

## DERIVATIVES OF THE DELTA FUNCTION

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As the Dirac delta function is essentially an infinitely high spike at a single point, it may seem odd that its derivatives can be defined. The derivatives are defined using the delta function's integral property

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0) \quad (1)$$

Consider the integral involving the  $n$ th derivative  $\delta^{(n)}(x)$  and apply integration by parts:

$$\int_{-\infty}^{\infty} f(x) \delta^{(n)}(x) dx = f(x) \delta^{(n-1)}(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) \delta^{(n-1)}(x) dx \quad (2)$$

The integrated term is taken to be zero, since the delta function itself is constant (at zero) for all  $x \neq 0$ , so all its derivatives are zero except at  $x = 0$ . Therefore  $\delta^{(n-1)}(x) = 0$  at the limits  $-\infty$  and  $\infty$ . We're therefore left with

$$\int_{-\infty}^{\infty} f(x) \delta^{(n)}(x) dx = - \int_{-\infty}^{\infty} f'(x) \delta^{(n-1)}(x) dx \quad (3)$$

Since this is true for all functions  $f(x)$ , the integrands must be equal, so we get

$$f(x) \delta^{(n)}(x) = -f'(x) \delta^{(n-1)}(x) \quad (4)$$

A common case is the first derivative, which satisfies

$$f(x) \delta'(x) = -f'(x) \delta(x) \quad (5)$$

If  $f(x) = x$ , we get the relation

$$x \delta'(x) = -\delta(x) \quad (6)$$

By iterating 4, we get

$$f(x) \delta^{(n)}(x) = (-1)^n \delta(x) f^{(n)}(x) \quad (7)$$

**Example 1.** Suppose  $f(x) = 4x^2 - 1$ . Then

$$\int_{-\infty}^{\infty} (4x^2 - 1) \delta'(x - 3) dx = - \int_{-\infty}^{\infty} 8x \delta(x - 3) dx \quad (8)$$

$$= -24 \quad (9)$$

**Example 2.** With  $f(x) = x^n$  we have, using 7

$$x^n \delta^{(n)}(x) = (-1)^n n! \delta(x) \quad (10)$$

Another use of the derivative of the delta function occurs frequently in quantum mechanics. In this case, we are faced with the integral

$$\int \delta'(x - x') f(x') dx' \quad (11)$$

where the prime in  $\delta'$  refers to a derivative with respect to  $x$ , not  $x'$ . Thus the variable in the derivative is *not* the same as the variable being integrated over, unlike the preceding cases. In this case, since only  $x$  (and not  $x'$ ) is visible outside the integral, we can move the derivative outside the integral and get

$$\int \delta'(x - x') f(x') dx' = \frac{d}{dx} \int \delta(x - x') f(x') dx' \quad (12)$$

$$= f'(x) \quad (13)$$

Notice that in this case, there is no minus sign attached to the  $f'$  unlike in 5.

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