

FITTING A NORMAL COPULA FOR A MULTIVARIATE DISTRIBUTION WITH BOTH DISCRETE AND CONTINUOUS MARGINALS

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ABSTRACT

We consider a multivariate distribution with both discrete and continuous marginals, for which the dependence is modeled by a normal copula (sometimes called the NORTA method), and provide an algorithm for fitting the copula in that situation. The fitting is done by matching (approximately) either the rank correlations or the product moment correlations for all pairs of marginals. Numerical illustrations are provided.

1 INTRODUCTION

Random vectors are essential to capture the dependence between various quantities in stochastic models. Capturing this dependence is often even more important than picking the correct marginal distributions. The usual way of defining a multivariate distribution is to first specify the marginal distributions, and then select a copula that determines the dependence structure between these marginals (Joe 1997, Nelsen 1999, Hörmann, Leydold, and Derflinger 2004).

A popular and convenient type of copula, especially when the dimension is large, is the normal copula, which depends only on the pairwise rank correlations between the marginals when these marginal are continuous (Mardia 1970). Modeling dependence by a normal copula is also known as the NORTA (*normal to anything*) method (Cario and Nelson 1997, Ghosh and Henderson 2003). It consists in specifying a multinormal distribution with given correlation matrix, where each marginal is standard normal (with mean 0 and variance 1). To generate a random vector with arbitrary marginal distributions, one generate a vector from this multinormal distribution, then transform each coordinate by applying first the standard normal distribution function, and then the inverse distribution function of the desired marginal. All the dependence is determined by the correlation matrix of the multinormal.

This correlation matrix is usually specified by specifying a target rank correlation for each pair of marginals (e.g., to match the sample rank correlation observed in some data set), and then finding the correlation between the corresponding pair of standard normals that would produce that target rank correlation for the marginals. We refer to the latter as the *pairwise correlation matching problem*. Solving this problem for each pair of marginals provides a proposal correlation matrix for the standard normal distribution. It often happens, particularly when the dimension is large, that this proposal matrix is not positive semi-definite (Ghosh and Henderson 2003), but in that case one can compute the “closest” positive semidefinite matrix (for a given norm on matrices) by solving an optimization problem that minimizes the distance to the proposal (Ghosh and Henderson 2003, Lurie and Goldberg 1998). Typically, this distance turns out to be small.

Our concern in this paper is to solve the pairwise correlation matching problem. When the two marginals are continuous, the solution is straightforward, because a simple analytic formula gives directly the correlation ρ between the standard normals as a function of the desired rank correlation r . We have $\rho = 2\sin(\pi r/6)$. But no direct formula is available if one (or both) marginal is discrete. The case where both marginals are discrete was studied by Avramidis, Channouf, and L'Ecuyer (2009), who proposed efficient approximation algorithms to match either the rank or the linear correlations. But their methods do not cover the case where one marginal is discrete and the other one is continuous. The aim of the present paper is to fill this gap.

Our proposed method builds on similar ideas as in Avramidis, Channouf, and L'Ecuyer (2009), with adaptations, and we end up with a different and simpler algorithm. In particular, the method involves the computation of the bivariate normal

distribution, which requires a double integration in [Avramidis, Channouf, and L'Ecuyer \(2009\)](#). But here we transform the two-dimensional integral to a one-dimensional one before solving it numerically. Moreover, whereas [Avramidis, Channouf, and L'Ecuyer \(2009\)](#) have double infinite sums to approximate, here we only have simple sums to handle.

Instead of specifying target rank correlations, one can alternatively specify the usual linear (product-moment) correlations between the marginals. These correlations do depend on the marginals and have other drawbacks discussed in [Hörmann, Leydold, and Derflinger \(2004\)](#) and [Avramidis, Channouf, and L'Ecuyer \(2009\)](#). Our recommendation is to use the rank correlations, which are generally preferable and easier to handle. But our methodology nevertheless covers the product-moment correlations as well.

The remainder is organized as follow. In Section 2, we recall basic facts and definitions on normal copulas and the correlation matching problem, for both the rank and linear correlations. In Section 3 we present our method for the mixed case. We give numerical examples in Section 4 and we conclude in Section 5.

2 THE NORMAL COPULA

2.1 Definition and the Matching Problem

We want to specify the probability distribution of a random vector $\mathbf{X} = (X_1, \dots, X_d)^t$ (we use column vectors and “t” means “transposed”) by specifying (i) the marginal distribution of each X_j and (ii) the dependence structure between them. There is an extremely rich variety of copulas that can be used to determine this dependence structure ([Joe 1997](#), [Nelsen 1999](#)). In this paper, we restrict ourselves to the normal copula, whose main advantage is its simplicity and its ability to match (exactly or approximately) the pairwise correlations.

Let F_j be the distribution function of X_j , defined by $F_j(x) = \mathbb{P}[X_j \leq x]$. Recall that $F_j(X_j)$ has the uniform distribution over $(0, 1)$ when X_j has a continuous distribution (F_j is continuous), but not necessarily when X_j is discrete. For $1 \leq i, j \leq d$, the product moment (linear) *correlation coefficient* between X_i and X_j is defined by

$$\rho_{i,j}^X = \text{Corr}(X_i, X_j) = \frac{\mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j]}{(\text{Var}[X_i]\text{Var}[X_j])^{1/2}}$$

and the *rank correlation* (also called the *Spearman's ρ*) is

$$r_{i,j}^X = \text{Corr}(F_i(X_i), F_j(X_j)).$$

A normal copula is defined by specifying a correlation matrix \mathbf{R}_ρ^Z with elements $\rho_{i,j}^Z = \text{Corr}(Z_i, Z_j)$, $1 \leq i, j \leq d$, for a multinormal vector $\mathbf{Z} = (Z_1, \dots, Z_d)^t$ with standard normal marginals. We then define

$$X_i = F_i^{-1}(\Phi(Z_i)), \quad i = 1, \dots, d, \tag{1}$$

where Φ is the standard normal cumulative distribution function and $F_i^{-1}(u) = \inf\{x : F_i(x) \geq u\}$. To generate a realization of \mathbf{X} , it suffices to generate a realization of \mathbf{Z} and to compute \mathbf{X} via (1). The multinormal vector \mathbf{Z} can be generated in the standard way: decompose \mathbf{R}_ρ^Z as $\mathbf{L}\mathbf{L}^t$ for some matrix \mathbf{L} , generate a vector \mathbf{W} of d independent standard normal random variables, and put $\mathbf{Z} = \mathbf{L}\mathbf{W}$. The correlation matrix \mathbf{R}_ρ^Z must be symmetric and its diagonal elements must be ones, so there are $d(d-1)/2$ elements to specify. The pairwise correlation matching problem studied in this paper consists in finding, for each pair $1 \leq i < j \leq d$, the value of $\rho_{i,j}^Z$ that produces a given target rank correlation $r_{i,j}^X$ or a given product-moment correlation $\rho_{i,j}^X$.

In the case where X_i has a continuous distribution, $F_i(X_i) = \Phi(Z_i)$ has the uniform distribution over $(0, 1)$. If both X_i and X_j have continuous distributions, then $r_{i,j}^X$ is the same as the rank correlation between Z_i and Z_j , and the matching problem is solved by the simple formula $\rho_{i,j}^Z = 2 \sin(\pi r_{i,j}^X / 6)$. In the remainder, we focus on the case where some X_i has a discrete distribution, in which case $F_i(F_i^{-1})$ is not the identity function. We consider a single pair (i, j) at a time. When $d > 2$, this pairwise correlation matching problem must be solved for all $d(d-1)/2$ pairs (i, j) with $i < j$.

2.2 Matching the Rank or the Product-Moment Correlations

The rank correlation $r_{i,j}^X$ satisfies

$$\begin{aligned} r_{i,j}^X &= \text{Corr}(F_i(X_i), F_j(X_j)) = \text{Corr}(F_i(F_i^{-1}(\Phi(Z_i))), F_j(F_j^{-1}(\Phi(Z_j)))) \\ &= \frac{\mathbb{E}\left[F_i(F_i^{-1}(\Phi(Z_i)))F_j(F_j^{-1}(\Phi(Z_j)))\right] - \mu_{F_i}\mu_{F_j}}{\sigma_{F_i}\sigma_{F_j}} = \frac{g_r(\rho_{i,j}^Z) - \mu_{F_i}\mu_{F_j}}{\sigma_{F_i}\sigma_{F_j}}, \end{aligned} \quad (2)$$

where

$$g_r(\rho_{i,j}^Z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_i(F_i^{-1}(\Phi(z_i)))F_j(F_j^{-1}(\Phi(z_j)))\phi(z_i, z_j, \rho_{i,j}^Z)dz_idz_j, \quad (3)$$

for $1 \leq i < j \leq d$, μ_{F_i} and $\sigma_{F_i}^2$ are the mean and variance of $F_i(X_i)$, and $\phi(\cdot, \cdot, \rho)$ is the density of the bivariate standard normal distribution with correlation ρ . Rearranging (2), we see that the correlation matching problem amounts to finding a root $\rho = \rho_{i,j}^Z$ of the equation:

$$g_r(\rho) - \mu_{F_i}\mu_{F_j} - r_{i,j}^X\sigma_{F_i}\sigma_{F_j} = 0, \quad (4)$$

for $1 \leq i < j \leq d$.

For the product-moment correlation $\rho_{i,j}^X$, we have

$$\begin{aligned} \rho_{i,j}^X &= \text{Corr}(X_i, X_j) = \text{Corr}(F_i^{-1}(\Phi(Z_i)), F_j^{-1}(\Phi(Z_j))) \\ &= \frac{\mathbb{E}\left[F_i^{-1}(\Phi(Z_i))F_j^{-1}(\Phi(Z_j))\right] - \mu_i\mu_j}{\sigma_i\sigma_j} = \frac{g_l(\rho_{i,j}^Z) - \mu_i\mu_j}{\sigma_i\sigma_j}, \end{aligned} \quad (5)$$

where

$$g_l(\rho_{i,j}^Z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_i^{-1}(\Phi(z_i))F_j^{-1}(\Phi(z_j))\phi(z_i, z_j, \rho_{i,j}^Z)dz_idz_j, \quad (6)$$

where μ_i and σ_i^2 are the mean and variance of X_i . The corresponding correlation matching problem amounts to finding a root $\rho = \rho_{i,j}^Z$ of

$$g_l(\rho) - \mu_i\mu_j - \rho_{i,j}^X\sigma_i\sigma_j = 0, \quad (7)$$

for $1 \leq i < j \leq d$.

[Avramidis, Channouf, and L'Ecuyer \(2009\)](#) provide efficient algorithms for the root-finding problems (4) and (7) when both marginal are discrete. In the next section, we provide an algorithm for the situation where one marginal is discrete and the other is continuous.

3 FITTING IN THE CASE OF DISCRETE AND CONTINUOUS MARGINALS

Here we assume that X_i has a continuous distribution over the real line and that X_j has a discrete distribution over the integers. We develop algorithms for solving (4) and (7) for this case. The method differs from that of [Avramidis, Channouf, and L'Ecuyer \(2009\)](#), although it is based on similar ideas.

Using the fact that

$$\phi(z_i, z_j, \rho_{i,j}^Z) = \phi(z_i)\phi\left(\frac{z_j - \rho_{i,j}^Z z_i}{1 - (\rho_{i,j}^Z)^2}\right)(1 - (\rho_{i,j}^Z)^2)^{-1/2}$$

where ϕ is the density of the standard normal, we can write (3) as

$$\begin{aligned}
g_r(\rho_{i,j}^Z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_i(F_i^{-1}(\Phi(z_i))) F_j(F_j^{-1}(\Phi(z_j))) \phi(z_i, z_j, \rho_{i,j}^Z) dz_i dz_j \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(z_i) F_j(F_j^{-1}(\Phi(z_j))) \phi(z_i) \phi\left(\frac{z_j - \rho_{i,j}^Z z_i}{\sqrt{1 - (\rho_{i,j}^Z)^2}}\right) (1 - (\rho_{i,j}^Z)^2)^{-1/2} dz_i dz_j \\
&= \int_{-\infty}^{\infty} \frac{\Phi(z_i) \phi(z_i)}{\sqrt{1 - (\rho_{i,j}^Z)^2}} \left[\int_{-\infty}^{\infty} F_j(F_j^{-1}(\Phi(z_j))) \phi\left(\frac{z_j - \rho_{i,j}^Z z_i}{\sqrt{1 - (\rho_{i,j}^Z)^2}}\right) dz_j \right] dz_i \\
&= \int_{-\infty}^{\infty} \frac{\Phi(z_i) \phi(z_i)}{\sqrt{1 - (\rho_{i,j}^Z)^2}} \left[\sum_{l=-\infty}^{\infty} f_{j,l} \int_{z_{j,l-1}}^{z_{j,l}} \phi\left(\frac{z_j - \rho_{i,j}^Z z_i}{\sqrt{1 - (\rho_{i,j}^Z)^2}}\right) dz_j \right] dz_i \\
&= \sum_{l=-\infty}^{\infty} f_{j,l} \left[\int_{-\infty}^{\infty} \Phi(z_i) \phi(z_i) \left(\Phi\left(\frac{z_{j,l} - \rho_{i,j}^Z z_i}{\sqrt{1 - (\rho_{i,j}^Z)^2}}\right) - \Phi\left(\frac{z_{j,l-1} - \rho_{i,j}^Z z_i}{\sqrt{1 - (\rho_{i,j}^Z)^2}}\right) \right) dz_i \right], \tag{8}
\end{aligned}$$

where $f_{j,l} = F_j(l)$ and $z_{j,l} = \Phi^{-1}(f_{j,l})$. Here we have transformed the integral involving the discrete distribution of X_j to a sum of terms where each term is a constant multiplied by an area under the normal density. Note that in [Avramidis, Channouf, and L'Ecuyer \(2009\)](#), we had to apply the same transformation with a bivariate normal density, because both marginals were discrete. We now apply the change of variable $u_i = \Phi(z_i)$ to obtain an integral over the interval $(0, 1)$:

$$g_r(\rho_{i,j}^Z) = \sum_{l=-\infty}^{\infty} f_{j,l} \left[\int_0^1 u_i \left(\Phi\left(\frac{z_{j,l} - \rho_{i,j}^Z \Phi^{-1}(u_i)}{\sqrt{1 - (\rho_{i,j}^Z)^2}}\right) - \Phi\left(\frac{z_{j,l-1} - \rho_{i,j}^Z \Phi^{-1}(u_i)}{\sqrt{1 - (\rho_{i,j}^Z)^2}}\right) \right) du_i \right]. \tag{9}$$

To find a root of (4) with g_r given in (9), as in the algorithm NII of [Avramidis, Channouf, and L'Ecuyer \(2009\)](#), we use the procedure `zero` of [Brent \(1971\)](#), also described in [Press, Teukolsky, Vetterling, and Flannery \(1992\)](#), and called Brent's method. It combines root bracketing, bisection, and inverse quadratic interpolation. To find the root of a function f , at each step of the algorithm, we have an interval $[a, b]$ that contains the root, so $f(a)$ and $f(b)$ have opposite signs. The algorithm then selects another point c in the interval, either by bisection, or by inverse quadratic interpolation at steps where this is safe. In our case, we are looking for a root of the function

$$f(\rho) = g_r(\rho) - \mu_{F_i} \mu_{F_j} - r_{i,j}^X \sigma_{F_i} \sigma_{F_j}, \tag{10}$$

with g_r given in (9). We know that the solution is in the interval $[-1, 0]$ when the target correlation is negative, and in $[0, 1]$ when it is positive, so we start with one of those intervals. The algorithm stops when it finds a number that is guaranteed to be at a distance smaller than ε from the true root, for a given (selected) tolerance $\varepsilon > 0$. We compute the integral in (9) numerically, which is not problematic because the integrand is fairly smooth as a function of u_i . In our implementation, we took the Gauss-Legendre quadrature, using the numerical integration classes from the Java library of [Flanagan \(2007\)](#). The sum in (9) is truncated to some index ℓ_0 such that $\mathbb{P}[|X_j| > \ell_0] < \varepsilon_1$. In our implementation, we took $\varepsilon_1 = 10^{-6}$. The usual discrete distributions are over the non-negative integers; then we only need to truncate on the right. The correlation-matching procedure for an arbitrary pair (i, j) is summarized in Algorithm 1.

Algorithm 1 Pairwise correlation-matching algorithm.

// This algorithm takes as input the marginal distributions F_i and F_j of X_i and X_j where F_i is continuous and F_j is discrete, their rank correlation $r_{i,j}^X$, the maximum number of iterations N_{\max} , and the desired tolerance ε .

It returns the product moment correlation $\rho = \rho_{i,j}^Z$ that solves (approximately) (10).

1. **Initialize** the search interval: $[a, b] = [0, 1]$ if $r_{i,j}^X > 0$, and $[a, b] = [-1, 0]$ otherwise;
 2. Apply Brent's method to find a root of (10), as described in [Press, Teukolsky, Vetterling, and Flannery \(1992\)](#);
 3. **Return** $\rho_{i,j}^Z$.
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Table 1: Results for the rank correlations.

| | ε | $r_{1,2}^X$ | $\rho_{1,2}^Z$ | CPU (s) | N_1 | $r_{1,2}^X(\rho_{1,2}^Z)$ | Rel. error (%) |
|---|---------------|-------------|----------------|---------|-------|---------------------------|----------------|
| X_1 Exponential $\lambda_1 = 25$ X_2 Binomial $n_2 = 100$ $p_2 = 0.65$ | 10^{-2} | -0.50 | -0.5189 | 0.0033 | 4 | -0.4995 | 0.106 |
| | | 0.05 | 0.0503 | 0.0023 | 3 | 0.0479 | 4.293 |
| | | 0.20 | 0.2099 | 0.0032 | 4 | 0.1999 | 0.025 |
| | | 0.90 | 0.9110 | 0.0023 | 3 | 0.9010 | 0.109 |
| | | 0.98 | 0.9866 | 0.0023 | 3 | 0.9827 | 0.275 |
| | 10^{-4} | -0.50 | -0.5194 | 0.0042 | 5 | -0.5000 | < 0.001 |
| | | 0.05 | 0.0526 | 0.0032 | 4 | 0.0500 | < 0.001 |
| | | 0.20 | 0.2099 | 0.0041 | 5 | 0.2000 | < 0.001 |
| | | 0.90 | 0.9101 | 0.0041 | 5 | 0.9000 | < 0.001 |
| | | 0.98 | 0.9841 | 0.0041 | 5 | 0.9800 | < 0.001 |
| X_1 Gamma $\alpha_1 = 20$ $\beta_1 = 30$ X_2 Negative bin. $s_2 = 60.21$ $p_2 = 0.611$ | 10^{-2} | -0.50 | -0.5183 | 0.0036 | 4 | -0.4997 | 0.053 |
| | | 0.05 | 0.0513 | 0.0026 | 3 | 0.0489 | 2.300 |
| | | 0.20 | 0.2102 | 0.0026 | 3 | 0.2006 | 0.280 |
| | | 0.90 | 0.9088 | 0.0047 | 5 | 0.9000 | < 0.001 |
| | | 0.96 | 0.9633 | 0.0047 | 5 | 0.9587 | 0.135 |
| | 10^{-4} | -0.50 | -0.5186 | 0.0046 | 5 | -0.5000 | < 0.001 |
| | | 0.05 | 0.0525 | 0.0036 | 4 | 0.0500 | < 0.001 |
| | | 0.20 | 0.2096 | 0.0036 | 4 | 0.2000 | < 0.001 |
| | | 0.90 | 0.9088 | 0.0047 | 5 | 0.9000 | < 0.001 |
| | | 0.96 | 0.9644 | 0.0058 | 6 | 0.9600 | < 0.001 |
| X_1 Normal $\mu_1 = 25$ $\sigma_1 = 10$ X_2 Poisson $\lambda = 25$ | 10^{-2} | -0.50 | -0.5185 | 0.0024 | 4 | -0.4992 | 0.163 |
| | | 0.05 | 0.0507 | 0.0017 | 3 | 0.0482 | 3.657 |
| | | 0.20 | 0.2098 | 0.0024 | 4 | 0.2000 | < 0.001 |
| | | 0.90 | 0.9118 | 0.0018 | 3 | 0.9018 | 0.201 |
| | | 0.98 | 0.9836 | 0.0024 | 4 | 0.9794 | 0.058 |
| | 10^{-4} | -0.50 | -0.5193 | 0.0030 | 5 | -0.5000 | < 0.001 |
| | | 0.05 | 0.0526 | 0.0024 | 4 | 0.0500 | < 0.001 |
| | | 0.20 | 0.2099 | 0.0024 | 4 | 0.2000 | < 0.001 |
| | | 0.90 | 0.9101 | 0.0031 | 5 | 0.9000 | < 0.001 |
| | | 0.98 | 0.9841 | 0.0039 | 6 | 0.9800 | < 0.001 |

For the product moment correlations $\rho_{i,j}^X$, $1 \leq i < j \leq d$, we obtain a similar formula by transforming Equation (6):

$$\begin{aligned}
 g_l(\rho_{i,j}^Z) &= \sum_{l=-\infty}^{\infty} y_{j,l} \left[\int_{-\infty}^{\infty} F_i^{-1}(\Phi(z_i)) \phi(z_i) \left(\Phi \left(\frac{z_{j,l} - \rho_{i,j}^Z z_i}{\sqrt{1 - (\rho_{i,j}^Z)^2}} \right) - \Phi \left(\frac{z_{j,l-1} - \rho_{i,j}^Z z_i}{\sqrt{1 - (\rho_{i,j}^Z)^2}} \right) \right) dz_i \right] \\
 &= \sum_{l=-\infty}^{\infty} y_{j,l} \left[\int_0^1 F_i^{-1}(u_i) \left(\Phi \left(\frac{z_{j,l} - \rho_{i,j}^Z \Phi^{-1}(u_i)}{\sqrt{1 - (\rho_{i,j}^Z)^2}} \right) - \Phi \left(\frac{z_{j,l-1} - \rho_{i,j}^Z \Phi^{-1}(u_i)}{\sqrt{1 - (\rho_{i,j}^Z)^2}} \right) \right) du_i \right], \quad (11)
 \end{aligned}$$

where $y_{j,l} = \mathbb{P}[X_j = l]$. To find a root of (7), with g_l given in (11), we use a similar algorithm as for (4).

4 NUMERICAL EXAMPLES

We report the performance of our algorithm for some numerical examples. We consider three pairs of distributions (three cases), as follows:

1. X_1 is exponential with mean $\lambda_1 = 25$ and X_2 is binomial with parameters $n_2 = 100$ and $p_2 = 0.5$.

Table 2: Results for the product moment correlations.

| | ϵ | $\rho_{1,2}^X$ | $\rho_{1,2}^Z$ | CPU (s) | N_1 | $r_{1,2}^X(\rho_{1,2}^Z)$ | Rel. error (%) |
|---|------------|----------------|----------------|---------|-------|---------------------------|----------------|
| X_1 Exponential $\lambda_1 = 25$ X_2 Binomial $n_2 = 100$ $p_2 = 0.5$ | 10^{-2} | -0.50 | -0.5365 | 0.0038 | 4 | -0.5002 | 0.044 |
| | | 0.05 | 0.0824 | 0.0049 | 5 | 0.0500 | < 0.001 |
| | | 0.20 | 0.2531 | 0.0038 | 4 | 0.2018 | 0.886 |
| | | 0.50 | 0.5884 | 0.0038 | 4 | 0.5000 | < 0.001 |
| | | 0.80 | 0.9277 | 0.0028 | 3 | 0.8017 | 0.211 |
| | 10^{-4} | -0.50 | -0.5362 | 0.0049 | 5 | -0.5000 | < 0.001 |
| | | 0.05 | 0.0824 | 0.0049 | 5 | 0.0500 | < 0.001 |
| | | 0.20 | 0.2511 | 0.0049 | 5 | 0.2000 | < 0.001 |
| | | 0.50 | 0.5885 | 0.0038 | 4 | 0.5000 | < 0.001 |
| | | 0.80 | 0.9258 | 0.0039 | 4 | 0.8000 | < 0.001 |
| X_1 Gamma $\alpha_1 = 20$ $\beta_1 = 30$ X_2 Negative bin. $s_2 = 60.21$ $p_2 = 0.611$ | 10^{-2} | -0.50 | -0.5067 | 0.0343 | 4 | -0.4998 | 0.044 |
| | | 0.05 | 0.0528 | 0.0253 | 3 | 0.0489 | 2.264 |
| | | 0.20 | 0.2062 | 0.0255 | 3 | 0.1998 | 0.101 |
| | | 0.50 | 0.5098 | 0.0341 | 4 | 0.4992 | 0.167 |
| | | 0.90 | 0.9149 | 0.0430 | 5 | 0.9000 | < 0.001 |
| | 10^{-4} | -0.50 | -0.5070 | 0.0424 | 5 | -0.5000 | < 0.001 |
| | | 0.05 | 0.0540 | 0.0342 | 4 | 0.0500 | < 0.001 |
| | | 0.20 | 0.2064 | 0.0428 | 5 | 0.2000 | < 0.001 |
| | | 0.50 | 0.5106 | 0.0428 | 5 | 0.5000 | < 0.001 |
| | | 0.90 | 0.9149 | 0.0428 | 5 | 0.9000 | < 0.001 |
| X_1 Normal $\mu_1 = 25$ $\sigma_1 = 10$ X_2 Poisson $\lambda = 25$ | 10^{-2} | -0.50 | -0.5080 | 0.0019 | 3 | -0.5024 | 0.487 |
| | | 0.05 | 0.0502 | 0.0019 | 3 | 0.0496 | 0.724 |
| | | 0.20 | 0.2009 | 0.0019 | 3 | 0.1986 | 0.704 |
| | | 0.90 | 0.9104 | 0.0027 | 4 | 0.8995 | 0.053 |
| | | 0.98 | 0.9896 | 0.0027 | 4 | 0.9777 | 0.236 |
| | 10^{-4} | -0.50 | -0.5055 | 0.0026 | 4 | -0.5000 | < 0.001 |
| | | 0.05 | 0.0506 | 0.0027 | 4 | 0.0500 | < 0.001 |
| | | 0.20 | 0.2024 | 0.0026 | 4 | 0.2000 | < 0.001 |
| | | 0.90 | 0.9109 | 0.0034 | 5 | 0.9000 | < 0.001 |
| | | 0.98 | 0.9919 | 0.0041 | 6 | 0.9800 | < 0.001 |

- X_1 is gamma with shape parameter $\alpha_1 = 20$ and scale parameter $\beta_1 = 30$ (so the mean is $\alpha_1\beta_1$), and X_2 is negative binomial with parameters $s_2 = 60.21$ and $p_2 = 0.611$.
- X_1 is normal with mean $\mu_1 = 25$ and variance $\sigma_1^2 = 100$, and X_2 is Poisson with parameter (mean) $\lambda_2 = 25$.

We tested the algorithm for both the rank and the product moment correlations, with several target values of $r_{1,2}^X$ and $\rho_{1,2}^X$, in the interval $(-1, 1)$, that provide a reasonable coverage of the different situations that might occur. We considered two choices of tolerances for the root-finding algorithm based on Brent's method: $\epsilon = 10^{-2}$ and $\epsilon = 10^{-4}$. In each case, we also computed the average CPU time over 10^4 replications, the number N_1 of iterations (that is, the number of evaluations of the function g_r or g_l), and the relative error. The results are in Tables 1 and 2.

The computing time is less than 5 milliseconds in almost all cases, except for the combination of gamma and negative binomial distributions when we specify the product moment correlations, which is about 10 times slower with the smallest tolerance. These times are faster than those reported by [Avramidis, Channouf, and L'Ecuyer \(2009\)](#) for the NI1 algorithm, by a factor of 10 or more for the similar examples.. When the tolerance is reduced from $\epsilon = 10^{-2}$ to $\epsilon = 10^{-4}$, N_1 increases by only 1 or 2, and the CPU times increase only marginally. The largest relative error is observed with a target rank correlation of 0.05 and $\epsilon = 10^{-2}$; in this case the target rank correlation is close to zero, so the "large" relative error corresponds to a small absolute error.

5 CONCLUSION

We proposed an efficient correlation matching algorithm for a normal copula in the case of a mixture of discrete and continuous marginal distributions. The method is faster than the algorithms proposed in Avramidis, Channouf, and L'Ecuyer (2009) for the case where both marginals are discrete.

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