

The Classification of Leaves on The Continuous Static Optimal Model

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2012 Mathematical Contest in Modeling (MCM) Summary Sheet

(Attach a copy of this page to each copy of your solution paper.)

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We build up two basic models: the first to find out an optimal number of leaves on a tiny spot and the second to derive a function for the leaf area to tree profile and leaf distribution. In the extended model, we analyze the possible influence of other environment's factors.

There are two primary assumptions: the tree could arrange the distribution of leaves in order to maximize the net absorption of energy and the outline of leaves and tree are considered regular. We utilize the number of leaves on a spot and the ratio of major axis to minor axis to classify the shape.

We could build up an equation for the net absorption to the number of leaves and the ratio of major axis to minor axis under the assumption. Then we take the radius of the cross-side of branches as another parameter in the function into account. After calculating the area of leaf from a view of tree profile and leaf distribution, we obtain a relationship among the leaf mass and the size characteristics of tree (height, density and etc.)

In the first model, we analyze the effects of the number of leaves and the ratio of leaves, leading to the conclusion that a value of the ratio corresponds to an optimal number which could maximize the net absorption. Furthermore, we test and verify the rationality of the first model by measuring some samples and comparing the analytical and actual results. Likewise, applying the major assumption, we establish the second model which provides us a clear and straightforward equation for the leaf area to tree profile and leaf distribution. The statistics quoted and the figures generated lend a strong support to our model. In the extended model, more factors such as temperature, humidity and light intensity, are taken into account; moreover, we proposed a solution to solve the problem of various density of the leaves' distribution by quantizing the density. Although we omit the influence of respiration and other factors, we establish an optimal system of the plant to reach the maximal net accumulation of energy.

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1 Introduction

The classification of the plants is a very complex issue especially about the shape of leaves, as **Figure 1** presents. Some researchers have proposed many approaches to assess and classify the leaves [1]. The idea about how to determine the shape of the leaf might once swirl at our heads. This classification could enable us to differentiate various species and even facilitate extraction of some plants. In this passage, we take these following problems into account:

- the reasons why leaves have the various shape and how to classify leaves,
- other factors which could influence the characteristics of leaf, such as tree profile, structure, mass, volume.



Figure 1: leaves [2]

In order to establish a reasonable system, we begin the analysis from the view of 2-dimension and evaluate the number of leaves projected by the actual 3-dimensional leaves on a tiny spot based on the optimal assumption of energy absorption and metabolism. And on the basis of these analysis, we use several parameters and modeling to classify leaves and describe the different influences on the shape exerted by tree profile, structure, etc.

2 Assumption

- All leaves grow in the same tree.

- We discuss the situation of the leaf and tree at a certain moment.
- The tree grows in spring, fade away in winter.
- All leaves at a certain moment confront the same environment which is suitable enough to make them healthy.
- The shape of leaves in the same tree are the same and considered as ellipse.
- All leaves on the same spot have the same value of γ .

3 Definition, Terminology and Symbols

Table 1
Model Parameter

Parameter	Meaning
S_h	the gross area covered by leaves in a tiny spot
S_n	the area of a regular n-sided polygon created by connecting the intersection of contiguous ellipses
p, q	the coordinate of the intersection of contiguous ellipses
S_{nl}	the area of the larger regular n-sided polygon created in the situation with a radius in the core of the branch
S_{ns}	the area of the smaller regular n-sided polygon created in the situation with a radius in the core of the branch
S_{el}	the area of each leaf
S_a	the area figured out in Figure 2
S_b	the area figure out in Figure 5
S_c	the area figure out in Figure 5
a	the major semi-axis of ellipse
b	the minor semi-axis of ellipse
n	the number of leaves on a tiny spot
α	coefficient signifying the average accumulation in energy per unit area
β	coefficient signifying the average consumption in energy per unit area
f	net absorption in energy
p, q	the coordinate of the intersection of contiguous ellipses
r	the value of the radius of the branches
γ	β/α
η	a/b

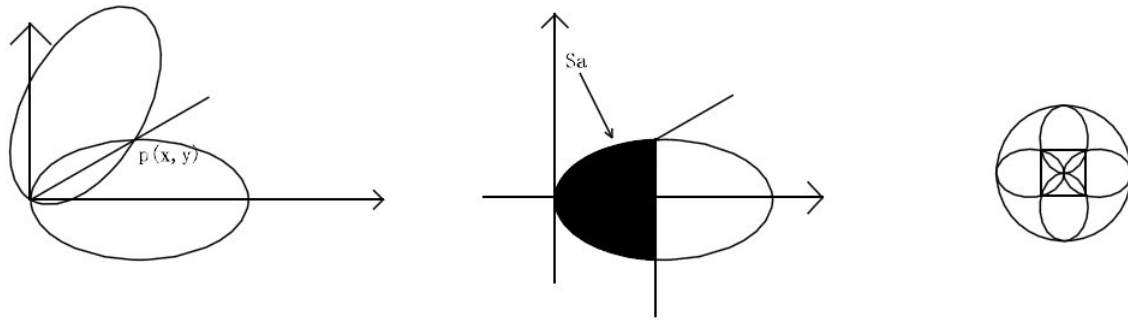


Figure 2: overlapping condition

4 Basic Model

4.1 Analysis of the problem

In order to classify leaves, we focus on two basic parameters (n, η) on a tiny spot. For the first, we hold the notion that the tree arranges the position of leaves which would maximize the absorption of energy. After that, we use two coefficients to present the accumulation and consumption of the energy. If we neglect the variance of different leaves on a tiny spot and consider the shape as ellipse, we could establish a rationale function and make the quantitative measurement available. Then, in order to illustrate the influence of the volume of the branches, we also add the radius of the branch as a parameter into the modeling on a tiny spot. Further up, supposing the profile of tree as a cone, we can establish the relationship among the distribution of leaves, tree profile and the area of leaves. Based on these inferences, we could get a equation which would generally describe the leaf mass on a tree. Back to the classification, with an assumption of energy, we could find a relation of n and η to sort all leaves.

4.2 Step 1: The condition of the overlapping shadows

According to the basic knowledge, we know that the plant's growing adhere to the law of wheel law. Therefore, we focus on the overlapping condition in a tiny spot (which is considered as a circle); the specific relationship of the leaf's position is shown in **Figure 2**; especially a regular n -sided polygon could be established by connecting the intersection of contiguous ellipses. We could first find this following relationship of the intersection's coordinate:

$$\begin{cases} \left(\frac{p-a}{a}\right)^2 + \left(\frac{q}{b}\right)^2 = 1 \\ \frac{q}{p} = \tan\left(\frac{\pi}{n}\right) \end{cases}$$

so that

$$p = \frac{\frac{2}{a}}{\frac{1}{a^2} + \frac{\tan^2 \frac{\pi}{n}}{b^2}}$$

$$q = \tan \frac{\pi}{n} \cdot x$$

on the other hand, as we know

$$S_a = 2 \int_0^x b \cdot \sqrt{1 - \left(\frac{p}{a} - 1\right)^2} dx = ab \cdot \left[\left(\frac{p}{a} - 1\right) \sqrt{1 - \left(\frac{p}{a} - 1\right)^2} + \pi - \arccos\left(\frac{p}{a} - 1\right) \right]$$

$$S_{el} = \pi ab$$

$$S_n = npq$$

furthermore, we have

$$S_h = S_n + n(S_{el} - S_a)$$

From the point of view that the plant could maximize the accumulation of energy, we could take f into consideration and find out the optimal number of leaves on a tiny spot. Combined with α, β , we have

$$f = \alpha S_h - \beta n S_{el} = \alpha (S_h - \gamma n S_{el}) = \alpha [npq + (1 - \gamma)\pi nab - nS_a] \quad (1)$$

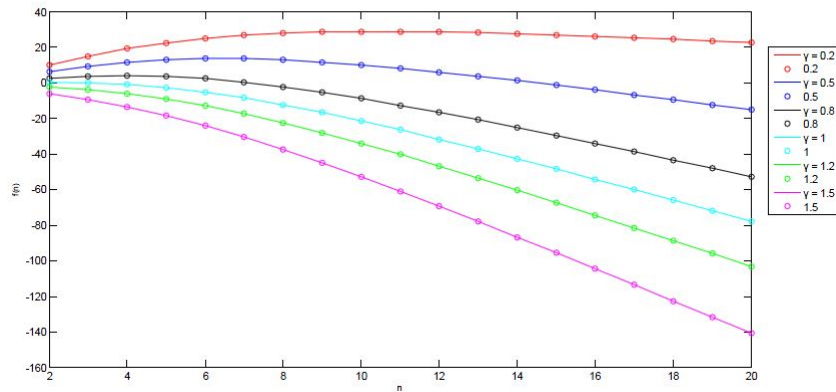


Figure 3: with fixed η and changing γ , the curve of f and n

In order to find out more correlation among f and other parameters such as $n, \frac{a}{b}, \gamma$, In the **Figure 3**, we assume $\eta = 2, b = 1$ and $\alpha = 1$. From any curve in the **Figure 3**, we know that the value of f increases at first and then decrease after one point. It tells us that it exists a optimal value of n to maximize the net absorption of energy on a tiny spot with fixed η and γ . Furthermore, the value of the zenith of the curve decreased with the γ increasing.

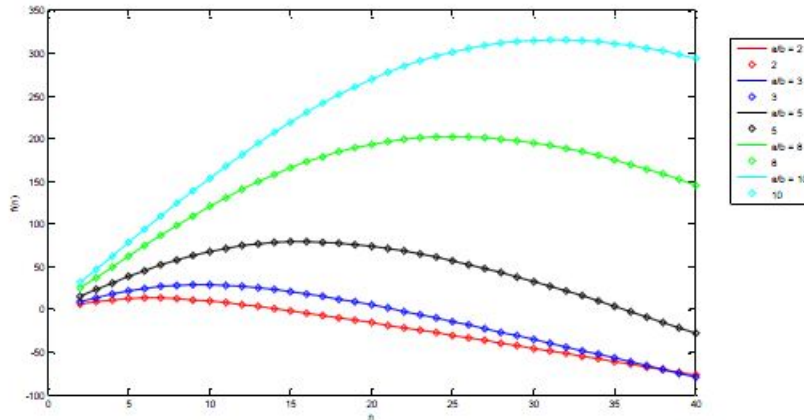


Figure 4: with fixed γ and changing η , the curve of f and n

In the **Figure 4**, we assume $\gamma = 0.5, b = 1$ and $\alpha = 1$ From any curve in the **Figure 4**, we could see that the net absorption also first increase and then decrease with fixed γ and η ; nevertheless, the value of the zenith of the curve increase with η increasing.

4.3 Step 2: the influence of the radius of the branches

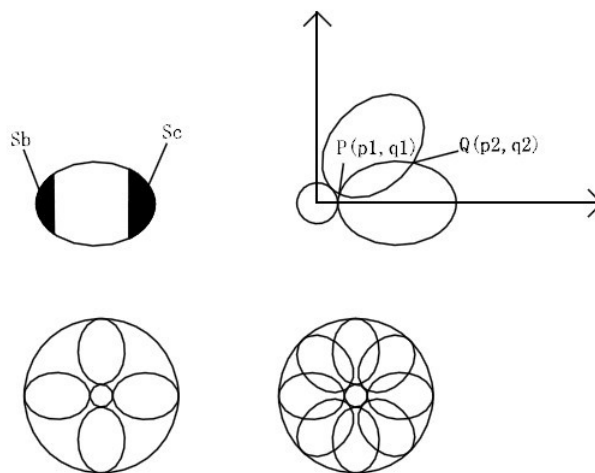


Figure 5: the overlapping condition with a circle in the core

Actually the branches' radius would also influence the distribution of leaves on a tiny spot; in the **Figure 5**, it is obvious that the ellipses which would inter-

sect once could not intersect with a circle in the core. Based on the assumption that the tree would maximize the possible accumulation of energy, we know that more leaves are needed or the area of each leaf would expand. Just like **Step 1**, we also have some correlations about the coordinate and area in the **Figure 5**:

$$\begin{cases} \left(\frac{p-a-r}{a}\right)^2 + \left(\frac{q}{b}\right)^2 = 1 \\ \frac{q}{p} = \tan\left(\frac{\pi}{n}\right) \end{cases}$$

Shown in the **Figure 5**, if the value of the axis of the ellipse doesn't change, the ellipse cross each other at two points, generating two regular n -sided polygons. Thus, we have:

$$S_b = 2 \int_r^{p_1} b \sqrt{1 - \left(\frac{p_1}{a} - \frac{a+r}{a}\right)^2} dx = ab \left(\frac{p_1 - a - r}{a} \sqrt{1 - \left(\frac{p_2 - a - r}{a}\right)^2} - \arccos \frac{p_1 - a - r}{a} + \pi\right)$$

$$S_c = 2 \int_{p_2}^{r+2a} b \sqrt{1 - \left(\frac{p_2}{a} - \frac{a+r}{a}\right)^2} dx = ab \left(-\frac{p_2 - a - r}{a} \sqrt{1 - \left(\frac{p_2 - a - r}{a}\right)^2} + \arccos \frac{p_2 - a - r}{a}\right)$$

And moreover,

$$\begin{aligned} S_{ns} &= np_1q_1 & S_{nl} &= np_2q_2 \\ S_h &= n(S_b + S_c) + S_{nl} - S_{ns} \end{aligned}$$

From this, we get

$$f = \alpha S_h - \beta n S_{el} = \alpha [S_{nl} - S_{ns} - rn S_{el} + n(S_b + S_c)] = \alpha [np_2q_2 - np_1q_1 - rn\pi ab + n(S_b + S_c)]$$

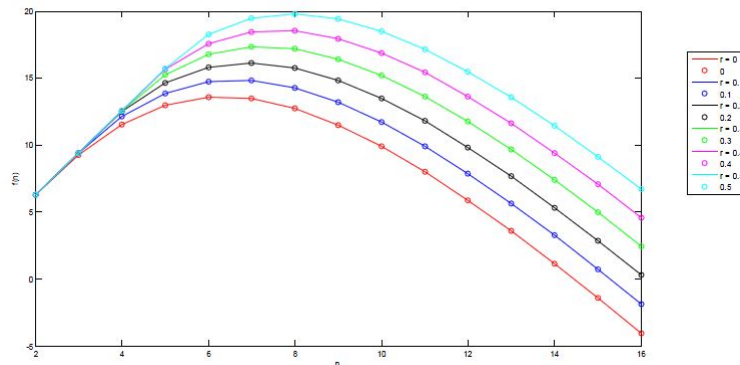


Figure 6: with fixed γ, η and changing r , the curve of f and n

In the **Figure 6**, we assume that $\eta = 2, b = 1$ and $\gamma = 0.5, \alpha = 0.1$ and make out curves to illustrate the relationship between f and n with a fixed r . As

plain as the picture shows, with a fix r , the value of f increase and then decrease; and the value of the zenith of n augments with the growth of r . It is apparently rationale that the number of leaves could increase in order to extend the area to absorb more energy. Especially, all curves converge at the inial point, where the value of n is 2. It is obvious that when η make $n = 2$ suitable to reach the maximum of absorption, the value of r could not exert any influence on the zenith of the curve, for 2 leaves cannot intersect each other in any cases.

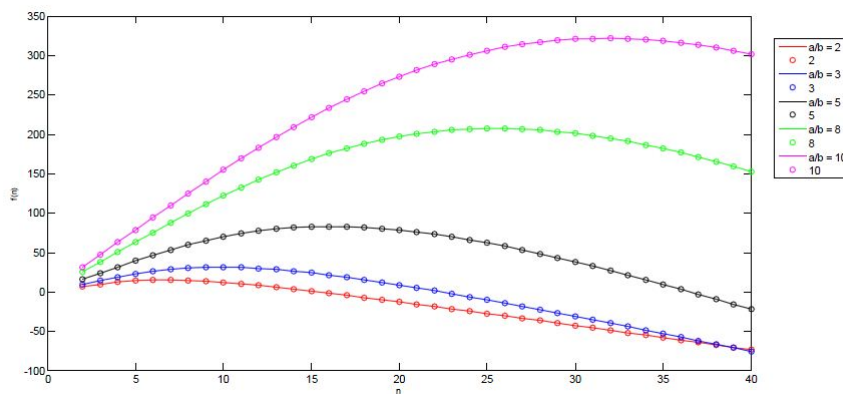


Figure 7: with fixed γ, r and changing η , the curve of f and n

In the **Figure 7**, we assume that $\gamma = 0.5, \alpha = 1$ and $b = 1, r = 0.1$ and make out curves to illustrate the relationship between f and n with a fix η . In any curve, the value of f first increase and then decrease; and as the value of η augments, the value of n in the zenith increase, either.

4.4 Step 3 :The distribution of leaves and tree profile

In this part, we need several additional assumptions and parameters:

- The profile is considered as a cone.
- All the leaves which could absorb energy are located in the surface of a tree.
- The area of the tree's surface is covered by leaves.

First of all, we have a few basic relations:

$$V = \frac{1}{3}\pi r^3 h,$$

$$S_{lateral} = \pi r \sqrt{h_2 + r^2}$$

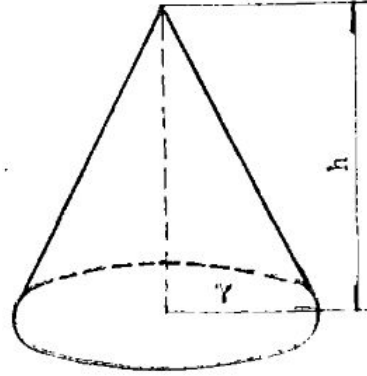


Figure 8: A conical tree [4](Source: Discussion About Using Volume–Weight Means To Estimate The Mass Of Dust On a tree by Caiqin Feng, Yuanping He)

	Additional Parameter
V	the volume of a tree
h	the height of a tree
r	the radius of the underside of a tree
t	$\frac{h}{r}$
N	the number of leaves per unit volume
F	the net accumulation of energy of a tree
$S_{lateral}$	the lateral area of a tree
S_{tot}	the whole area of the leaves in a tree
M_{el}	the leaf mass of a tree
d	the thickness of a leaf
ρ	the density of a leaf

Alike the former steps, we still adhere to the assumption that the tree could raise the accumulation of energy; so the value of F could be maximized.

$$F = \alpha S_{lateral} - \beta S_{tot} = \alpha \pi r \sqrt{h^2 + r^2} - \beta N \cdot \frac{1}{3} \pi r^3 h \cdot \pi a b$$

Especially, we use t as a parameter to indicate the tree profile. Therefore:

$$F(t) = \alpha \pi r^2 \sqrt{t^2 + 1} - \beta N t \cdot \frac{1}{3} \pi r^4 \cdot \pi a b$$

When the F reaches the maximum,

$$F'(t) = \alpha \pi r^2 \cdot \frac{t}{\sqrt{t^2 + 1}} - \beta N \frac{1}{3} \pi r^4 \pi a b = 0$$

Consequently:

$$\frac{t}{\sqrt{t^2 + 1}} = \frac{1}{3} \pi r^2 N a b \cdot \gamma = \frac{1}{3} \gamma r^2 N S_{el}$$

Finally:

$$S_{el} = \frac{3 \frac{t}{\sqrt{t^2+1}}}{\gamma r^2 N} \quad (2)$$

As the equation displays, the area of a leaf depend on the distribution of leaves and tree profile. With a fixed t , if N become bigger, the area of a leaf would be reduced. On the other side, with a fixed N , if t become bigger, the area of a leaf would enlarge.

4.5 Step 4: The correlation between the leaf mass and the size characteristics

Basic equation:

$$M_{el} = S_{el} \cdot N \cdot d \cdot V \rho$$

Combine with (2); we get:

$$M_{el} = \frac{3 \frac{t}{\sqrt{t^2+1}}}{\gamma r^2 N} \cdot N \cdot d V \rho = \frac{t \pi \gamma h \rho}{\sqrt{t^2 + 1} r} \quad (3)$$

According to (3), there is a strong association between the leaf mass and the size characteristics of the tree (height, the radius of the underside, the density). And with a fixed ρ and γ , the raise of h could lead to a quicker ratio of magnification of M_{el} than the raise of r . And with other variables fixed, a reduced γ could result in the diminution of M_{el} .

4.6 Step 5: The classification and explanations of different shapes

Table 3

The correlation of n and η ($\gamma = 0.5$)	
The optimal value of n	The range of η
3	1.00~1.06
4	1.07~1.41
5	1.42~1.73
6	1.74~2.06
7	2.07~2.38
8	2.39~2.70
9	2.71~3.02
10	3.03~3.33

In the **Table 3**, every value of n is related to a range of η . In other words, in the listed range of η , the matching value of n is the optimal number to minimize the overlapping shadows and maximize the net absorption of energy. This table

illustrate a classification of leaves on a tiny spot; with a definite value of η , we could find out a optimal number of leaves on a tiny spot, sorting out different situations of placing leaves.

In addition, the value of γ in various trees or even disparate spots in the same could differ from each other to some degree. Hence, the shape of leaves could vary accordingly to enhance the net absorption of energy.

5 Validating the model

Table 4
The size of some leaves

The value of $2b$	The value of $2a$	The value of η
3.6	5.6	1.56
3.5	7.8	2.23
4.0	6.4	1.60
3.6	6.0	1.67
3.0	4.6	1.53
4.5	6.6	1.47

For the first, we check out the feasibility of the calculation of n . We collect some leaves in a tiny part of *Cinnamomum camphora* and measure the size of the leaves. From this statistic, we know that:

$$\bar{\eta} = \frac{1.56 + 2.23 + 1.60 + 1.67 + 1.53 + 1.47}{6} = 1.66$$

Hence, according to $n = 6$ and $\bar{\eta} = 1.66$ and (1), we could get:

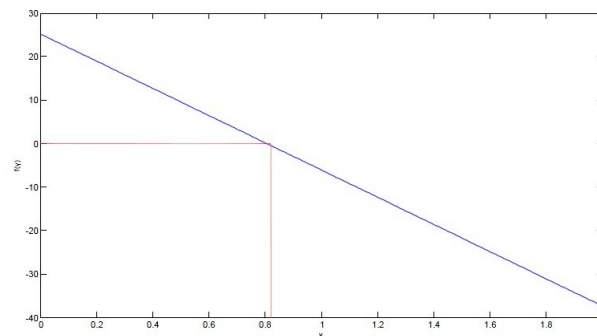


Figure 9: The relation of f and γ

As the picture illuminate, when f reach 0: $\gamma = 0.82$; and this value could reflect the ratio of $S_{lateral}$ to S_{tot} . And after observing some samples of trees, we

estimate that this ratio range from 0.7 to 0.9. Consequently, the model is largely correct.

Table 5
The relation of t and $\eta(b = 1)$

The value of t	The value of η
1.2359	1.8059
1.3154	1.7915
1.0511	1.8325
1.1830	1.8258
1.2118	1.6935
1.0653	1.8352
1.3333	1.8299
1.5132	1.6935
1.0821	1.7883
1.1176	1.8537

Then we check up the second model. From [5], we could get the **Table 5**. From (2) and other relevant parameters, we have:

$$\eta = \frac{3t}{\pi b^2 \gamma N r^2 \sqrt{t^2 + 1}} \quad (4)$$

We can associate (4) with **Table 5**:

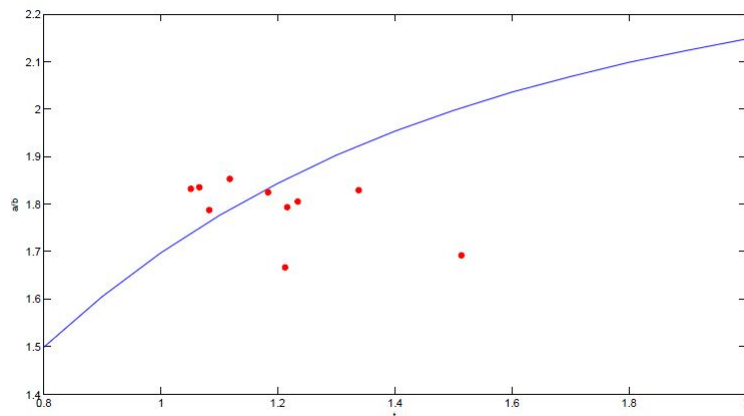


Figure 10: The combination of (4) and Table 5

From the picture, generally our equation accords with the statistics provided by Zhengtao Lv [5]. Therefore, the second model can provide a comparative accurate association between the leaf shape and tree profile.

6 The Extended Model

Actually, the real situation is more complicated than the model we have discussed. We could take more factors into accounts:

1. α, β, γ could be affected by the situation of the environment: such as the temperature T , light intensity I , humidity H . Hence, we could extend the equation of γ :

$$\gamma(T, I, H, \vec{x}) = \frac{\alpha(T, I, H, \vec{x})}{\beta(T, I, H, \vec{x})}$$

2. As the actual distribution of leaves could vary greatly at different position, we can associate N with the height: $N(\vec{x})$.
3. The shape of a specific leaf might be irregular; in order to be more precise, S_{el} needs to be calculated respectively.

6.1 The solution

When we apply these expansions, we need to use integral calculus as the value of N become successive. Nevertheless, the basic idea of f and F is the same as the basic model. As for S_{el} , we could also make the allowance of the distinction and form an average weighted outcome. This extended model could link the shape of leaves with more factors, and the result might come closer to the fact.

7 Strengths and Weaknesses

- As we focus on the situation at a certain moment, we could apply the model to describe the change of leaves throughout the year.
- Although we just assess the situation from the view in 2-dimension, the value of γ remains the same with the value of γ in 3-dimensional space.
- The expression of F does not include the impact of the nutrients absorbed by the root of a tree and the consumption of respiration of tree trunks and branches.
- The first model ignores some irregular shapes of leaves.
- η could be influenced by the angle between the leaves and the branches in 3-dimensional space.

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Appendices

Dear Editor:

Participating in the MCM is an amazing experience for us: it is torturing when we encountered seemingly unconquered puzzle, it is such a splendid feeling when we establish the modeling brick by brick. In this progress, key findings serves as the beacon illuminating the direction of the way to the final point.

First of all, after reading some materials of botany, we realized that the leaf growth of the plants in the space adhered to various forms of phyllotaxis. As a consequence, we associate this trend with the assumption that plant could place leaves in the most suitable position in order to obtain the maximal net absorption of energy. Meanwhile, we decided to commence the modeling in 2-dimension; surprisingly, the ratio of the accumulation of energy to the consumption would not change after the leaf shape is casted to the plane. With these two key findings, we determined to build up an equation on a tiny spot based on the assumption to figure out the correlation between the number and the ratio of leaves. On top of that, the correlation comes out as a clear equation, signifying that there is an optimal value of n . Every value of η corresponds with an according n . It is so encouraging to find out this result which could largely fit the practical situation. What's more, when we added the radius of the end-face of tree trunk into account, this trend remained. This is another key findings. Finally, we use the number and the ratio as the measurement to classify the leaves on different spots.

It is a hard time for us to seek out a right direction to establish a proper model to illustrate the association between the leaf shape and tree profile. Tree profile is such a complicated problem. However, the crucial point is that Caiqin Feng [4] sorted it into five kinds which greatly simplifies the complexity of this problem; and we started trying to calculate the leaf shape under the situation that tree profile is considered as cone. Still from the view of energy, a clear and beautiful relationship came into our eyes; all things suddenly became enlightened. Moreover, we validated this model by computing the statistics provided by Zhengtao Lv [5]. And the other four shapes could also lead to the similar outcomes. This consequence further verified the assumptions and these two model. Finally, based on these conclusions, we worked out another equation for the leaf mass to other factors.

Everyone might feel hopeless when trapped in some problems; nonetheless, it is very clear for us to connect these dots when we looked backwards. All these key findings link everything together, making everything clear.

Your Sincerely