

# Introduction to randomized quasi-Monte Carlo methods in simulation

Abstract, outline, and references

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## Abstract

Randomized quasi-Monte Carlo (RQMC) provides unbiased estimators whose variance converges at a faster rate than standard Monte Carlo when estimating an integral, under appropriate conditions. Variants of RQMC have been designed recently for the simulation of Markov chains, for function approximation and optimization, for density estimation, for solving partial differential equations, etc. In this tutorial, we will review the basic principles and main results on RQMC, discuss their practical aspects, and give numerical illustrations showing that they can reduce the variance by huge factors. We will look at how RQMC point sets are constructed, how we measure their uniformity, why they can work even for high-dimensional integrals, and how can they work when simulating Markov chains over a large number of steps. We will show how these methods can be applied in practice by using a Java software library that supports RQMC.

# Outline

- ▶ Monte Carlo, quasi-Monte Carlo, randomized quasi-Monte Carlo
- ▶ QMC point sets (lattices, digital nets) and their randomizations
- ▶ Software and numerical illustrations with SSJ
- ▶ Variance bounds and convergence rates
- ▶ Underlying theory: Why and when does it work?
- ▶ Transforming the function to help RQMC
- ▶ Optimizing vs randomizing the point set constructions
- ▶ More numerical examples
- ▶ RQMC for Markov chains
- ▶ RQMC for density estimation
- ▶ QMC/RQMC for function approximation

## Software Tools in Java for QMC/RQMC

For all my experiments with MC, QMC, RQMC, I use the Java library

**SSJ: Stochastic Simulation in Java.**

The official GitHub release is here: <https://github.com/umontreal-simul/ssj>. All the installation instructions are there.

My own working version is at <https://github.com/pierrelecuyer/ssj>. It is more recent and has some additional material, but some parts are under construction and not fully documented.

Personally, I use Eclipse (<https://www.eclipse.org/>) with Maven for Java and SSJ.

A tutorial on SSJ with commented examples:

<http://umontreal-simul.github.io/ssj/docs/master/examples.html>.

## Other QMC/RQMC Software

**LatNet Builder:** A tool to search for good parameters for QMC/RQMC point set constructions: <https://github.com/umontreal-simul/latnetbuilder>.

**qmcpy:** QMC software in Python: <https://pypi.org/project/qmcpy/> and <https://qmcsoftware.github.io/QMCSoftware/>.

### **QMC in MATLAB:**

<https://in.mathworks.com/help/stats/generating-quasi-random-numbers.html>.

**R package qrng:** <https://cran.r-project.org/web/packages/qrng/qrng.pdf>.

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