

A Flexible Window Function for Spectral Analysis

“DSP Tips and Tricks” introduces practical design and implementation signal processing algorithms that you may wish to incorporate into your designs. We welcome readers to submit their contributions. Contact Associate Editors Rick Lyons (R.Lyons@ieee.org) or C. Britton Rorabaugh (dspboss@aol.com).

Regarding window functions used in spectral analysis, the most important performance measures are 3-dB bandwidth and sidelobe attenuation. For many window functions, Hanning and Hamming for example, we have no control over a window’s 3-dB bandwidth and sidelobe attenuation for a given window length. For other window functions—Kaiser, Gaussian, and Chebyshev—we can reduce those windows’ 3-dB bandwidth to get improved spectral resolution. However, with these later window functions (what we refer to as “conventional windows”), spectral resolution improvement comes at the expense of sidelobe attenuation reduction that degrades our ability to avoid undesirable spectral leakage. Likewise we can increase those windows’ sidelobe attenuation, but only by sacrificing desirable spectral resolution. This article describes a novel window function that enables us to control both its 3-dB bandwidth (spectral resolution) and sidelobe attenuation (spectral leakage) independently.

The 3-dB bandwidth, sidelobe attenuation, and roll-off rate are used to measure the performance of windows for power spectral density (PSD) estimation [1]–[3]. Improved frequency

resolution of the estimated PSD can be obtained if we reduce a window’s 3-dB bandwidth. The sidelobe attenuation means the difference between magnitude of the mainlobe and the maximum magnitude of the sidelobes. The sidelobe roll-off rate is the asymptotic decay rate of sidelobe peaks. Undesirable spectral leakage [4]–[6] can be reduced by increasing sidelobe attenuation and roll-off rate. Therefore, an ideal window for PSD estimation has zero bandwidth and infinite sidelobe attenuation such as an impulse function in frequency domain.

The conventional windows are able to control 3-dB bandwidth or sidelobe attenuation by only one parameter in general [1], [7]–[12]. Thus, they cannot control these two characteristics independently. In other words, if we reduce a window function’s 3-dB bandwidth, the sidelobe attenuation is also reduced, and vice versa [5], [6]. This behavior is the cause of the tradeoff problem between good frequency resolution and acceptable spectral leakage in the estimated PSD. The Butterworth window does not have this problem because it allows control of the 3-dB bandwidth and sidelobe attenuation independently.

Butterworth windows are used as antialiasing filters to reduce the noise in the reconstructed image in previous research [13]. They are also used to remove the edge effect of the matched filter output in pattern matching algorithm [14]. The transfer function of a Butterworth filter is adopted as a window in those applications. However, in this article, a portion of the impulse response of a Butterworth filter is called the Butterworth window and its characteristics in PSD estimation are analyzed.

BUTTERWORTH WINDOW

The Butterworth window can be obtained by the standard Butterworth filter design procedure. Important to us is the frequency magnitude response $|H(f)|$ of a Butterworth filter, denoted by [2] and [6]

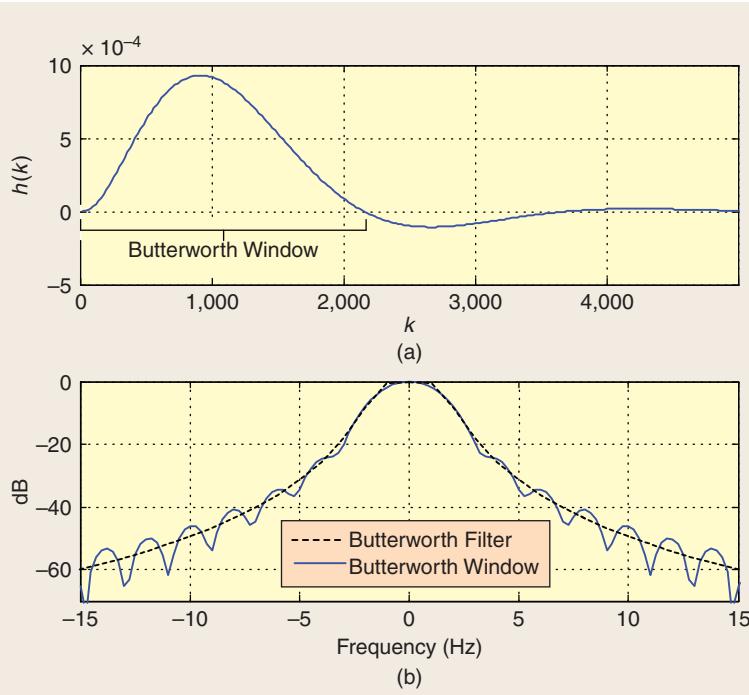
$$|H(f)| = 1/\sqrt{1 + \left(\frac{f}{f_c}\right)^{2N}}, \quad (1)$$

where f is frequency in hertz.

The Butterworth filter is characterized by two independent parameters, 3-dB cutoff frequency f_c and filter order N . The cutoff frequency and order of the Butterworth filter serve as parameters that control the bandwidth and sidelobe attenuation of the Butterworth window. The cutoff frequency of a filter has the identical meaning with the bandwidth of a window. However, the cutoff frequency is represented as a half of the bandwidth since the bandwidth of a window refers to two-sided frequency from negative to positive, while the cutoff frequency of a filter refers to only one-sided positive frequency. Our desired window spectrum is identical to the frequency response of the Butterworth filter. Thus, the inverse Fourier transform is applied to the Butterworth filter’s frequency response, in (1), to obtain the filter’s impulse response, and a portion of that response becomes the Butterworth window in the time domain.

SIMULATION AND PERFORMANCE ANALYSIS

In our simulation, the frequency and impulse responses of Butterworth filters are investigated to design the Butterworth window by varying the cutoff frequency f_c and filter order N . The sampling frequency f_s is set to 2,048 Hz. The magnitude levels of the impulse response of a



[FIG1] Butterworth filter and window: (a) filter impulse response and window function and (b) the filter magnitude response and window spectrum.

Butterworth filter are nearly zero after a certain point—almost all information that determines the filter's frequency response is in the portion before that zero-magnitude point of the impulse response. Therefore, it is expected that the suitable length of a Butterworth window can be determined by only a part of the infinite-time duration impulse response of a Butterworth filter.

Figure 1(a) shows the time-domain impulse response $h(k)$ of a unity-gain low-pass Butterworth filter when

[TABLE 1] SUITABLE LENGTHS OF BUTTERWORTH WINDOWS
(f_s = SAMPLING FREQUENCY, f_c = CUTOFF FREQUENCY).

FILTER ORDER, N	WINDOW LENGTH (SAMPLES)
1	$\left\lfloor 0.660 \cdot \frac{f_s}{f_c} \right\rfloor$
2	$\left\lfloor 0.705 \cdot \frac{f_s}{f_c} \right\rfloor$
3	$\left\lfloor 0.784 \cdot \frac{f_s}{f_c} \right\rfloor$
4	$\left\lfloor 0.890 \cdot \frac{f_s}{f_c} \right\rfloor$
5	$\left\lfloor 1.005 \cdot \frac{f_s}{f_c} \right\rfloor$

$f_c = 0.75$ Hz and $N = 3$. In that figure, we show the initial positive-only portion of the impulse response that becomes our desired Butterworth window. The 2,139th sample of $h(k)$ is the point that the magnitude of the impulse response of the Butterworth filter becomes zero for the first time.

The solid curve in Figure 1(b) is the frequency spectrum of the 2,139-sample Butterworth window. The 3-dB bandwidth and sidelobe attenuation of this window are 1.3 Hz and 24.3 dB, respectively. In Figure 1(b), for comparison, we show the frequency magnitude response of the Butterworth filter as the dashed curve. We see that there is no significant difference between the magnitude

response of the Butterworth filter and the spectrum of the Butterworth window. Therefore, a suitable length of the Butterworth window may be considered to be up to the point where the magnitude of the impulse response of filter becomes zero for the first time.

Based on the order N , the sampling frequency f_s , and the cutoff frequency f_c of the Butterworth filter, we have empirically determined the suitable lengths of the Butterworth windows to be those given in Table 1. Here the $\lfloor x \rfloor$ notation means the integer part of x .

The frequency characteristics of Butterworth windows with $f_c = 0.75$ Hz are shown in Table 2. The sidelobe attenuation is increased from ten to 30.4 dB as the filter order is increased from one to five, while the 3-dB bandwidth is fixed at about 1.5 Hz.

The PSD of an example signal is estimated by Butterworth windows to confirm the performance. The signal $x(t)$ used for our simulation is

$$\begin{aligned}
 x(t) = & 0.84\cos(2\pi \cdot 52 \cdot t) \\
 & + 0.8\cos(2\pi \cdot 65.5 \cdot t) \\
 & + 0.3\cos(2\pi \cdot 85 \cdot t) \\
 & + 1.1\cos(2\pi \cdot 105 \cdot t) \\
 & + 0.35\cos(2\pi \cdot 140 \cdot t) \\
 & + 0.98\cos(2\pi \cdot 159 \cdot t) \\
 & + 0.6\cos(2\pi \cdot 174 \cdot t) \\
 & + 0.8\cos(2\pi \cdot 190 \cdot t) \\
 & + \cos(2\pi \cdot 205 \cdot t). \quad (2)
 \end{aligned}$$

The solid lines in Figure 2(a) show the ideal PSD of $x(t)$. The dotted curve in Figure 2(a) shows the estimated PSD of a 2,139-sample rectangular windowed $x(t)$, using Welch's method [15], where that window's insufficient sidelobe attenuation (spectral leakage) produces

[TABLE 2] FREQUENCY CHARACTERISTICS OF THE BUTTERWORTH WINDOWS.

FILTER ORDER	3-DB BANDWIDTH	SIDELOBE ATTENUATION	ROLL-OFF RATE
1	1.5 Hz	10.0 dB	
2	1.4 Hz	18.2 dB	
3	1.3 Hz	24.3 dB	-12 dB/OCT.
4	1.3 Hz	28.8 dB	
5	1.3 Hz	30.4 dB	

false spectral components, particularly on either side of a relatively high-level ideal PSD spectral component.

Figure 2(b) shows the estimated PSDs of $x(t)$ using various 2,139-sample Butterworth windows, where we see that reducing the cutoff frequency and increasing the order can reduce spectral leakage without the undesirable spectral mainlobe broadening (loss of resolution) experienced by the conventional window functions. It means the tradeoff problem between resolution and spectral leakage is solved. This beneficial behavior is illustrated in Table 3, where Butterworth windows are compared to the conventional window functions. In that table, we see that Butterworth windows can increase their sidelobe attenuation without the undesirable mainlobe broadening exhibited by the conventional window functions. Reference [16] provides spectral plots comparing Butterworth windows to the conventional window functions in Table 3.

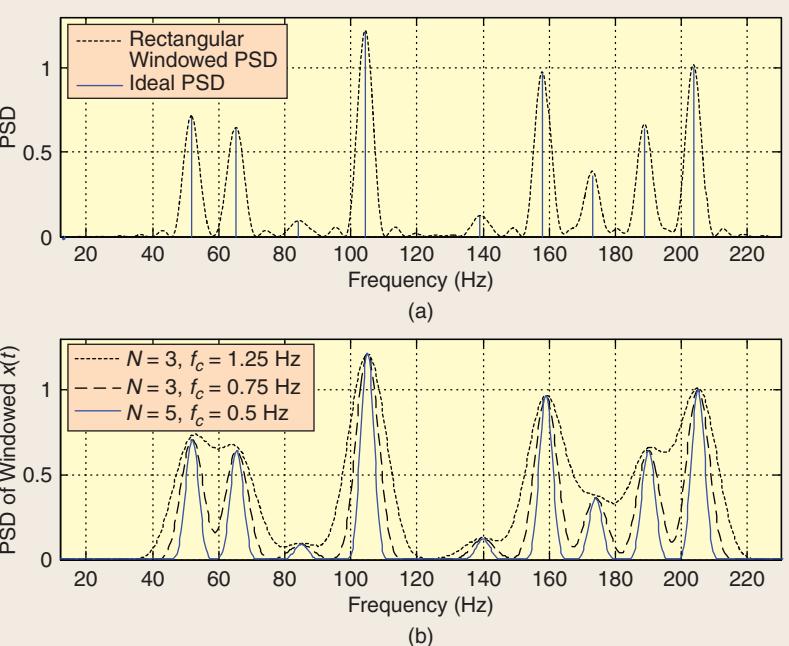
IMPLEMENTATION ISSUES

- The computational time of PSD estimation is not related to window type, but rather the window length and estimation method. So Butterworth windows have the same computational workload as the conventional window functions.
- To use the computationally efficient radix-2 fast Fourier transform algorithm, we suggest that the time-domain samples of Butterworth window should be zero padded to make the window length an integer power of two. As an alternative to zero padding, we can restrict the Butterworth window's f_c cutoff frequency to be

$$f_c = \frac{Kf_s}{2^{M+1}}, \quad (3)$$

which leads to Butterworth windows that are 2^M in length, where K is one of the scaling constants from Table 1, and M is an integer.

- Because Butterworth windows are not symmetrical, any specialized spectral analysis scheme that requires the imaginary part of a window function's



[FIG2] Ideal and windowed PSD: (a) rectangular windowed and ideal PSDs and (b) PSDs using various Butterworth windows.

spectrum to be all zero will not work with the Butterworth windows.

CONCLUSIONS

We've shown that the Butterworth window can be obtained by the conventional Butterworth filter design procedure. This window is able to control the 3-dB bandwidth and sidelobe attenuation independently by two parameters, the cutoff frequency and the order of the filter. As such, the sidelobe attenuation can be varied even if the 3-dB bandwidth is fixed, and vice versa. Therefore the tradeoff problem between the frequency resolution and spectral leakage in the estimated PSD, unavoidable with the conventional windows, can be solved by the Butterworth window.

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[TABLE 3] COMPARISON OF FREQUENCY CHARACTERISTICS.

WINDOW	3-DB BANDWIDTH	SIDELOBE ATTENUATION	
RECTANGULAR	0.879 Hz	13.3-dB	
TRIANGULAR	1.270 Hz	26.5 dB	
HANNING	1.367 Hz	31.3-dB	
KAI SER	$\alpha = 2$	0.980 Hz	18.5 dB
	$\alpha = 4$	1.172 Hz	30.4 dB
CHEBYSHEV	$\beta = 1$	0.890 Hz	20.1 dB
	$\beta = 2$	1.172 Hz	40.5 dB
BUTTERWORTH	$N = 2$	0.793 Hz	18.2 dB
$(f_c = 0.439 \text{ Hz})$	$N = 3$	0.740 Hz	24.3-dB
	$N = 4$	0.731 Hz	28.8 dB
BUTTERWORTH	$f_c = 0.439 \text{ Hz}$	0.731 Hz	28.8 dB
$(N = 4)$	$f_c = 1.500 \text{ Hz}$	2.815 Hz	28.8 dB
	$f_c = 2.500 \text{ Hz}$	4.720 Hz	28.8 dB

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REFERENCES

- [1] F. J. Harris, "On the use of windows for harmonic analysis with the discrete Fourier transform," *Proc. IEEE*, vol. 66, no. 1, pp. 51–83, Jan. 1978.
- [2] S. Kay and S. Marple, "Spectrum analysis—a modern perspective," *Proc. IEEE*, vol. 69, no. 11, pp. 1380–1419, Nov. 1981.
- [3] M. H. Hayes, *Statistical Signal Processing and Modeling*, New York: Wiley, 1996.
- [4] B. P. Lathi, *Modern Digital and Analog Communication Systems*, 3rd ed., New York: Oxford Univ. Press, 1998.
- [5] N. Geckinli and D. Yavuz, "Some novel windows and a concise tutorial comparison of window families," *IEEE Trans. Acous. Speech Signal Processing*, vol. ASSP-26, no. 6, pp. 501–507, Dec. 1978.
- [6] S. Orfanidis, *Introduction to Signal Processing*, Englewood Cliffs, NJ: Prentice-Hall, 1996.
- [7] S. Mitra and J. Kaiser, *Handbook for Digital Signal Processing*, New York: Wiley, 1993.
- [8] S. Bergen and A. Antoniou, "Design of ultraspherical window functions with prescribed spectral characteristics," *EURASIP J. Appl. Signal Process.*, vol. 2004, no. 13, pp. 2053–2065, Oct. 2004.
- [9] A. Nuttall, "Some windows with very good sidelobe behavior," *IEEE Trans. Acous. Speech Signal Processing*, vol. 29, no. 1, pp. 84–91, Feb. 1981.
- [10] G. Andria, M. Savino, and A. Trotta, "Windows and interpolation algorithms to improve electrical measurement accuracy," *IEEE Trans. Instrum. Meas.*, vol. 38, no. 4, pp. 856–863, Aug. 1989.
- [11] G. Andria, M. Savino, and A. Trotta, "FFT-based algorithms oriented to measurements on multi-frequency signals," *Measurement*, vol. 12, no. 1, pp. 25–42, Dec. 1993.
- [12] J. Gautam, A. Kumar, and R. Saxena, "On the modified Bartlett-Hanning window (family)," *IEEE Trans. Signal Processing*, vol. 44, no. 8, pp. 2098–2102, Aug. 1996.
- [13] H. Baghaei, W. Wong, H. Li, J. Uribe, Y. Wang, M. Aykac, Y. Liu, and T. Xing, "Evaluation of the effect of filter apodization for volume PET imaging using the 3-D RP algorithm," *IEEE Trans. Nucl. Sci.*, vol. 50, no. 1, pp. 3–8, Feb. 2003.
- [14] S. Su, C. Yeh, and C. Kuo, "Structural analysis of genomic sequences with matched filtering," in *Proc. 25th Ann. Int. Conf. IEEE Engineering in Medicine and Biology Society*, Sept. 2003, vol. 3, pp. 2893–2896.
- [15] P. Welch, "The use of fast Fourier transform for the estimation of power spectra: A method based on time averaging over short, modified periodograms," *IEEE Trans. Audio Electroacous.*, vol. AU-15, no. 2, pp. 70–73, June 1967.
- [16] T. Yoon and E. Joo, "Butterworth window for power spectral density estimation," *ETRI J.*, vol. 31, no. 3, pp. 292–297, June 2009.

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spotlight REPORT

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of which are donating hardware and some funding. "We have no intellectual property. Everything is open-source. We're not allowed to patent anything. Companies like that; it makes it easier to collaborate with them."

WHERE IS EVERYBODY?

Has anyone found anything? Nothing has been confirmed, but there have been some false terrestrial alarms and artificially produced extraterrestrial signals.

Probably the best known signal picked up by SETI researchers so far was at Ohio State in 1977 when they received what became known as the "Wow!" signal. As an Ohio State astronomer was checking printouts, he found a sharp, clear signal that even turned on and off during the period it was observed. The astronomer wrote "Wow!" in the margin of the printout. But the signal was never heard again.

Why wouldn't aliens respond to the signals we have already been broadcasting for years?

Today, some SETI scientists are less certain about the narrowband approach to the search for alien civilizations.

Shostak says that several years ago, when wideband techniques, such as spread spectrum, were coming into wide use, SETI scientists began to wonder if aliens were sending out spread-spectrum signals. "In which case, we're not going to find them." While the wideband

issue continues to be a concern, he thinks there might still be a very narrowband component to the signal just to get your attention.

Shostak notes that we have been broadcasting seriously into space for maybe 70 years and that means those early broadcasts are 70 light years out. "So, if you're running SETI experiments today and looking for responses, then the aliens can't be more than 35 light years away. You need enough time for 'Howdie Doody' to get to them and for them to get back to us." Shostak estimates those parameters give us access to a couple of thousand stars at most—out of a couple of hundred billion stars in the galaxy.

And while most astronomers are interested in finding other planets, Shostak believes that recent reports from European astronomers identifying 32 new planets orbiting stars outside our solar system doesn't improve the chances that we'll find intelligent life on those planets. "The number of stars in the Milky Way is on the order of 100 billion. So another 32 planets doesn't mean much to the SETI community."

"What you want to know," he says, "is what fraction of these planets are most like Earth. Unfortunately, these planets are never like Earth because those are hard to find. It will take a couple of years to find Earth-like planets, but within about a thousand days [by approximately

the end of 2012], we will know what percent of stars have planets that are sort of like Earth, with liquid water and atmospheres. That's what the Kepler mission is designed to do—find Earth-like planets"

WHAT'S NEXT?

The search goes on. In addition to the vastly improved radio searches—mainly the ATA—SETI is looking for signals that might be sent at visible wavelengths or in the infrared. Experiments at UC-Berkeley at Santa Cruz, Berkeley, and Harvard are using relatively large, conventional mirror telescopes to hunt for very brief flashes of light (presumably from high-powered lasers) that other civilizations might be beaming our way.

At the same time, Dr. Jill Tarter, the director of the SETI Institute and the winner of the 2009 Technology, Entertainment, and Design Award, says she plans to open-source the ATA's detection algorithms to advance the search so engineers around the world can help improve them and, in the process, develop a new generation of SETI enthusiasts.

In fact, Tarter has a wish list that includes a massive outreach to grow the TeamSETI network, including finding and recruiting engineers with expertise in digital signal processing. Another item high on her list is funding—to acquire additional telescopes for the Allen array.

[SP]