic systems can be modelled without an arbitrary *a priori* time and state discretization, as was often the case in previous stochastic dynamic programming implementations for maintenance applications.<sup>3,13,14</sup>

## 2. MODELLING THE REPAIR vs. REPLACEMENT PROBLEM

Consider a failure-prone system with an operating cost rate  $c_o(s)$  and a failure rate  $\lambda(s)$  at age  $s$ . At any instant, the system is either operating or has failed. If it fails at age  $s$ , a failure cost  $c_f(s)$  is incurred, and the system is either replaced at a cost  $c_r(s)$ , or repaired at a cost  $r$ , which is the sample value of a random variable whose probability law is defined by the conditional probability measure  $R(\cdot | s)$ . We assume that the decision to replace or repair is taken after having observed the actual value of  $r$ . A (minimal) repair brings the system back to an operational state without changing its age  $s$ , whereas a replacement brings the age of the system back to 0. At any age  $s$ , one may preventively replace the system, at a cost  $c_r(s)$ , in order to avoid possible failure costs. The system operates on an infinite time horizon, and the costs are discounted at a rate  $\rho > 0$ .

The *state* of the system at time  $t$  is, by convention, its age at time  $t^+$ ; i.e. at any intervention time (replacement or repair) the state  $s$  represents the age of the system immediately after the intervention. An intervention is triggered either by a failure or by a planned preventive replacement. The state space is thus  $S = [0, \infty)$ . A controller *action* can be represented by a pair  $a = (d, \gamma)$ , where  $d \in [0, \infty)$  is the time delay till the next planned preventive replacement, and  $\gamma: S \times [0, \infty) \rightarrow \{0, 1\}$  is a (measurable) binary function, called the *repair vs. replacement rule*, such that if a failure occurs at age  $s$  and the observed repair cost is  $r$ , a *replacement* is decided if  $\gamma(s, r) = 1$ , and a *repair* is decided if  $\gamma(s, r) = 0$ . Let  $A$  denote the set of all possible controller actions.

The evolution of the system is then described as follows. Call *stage*  $n$  the occurrence of the  $n$ th intervention. At the initial stage 0, the system is in state 0. Entering stage  $n$  at time  $\tau_n$ ,  $n \geq 0$ , the controller observes the age  $s_n$  of the system and then chooses an action  $a_n = (d_n, \gamma_n)$ . Given the failure process, the state  $s_n$  and the chosen action, a random transition defines (i) the next intervention time  $\tau_{n+1}$ , which corresponds to stage  $n + 1$  and is the minimum between the time  $\tau_n + \zeta$  of the next failure and the time  $\tau_n + d_n$  of the next planned preventive replacement; and (ii) the state  $s_{n+1}$  of the system at stage  $n + 1$ , which is zero if the system is replaced or  $s_n + \zeta$  if the system is repaired.

This system is thus modelled as a *Markov renewal decision process*<sup>9-11,15</sup> with continuous state and action sets. In order to complete the definition of this process, we introduce the survival function

$$\bar{F}(\zeta) = e^{-\int_0^\zeta \lambda(t) dt} \quad (1)$$

the conditional survival function given state  $s$ ,

$$\bar{F}(s + \zeta | s) = \frac{\bar{F}(s + \zeta)}{\bar{F}(s)} \quad (2)$$

and the conditional probability density function of the age of failure

$$f(s + \zeta | s) = -\frac{\partial}{\partial \zeta} \bar{F}(s + \zeta | s) = \lambda(s + \zeta) \bar{F}(s + \zeta | s), \quad f(\zeta) = f(\zeta | 0) \quad (3)$$

One can then define the expected *one-stage transition cost* given state  $s$  and action  $a = (d, \gamma)$  as

$$\begin{aligned} g(s, d, \gamma) = & \int_0^d \left[ \int_0^\zeta e^{-\rho t} c_o(s+t) dt \right. \\ & \left. + e^{-\rho \zeta} \left( c_r(s + \zeta) + \int_0^\infty w(s + \zeta, r) R(dr | s + \zeta) \right) \right] f(s + \zeta | s) d\zeta \\ & + \left[ \int_0^d e^{-\rho t} c_o(s+t) dt + e^{-\rho d} c_r(s+d) \right] \bar{F}(s+d | s) \end{aligned} \quad (4)$$

where

$$w(s + \zeta, r) = \begin{cases} c_r(s + \zeta), & \text{if } \gamma(s + \zeta, r) = 1 \\ r, & \text{otherwise} \end{cases} \quad (5)$$

A (stationary) *policy* is a map  $\mu: S \rightarrow A$  which associates with a state  $s$  an action  $a = \mu(s)$ . For the ease of exposition, we omit all mathematical details relating to the measurability questions and refer the reader to the Appendix, where these questions are addressed. The term *admissible* will always refer to the minimum measurability properties needed for the existence of the integrals (including those defining the mathematical expectations). An admissible policy  $\mu$  generates a Markov renewal stochastic process with value in  $S \times A$ , with an expected total discounted cost, given initial state  $s_0 = s$ :

$$V_\mu(s) = E_{\mu, s} \left[ \sum_{n=0}^{\infty} e^{-\rho \tau_n} g(s_n, a_n) \right] \quad (6)$$

The problem is to find an optimal policy  $\mu^*$ , such that

$$V_{\mu^*}(s) = V_*(s) \stackrel{\text{def.}}{=} \inf_{\mu} V_\mu(s) \quad (7)$$

or an  $\varepsilon$ -optimal policy  $\mu$ , such that for a given  $\varepsilon > 0$ ,

$$V_\mu(s) \leq V_*(s) + \varepsilon \quad (8)$$

for all  $s$  in  $S$ .

### Remark

An equivalent model could be defined with the pair  $(s, r)$  taken as the state variable,  $s$  being the age of the system at  $t^-$ , i.e. just before the intervention, and  $r$  being the observed repair cost at failure time. This would simplify the action space, but increase the dimension of the state set (from one-dimensional to two-dimensional); two-dimensional cost-to-go functions (instead of one-dimensional) would have to be computed and approximated at each iteration of the dynamic programming algorithm, thus squaring the amount of work. The increase in complexity of the action space has no significant impact on the amount of computation, since as we will see later (equation (18)), the optimizing  $\gamma$  at each iteration always has a trivial form and does not need to be computed explicitly. Our choice also makes easier the theoretical developments of the following sections.

### 3. THE DYNAMIC PROGRAMMING FORMALISM

When the state set is countable, it is well known that under some regularity conditions, a dynamic programming approach can be used for the solution of an infinite horizon discounted Markov renewal decision process (or semi-Markov decision process).<sup>3,15</sup> For the problem defined in Section 2, one has two additional difficulties, namely a continuous state space and absence of the classical regularity conditions,<sup>8,15</sup> which ask for the expected time between any two successive transitions to be bounded away from zero. A dynamic programming formalism adapted to this situation has been developed recently<sup>8-11</sup> and is mathematically more intricate because of delicate measurability problems. In this section, we use a dynamic programming approach in operator form and provide conditions under which successive applications of the operator converge to its fixed point.

For a given action  $a = (d, \gamma)$ , define the operator  $H(V)(\cdot, d, \gamma)$  which transforms any admissible *cost-to-go* function  $V: S \rightarrow [0, \infty)$  into another admissible cost-to-go function:

$$H(V)(s, d, \gamma) = g(s, d, \gamma) + \int_0^d \int_0^\infty V([1 - \gamma(s + \zeta, r)](s + \zeta))R(dr | s + \zeta)e^{-\rho\zeta}f(s + \zeta | s) d\zeta + V(0)e^{-\rho d}\bar{F}(s + d | s) \tag{9}$$

With any admissible policy  $\mu$ , we associate the operator  $T_\mu$  defined by

$$T_\mu(V)(s) = H(V)(s, \mu(s)) \tag{10}$$

Finally, we define the dynamic programming operator  $T$  by

$$T(V)(s) = \inf_\mu T_\mu(V)(s) \tag{11}$$

Value iteration (also called successive approximations) consists in applying the operator  $T$  repeatedly on an approximation of the true cost-to-go function. Starting from a bounded and admissible initial cost-to-go function, the value iteration algorithm converges to  $V_*$  under adequate regularity conditions. We will see in Proposition 1 that Condition 1 below is sufficient for  $T^n(V_0)$  to converge uniformly to  $V_*$  as  $n$  tends to infinity, where  $V_0$  denotes an initial cost-to-go function and  $T^n$  is the  $n$ -fold composition of the operator  $T$ .

#### Condition 1

- (i) There is a constant  $L > 0$  such that, for all  $s$  in  $S$ ,

$$\min\left(c_r(s), [c_0(s)/\lambda(s)] + c_r(s) + \int_0^\infty \min(r, c_r(s))R(dr | s)\right) \geq L \tag{12}$$

- (ii) There exists an admissible policy  $\mu_1$  and two constants  $U \geq 0$  and  $0 \leq b < 1$  such that for all  $s \geq 0$ ,

$$g(s, \mu_1(s)) \leq U \text{ and } \alpha(s, \mu_1(s)) \leq b \tag{13}$$

where

$$\alpha(s, d, \gamma) = \int_0^d e^{-\rho\zeta}f(s + \zeta | s) d\zeta + e^{-\rho d}\bar{F}(s + d | s) \tag{14}$$

denotes the one-stage expected discount factor when one chooses action  $(d, \gamma)$  in state  $s$ . ■

By a straightforward calculation (see the Appendix), the one-stage cost can be rewritten differently as

$$g(s, d, \gamma) = \int_0^d \left[ c_0(s + \zeta) + \lambda(s + \zeta) \left( c_r(s + \zeta) + \int_0^\infty w(s + \zeta, r) R(dr | s + \zeta) \right) \right] e^{-\rho \zeta} \bar{F}(s + \zeta | s) d\zeta + c_r(s + d) e^{-\rho d} \bar{F}(s + d | s). \tag{15}$$

which may facilitate the interpretation of Condition 1. Equation (13) states that for some policy  $\mu_1$ , the one-stage cost is bounded and the expected duration between any two stages is bounded away from zero. Equation (12) implies that the one-stage expected cost divided by the one-stage expected discount factor is bounded away from zero. More specifically, for each  $(s, d, \gamma)$  in  $S \times A$ , we have

$$g(s, d, \gamma) \geq \alpha(s, d, \gamma)L \tag{16}$$

This shows that Condition 1 also implies that this model is a so-called *version LC* of the Markov renewal decision process model introduced in References 9–11 for which the following result holds:

*Proposition 1*

Under Condition 1, let  $B$  be the set of admissible cost-to-go functions  $V$  such that  $0 \leq V \leq U/(1 - b)$ . Then

- (i)  $V_*$  is in  $B$ , and  $T(V) = V$  iff  $V = V_*$ .
- (ii)  $\lim_{n \rightarrow \infty} \sup_{s \in S} |T^n(V)(s) - V_*(s)| = 0$  for all  $V$  in  $B$ .
- (iii) A policy  $\mu$  is optimal iff  $T_\mu(V_*) = V_*$ .

*Proof.* See References 10 and 11. ■

The repair vs. replacement problem has thus been modelled in such a way that it lends itself to a value iteration dynamic programming approach for its solution. Clearly this modelling approach considers an *a priori* very large action space. In the next section, it is shown that, for an interesting class of systems, the policy space can be reduced to a much smaller class of decision rules, namely the *repair-limit* rules. For these forthcoming developments, it will be convenient to define for each  $d$  in  $[0, \infty)$  the operator  $J(V)(\cdot, d)$  by

$$J(V)(s, d) = H(V)(s, d, \gamma_V) = \inf_{\gamma} H(V)(s, d, \gamma) \tag{17}$$

where

$$\gamma_V(s + \zeta, r) = \begin{cases} 1, & \text{if } r + V(s + \zeta) \geq c_r(s + \zeta) + V(0) \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

One thus has

$$T(V)(s) = \inf_{0 \leq d \leq \infty} J(V)(s, d) \tag{19}$$

#### 4. OPTIMALITY CONDITIONS FOR REPAIR-LIMIT POLICES

A *repair-limit* policy is a rule under which the decision maker replaces the component either at a predetermined age  $d_*$  or when it fails at age  $s$  and the repair cost is greater than a predeter-

mined age-dependent limit  $L_*(s)$ ; any other failure is corrected by a *minimal repair*. More formally, a *repair-limit policy* is a policy  $\mu$  such that

$$\mu(s) = \begin{cases} (d_* - s, \gamma_*) & \text{if } s < d_* \\ (0, \gamma_*) & \text{otherwise} \end{cases} \tag{20}$$

where

$$\gamma_*(s, r) = \begin{cases} 1, & \text{if } r > L_*(s) \text{ or } s \geq d_* \\ 0, & \text{otherwise} \end{cases} \tag{21}$$

and where  $d_*$  is a fixed non-negative real number and  $L_*: [0, d_*] \rightarrow [0, \infty)$  is a given measurable function. This definition generalizes the one introduced by Drinkwater and Hastings.<sup>7</sup> In this section, we characterize a class of *repair vs. replacement* problems for which an optimal policy is of the *repair-limit* type.

*Condition 2*

Assume that  $c_r(\cdot)$  and  $\lambda(\cdot)$  are non-decreasing functions of the age  $s$ , that the derivative  $c'_r(\cdot)$  of  $c_r(\cdot)$  exists and that, for any non-decreasing function  $q: [0, \infty) \rightarrow [0, \infty)$ , bounded above by  $c_r(\cdot)$ , the following functionals:

$$\Lambda_1(s, q(s)) = c_0(s) + \lambda(s) \left\{ c_r(s) + \int_0^{c_r(s) - q(s)} [q(s) + r - c_r(s)] R(dr | s) \right\} \tag{22}$$

$$\Lambda_2(s, q(s)) = \Lambda_1(s, q(s)) + c'_r(s) - \rho c_r(s) \tag{23}$$

are non-decreasing in  $s$ . ■

We claim that the systems for which Condition 2 holds admit optimal policies which are *repair-limit*. More precisely, if one calls  $B_+$  the class of non-decreasing cost-to-go functions  $V(\cdot)$  in  $B$ , and if one defines the operators  $G_1: B_+ \rightarrow B_+$  and  $G_2: B_+ \rightarrow B_+$  by

$$G_1(V)(s) = \Lambda_1(s, \min(c_r(s), V(s) - V(0))) \tag{24}$$

$$G_2(V)(s) = \Lambda_2(s, \min(c_r(s), V(s) - V(0))) - \rho V(0) \tag{25}$$

and the functional

$$l(V) = \inf\{s \geq 0 \mid G_2(V)(s) \geq 0\} \tag{26}$$

then the following result, whose proof is given in the Appendix, holds.

*Proposition 2*

Under Condition 2, one has

(i) for each  $V$  in  $B_+$ ,  $T(V)$  is in  $B_+$  and

$$T(V)(s) = \begin{cases} J(V)(s, l(V) - s) & \text{if } s < l(V) \\ J(V)(s, 0) = c_r(s) + V(0) & \text{if } s \geq l(V) \end{cases} \tag{27}$$

(ii)  $V_*$  is in  $B_+$  and the policy  $\mu_*$  defined by

$$\mu_*(s) = \begin{cases} (d_* - s, \gamma_*) & \text{if } s < d_* \\ (0, \gamma_*) & \text{otherwise} \end{cases} \tag{28}$$

where  $d_* = l(V_*)$  and

$$\gamma_*(s, r) = \begin{cases} 1, & \text{if } r \geq L_*(s) \stackrel{\text{def}}{=} V_*(0) + c_r(s) - V_*(s) \\ 0, & \text{otherwise} \end{cases} \quad (29)$$

is optimal. ■

This result expresses the optimal maintenance policy as a *repair-limit* rule. Of course, both the age limit  $d_*$  and the repair limit function  $L_*(s)$  depend on the optimal cost-to-go function  $V_*(\cdot)$ . Therefore, the above theorem does not specify completely the optimal policy; it only gives information on the form of the optimal policy. This information will be exploited in the algorithm proposed in Section 4.  $V_* \in B_+$  means that the optimal expected cost-to-go is higher for an older system (which is quite reasonable intuitively).

Before leaving this section, we can show that the class of systems which satisfy Condition 2 encompasses the Drinkwater and Hastings model<sup>7</sup> as well as the classical *age replacement* model.<sup>2</sup>

#### Example 1

For the model considered in Reference 7, one has  $c_r = c_o = 0$ ,  $c_r$  is a constant,  $\lambda(s)$  is non-decreasing in  $s$ , and  $R(\cdot | s)$  is an exponential distribution with mean  $m(s)$ , where  $m(s)$  is non-decreasing in  $s$ . Equation (22) then becomes

$$\Lambda_1(s, q(s)) = \lambda(s) [q(s) - c_r + m(s)(1 - e^{-(q(s) - c_r)/m(s)})] \quad (30)$$

and Condition 2 is easily seen to hold. ■

#### Example 2

Consider a model where minimal repair is not allowed, or where the repair cost is always (with probability 1) larger than the replacement cost; that is  $R([0, c_r(s)] | s) = 0$  for all  $s$ . The integral in equation (22) then vanishes for all  $s$  and  $q$ , and one obtains

$$\Lambda_1(s, q(s)) = c_o(s) + \lambda(s)c_r(s).$$

The optimal policy in that case, under Condition 2, is *age replacement*.<sup>2,4</sup> ■

## 5. COMPUTING AN OPTIMAL REPAIR-LIMIT POLICY

We now provide an efficient computational algorithm for the case where Conditions 1 and 2 are satisfied. It is an adaptation of the more general algorithm proposed in Reference 12, and it operates as follows.

#### Algorithm

1. Compute  $U$ ,  $L$  and  $b$  satisfying Condition 1, and  $h := U/(1 - b)$ .
2. Choose  $\varepsilon > 0$  and  $V$  in  $B_+$ .
3. Let  $V_-(\cdot) := 0$ ,  $V_+(\cdot) := h$ .

#### repeat

4. Compute  $l(V)$  satisfying (26).
5. Compute  $T(V)(s) = J(V)(s, l(V) - s)$  at a finite number of points in the interval  $[0, l(V)]$ , including 0 and  $l(V)$ .

6. Define  $W(\cdot)$  in  $B_+$  as an approximation to  $T(V)(\cdot)$  on  $[0, l(V)]$  and equal to  $c_r(\cdot) + V(0)$  on  $(l(V), \infty)$ .

7. Compute (or estimate) non-negative real values  $\delta^-, \delta^+, \gamma^-$  and  $\gamma^+$  such that for all  $s$ ,

$$-\delta^- \leq T(V)(s) - W(s) \leq \delta^+$$

$$-\gamma^- \leq W(s) - V(s) \leq \gamma^+$$

8. Let

$$V_-(\cdot) := \max(V_-(\cdot), W(\cdot) - (\delta^- + \gamma^-)4h/L)$$

$$V_+(\cdot) := \min(V_+(\cdot), W(\cdot) + (\delta^+ + \gamma^+)4h/L).$$

9. Let  $V(\cdot) := W(\cdot)$ .

10. Display any desired values.

*until*  $V_+(s) - V_-(s) < \varepsilon$  for all  $s$ .

11. Display  $l(V)(\cdot)$ ,  $W(\cdot)$  and any other desired statistics.

12. Stop. ■

### Proposition 3

After any iteration of the algorithm, one has  $V_-(\cdot) \leq V_*(\cdot) \leq V_+(\cdot)$ . Further, if the error in the approximation of  $T(V)$  converges to 0, that is if the successive values of  $\delta^-$  and  $\delta^+$  converge to 0, then  $V_+(\cdot) - V_-(\cdot)$  converges to 0 uniformly and the algorithm terminates after a finite number of iterations.

*Proof.* It follows from Theorem 3.3 in Reference 12. ■

Some remarks should be made concerning this algorithm. First, the choice of the policy  $\mu_1$  is somewhat arbitrary, but in order to obtain tighter bounds for  $V_*$ , one should choose it to minimize  $h$ . Secondly, any method of interpolation or approximation may be used to obtain  $W(\cdot)$  in step 6, and the method can also vary between iterations. Interpolation could be used if  $T(V)(s)$  is computed with good precision; otherwise, approximation (e.g. least squares) is recommended. Thirdly, as explained in Reference 12, the bounds in step 7 can be exact and conservative, derived by using the monotonicity properties of  $V(\cdot)$ ,  $T(V)(\cdot)$  and  $W(\cdot)$ , or can be estimations. For instance, when  $T(V)$  is reasonably smooth (which is usually the case), one can perform step 7 by recomputing  $T(V)$  on a much finer regular mesh, computing the actual error of approximation at these new points, and taking the smallest and largest of these errors as estimates of  $\delta^-$  and  $\delta^+$ , respectively.

## 6. A NUMERICAL ILLUSTRATION

Let  $\rho = 0.1$ ,  $c_r(s) = 20$ ,  $c_f(s) = 5$ ,  $\lambda(s) = \min\{0.02s, 0.2\}$ ,  $c_0(s) = \min\{s + e^{-s}, 10 + e^{-10}\}$  (a system of age greater than 10 is assumed to be equivalent to a system of age 10), and the distribution function of the repair cost be defined by  $R([0, r] | s) = 1 - e^{-r/2}$  for all  $s \geq 0$  and  $r \geq 0$ . The failure rate  $\lambda(s)$  corresponds (for  $s \leq 10$ ) to a Weibull distribution, and the repair cost has an exponential distribution with mean 2. Consider the policy  $\mu_1$  defined by

$$\mu_1(s) = (d_1, \gamma_1)$$

where  $d_1 = 10$  and  $\gamma_1(\cdot) \equiv 1$ . Equation (13) is verified for  $b = 0.6833$ ,  $U = 48.5$ , and one has



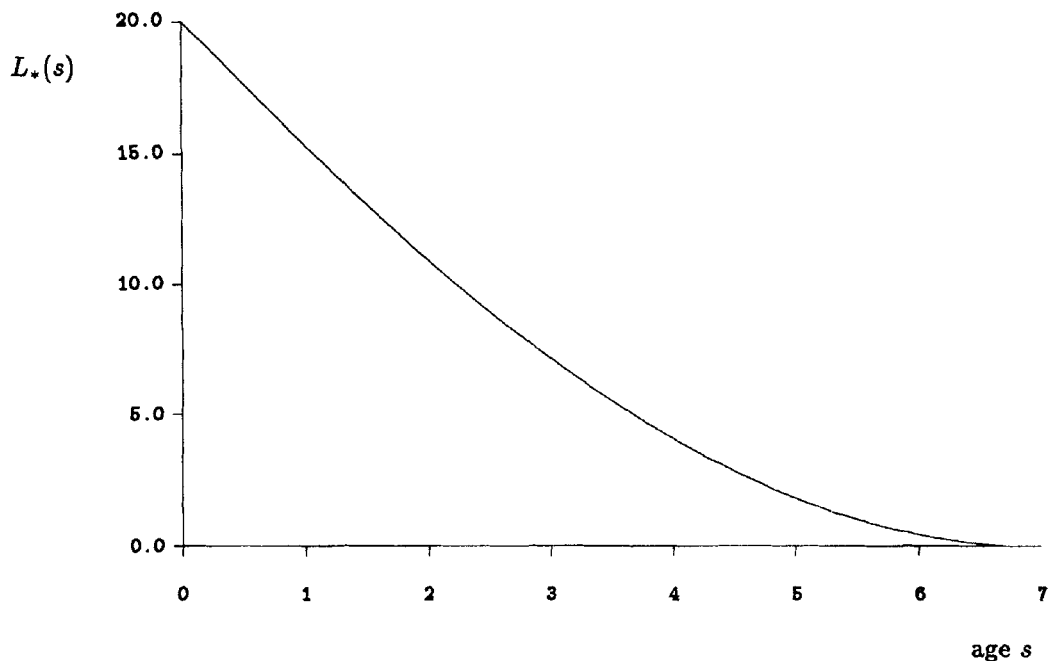


Figure 1. The optimal repair-limit function

$h = U/(1 - b) = 153 \cdot 14$ , whereas equation (12) is verified for  $L = 20$ . Condition 2 is easily verified, and the algorithm proposed in the previous section can be used.

A computer implementation has been done, in FORTRAN, and the program allows changing the evaluation points of  $T(V)$  or the method of approximation after any iteration. The initial  $V(\cdot)$  is set to zero in step 2. In step 4, a sequential search is used to localize  $l(V)$  approximately; then binary search is used to find its exact position. In step 5,  $T(V)$  is computed at  $n$  points, evenly spaced on the interval  $[0, l(V)]$ . The integrals are evaluated using Simpson's rule with an integration step small enough to obtain a negligible quadrature error.  $W(\cdot)$  is defined on  $[0, l(V)]$  by spline interpolation for splines of arbitrary order where the knots are chosen according to equation (10) on p. 219 of Reference 16.

After 30 iterations with  $n = 10$  evaluation points and splines of order  $k = 2$  (i.e. piecewise-linear interpolation), one obtains  $l(V) = 6 \cdot 7047$  and  $V(0) = 56 \cdot 35$ . After five more iterations with  $k = 2$  and  $n = 25$ , one obtains  $l(V) = 6 \cdot 7056$ ; and five more iterations with  $k = 4$  (cubic splines) and  $n = 25$  yield  $l(V) = 6 \cdot 7058$  and  $V(0) = 56 \cdot 365$ . Finally, after 10 more iterations with cubic splines and  $n = 50$ , one still has the same values.  $T(V)$  can then be recomputed on a much finer grid (500 uniformly spaced points) to estimate  $\delta^-$  and  $\delta^+$ , which yields the estimate bounds on  $V_*$ :

$$W(\cdot) - 0 \cdot 00023 = V_-(\cdot) \leq V_*(\cdot) \leq V_+(\cdot) = W(\cdot) + 0 \cdot 00009$$

The relative error on  $V_*$  is thus negligible. The repair-limit at age  $s$  is  $L_*(s) = c_r(s) - V_*(s) + V_*(0)$ , and this function is displayed in Figure 1. Here, the repair-limit is decreasing in  $s$ , but notice that this is not true in general when  $c_r(\cdot)$  is not constant.

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## APPENDIX

In this Appendix, we state the measurability conditions sufficient for the dynamic programming formalism to be well defined and for Proposition 1 to hold. We also derive equation (15), and we prove Proposition 2.

### *Measurability conditions*

Assume  $c_f(\cdot)$ ,  $c_r(\cdot)$ ,  $c_o(\cdot)$  and  $\lambda(\cdot)$  to be four Borel measurable functions from  $[0, \infty)$  to  $[0, \infty)$ .  $R$  is assumed to be a continuous Borel measurable stochastic kernel<sup>8,9</sup> on  $[0, \infty)$  given  $[0, \infty)$ , i.e. a family  $\{R(\cdot | s), s \geq 0\}$  of probability measures on  $[0, \infty)$  such that for every Borel set  $B \subseteq [0, \infty)$ ,  $R(B | \cdot)$  is a continuous (and Borel measurable) function. The *state space* is  $S = [0, \infty)$  and the *action space* is  $A = [0, \infty) \times Z$  where  $Z$  is the class of Borel measurable binary functions  $\gamma: S \times [0, \infty) \rightarrow [0, 1]$ . An *admissible policy* is a universally measurable<sup>8,9</sup> mapping  $\mu: S \rightarrow A$ . An *admissible cost-to-go function* is a bounded lower semi-analytic function  $V: S \rightarrow [0, \infty)$ . For any such function, the expressions in (9)–(11) are well defined.<sup>8,9</sup>

Notice that more general kinds of policies could have been considered where  $\mu$  is random and may depend on all past history. However, it could be shown<sup>8,10</sup> that such an extension provides no additional gain in the optimal cost function. ■

### *Derivation of Equation (15)*

Using Fubini's theorem, one has

$$\begin{aligned} \int_0^d \int_0^\zeta e^{-\rho t} c_o(s+t) dt f(s+\zeta | s) d\zeta &= \int_0^d \int_t^d e^{-\rho t} c_o(s+t) f(s+\zeta | s) d\zeta dt \\ &= \int_0^d e^{-\rho t} c_o(s+t) [\bar{F}(s+t | s) - \bar{F}(s+d | s)] dt \end{aligned}$$

and from (4), using the fact that  $f(s+\zeta | s) = \lambda(s+\zeta) \bar{F}(s+\zeta | s)$ , one obtains (15). ■

### *Proof of Proposition 2*

Let  $V \in B_+$ . One has

$$\begin{aligned} \frac{\partial}{\partial d} J(V)(s, d) &= e^{-\rho d} \bar{F}(s+d | s) \left\{ c_o(s+d) + \lambda(s+d) \left[ c_f(s+d) \right. \right. \\ &\quad \left. \left. + \int_0^\infty \min\{r + V(s+d), c_r(s+d) + V(0)\} R(dr | s+d) \right] \right. \\ &\quad \left. + c'_f(s+d) - [c_r(s+d) + V(0)] [\lambda(s+d) + \rho] \right\} \\ &= e^{-\rho d} \bar{F}(s+d | s) G_2(V)(s+d). \end{aligned} \tag{31}$$

From equation (23) in Condition 2,  $G_2(V)(s)$  is non-decreasing in  $s$ . It is thus negative for

$s < l(V)$ , non-negative for  $s > l(V)$ , and one has

$$\frac{\partial}{\partial d} J(V)(s, d) \begin{cases} < 0, & \text{if } d < l(V) - s \\ \geq 0, & \text{if } d > l(V) - s \end{cases}$$

$J(V)(s, d)$  is thus decreasing in  $d$  on the interval  $[0, \max(0, l(V) - s)]$ , non-decreasing thereafter, and the infimum in (19) is attained for  $d = \max(0, l(V) - s)$ . Notice that  $d = \infty$  if  $l(V) = \infty$ .

It remains to show that  $T(V)$  is non-decreasing on  $(0, l(V))$ . Let  $0 \leq s_1 < s_2 < l(V)$  and  $d_2 = l(V) - s_2$ . One has

$$\begin{aligned} T(V)(s_2) &= J(V)(s_2, d_2) \\ &= \int_{s_2}^{s_2 + d_2} e^{-\rho(s - s_2)} \bar{F}(s | s_2) \{ G_1(V)(s) + \lambda(s) [c_r(s) + V(0)] \} ds \\ &\quad + e^{-\rho d_2} \bar{F}(s_2 + d_2 | s_2) [c_r(s_2 + d_2) + V(0)] \end{aligned} \quad (32)$$

For each  $s$  in  $[s_2, l(V)]$ , let  $Q(s)$  be the value in  $[s_1, s)$  satisfying

$$\int_{s_1}^{Q(s)} [\lambda(t) + \rho] dt = \int_{s_2}^s [\lambda(t) + \rho] dt \quad (33)$$

Such a value uniquely exists, since  $\int_{s_1}^{s'} [\lambda(t) + \rho] dt$  is continuous and strictly increasing in  $s'$ ,

and

$$\int_{s_1}^s [\lambda(t) + \rho] dt > \int_{s_2}^s [\lambda(t) + \rho] dt \geq 0 = \int_{s_1}^{s_1} [\lambda(t) + \rho] dt$$

One then has

$$\begin{aligned} \bar{F}(s | s_2) e^{-\rho(s - s_2)} &= e^{-\rho(s - s_2) - \int_{s_1}^s \lambda(t) dt} \\ &= e^{-\int_{s_1}^s [\lambda(t) + \rho] dt} = e^{-\int_{s_1}^{Q(s)} [\lambda(t) + \rho] dt} \\ &= e^{-\rho[Q(s) - s_1] - \int_{s_1}^{Q(s)} \lambda(t) dt} \\ &= \bar{F}(Q(s) | s_1) e^{-\rho[Q(s) - s_1]} \end{aligned} \quad (34)$$

Differentiating both sides of (33) with respect to  $s$  and using the fact that  $\lambda(\cdot)$  is non-decreasing, one obtains

$$\frac{\partial Q(s)}{\partial s} = \frac{\lambda(s) + \rho}{\lambda(Q(s)) + \rho} \geq 1 \quad (35)$$

Let  $d_1 = Q(l(V)) - s_1$ . From (32), (34), and (35), since  $G_1(V)(\cdot)$ ,  $\lambda(\cdot)$  and  $c_r(\cdot)$  are non-decreasing on  $[0, l(V)]$ , and with the change of variable  $t = Q(s)$ , one obtains

$$\begin{aligned} T(V)(s_2) &\geq \int_{s_2}^{s_2 + d_2} e^{-\rho(Q(s) - s_1)} \bar{F}(Q(s) | s_1) \{ G_1(V)(Q(s)) \\ &\quad + \lambda(Q(s)) [c_r(Q(s)) + V(0)] \} \{ [\lambda(s) + \rho] / [\lambda(Q(s)) + \rho] \} ds \\ &\quad + e^{-\rho d_1} \bar{F}(s_1 + d_1 | s_1) [c_r(s_1 + d_1) + V(0)] \end{aligned}$$

$$\begin{aligned}
&= \int_{s_1}^{s_1+d_1} e^{-\rho(t-s_1)} \bar{F}(t|s_1) \{G_1(V)(t) + \lambda(t)[c_r(t) + V(0)]\} dt \\
&\quad + e^{-\rho d_1} \bar{F}(s_1+d_1|s_1) [c_r(s_1+d_1) + V(0)] \\
&= J(V)(s_1, d_1) \\
&\geq T(V)(s_1).
\end{aligned} \tag{36}$$

Thus, for each  $V$  in  $B_+$ ,  $T(V)$  is in  $B_+$ , and since  $B_+$  is closed,  $V_*(\cdot) = \lim_{n \rightarrow \infty} T^n(V)(\cdot)$  is also in  $B_+$ . Taking  $d_* = l(V_*)$ , one then has

$$J(V_*)(s, \max(0, d_* - s)) = \inf_{0 \leq d \leq \infty} J(V_*)(s, d) \tag{37}$$

for all  $s$  in  $S$ , and from Proposition 1 (iii), this completes the proof. ■

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