Singleton Types Here, Singleton Types There, Singleton Types Everywhere

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Dependent types for programmers

Type based formal methods for security

Type based formal methods for reliability

Proof systems for program verification

Proof systems for meta theory
Love & hate

Dependent types and programming languages’s troubled relationship:

Dependent types are sexy and elegant
  They are to plain types, what plain types are to dynamic typing

They mix very poorly with real-world features
  ⇒ Poor man’s dependent types

Compiler writers do not know what to do with them
  ⇒ Don’t compile
  ⇒ Drop types
  ⇒ Drop dependencies
Contributions

Tame dependent types for compilers

Type-preserving closure conversion of CC

Singleton types as powerful as dependent types

Way to make singleton types prettier
Singleton types

Poor man’s dependent types which enjoy a phase distinction

\[ \text{aref} : n < m \Rightarrow \text{Snat} \ n \rightarrow \text{Array} \ \alpha \ m \rightarrow \alpha \]

What about operations on integers?

\[ \text{Nat} = \exists n.\text{Snat} \ n \]

\[ + : \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat} \]

or

\[ + : \text{Snat} \ n \rightarrow \text{Snat} \ m \rightarrow \text{Snat} \ (n + m) \]

¡Two additions!
Poisonous duplication

Everything tends to get sucked up in the copy machine

- First basic data types
- Then operations on them

Proofs at type level want extensions to the type language

- The $n < m$ cannot always be proved automatically
- If the type language does not help, use ... singleton types!
**Dependent types**

\[ \text{aref} : \Pi t : \star, n : \text{Nat}, m : \text{Nat}. (n < m) \rightarrow \text{Array } t m \rightarrow t \]

No funny Snat, no duplication: simpler, cleaner

Types, values, proofs are neatly intertwined

A happy family . . . or a big mess

No more *phase distinction*
Compiling dependent types

Drop types altogether

Drop dependencies

\[ \text{aref} : \forall \alpha : \Omega. \text{Nat} \rightarrow \text{Nat} \rightarrow \text{LT} \rightarrow \text{Array } \alpha \rightarrow \alpha \]

⇒ Back to plain types

⇒ “Type preserving” but without preserving type info
There’s hope
Conversion to singleton types

Automate the duplication, so the programmer does not have to see it

Split each element into a value and a type

\[ n : \text{Nat} \quad \Rightarrow \quad \check{n} : \text{Snat} \ 
\hat{n} \]

We can choose \( \hat{n} \equiv n \)

For other types:

\[ e : \tau \quad \Rightarrow \quad C[e] : S[\tau] \ e \]

Destination type language \( \supseteq \) source language

We compile the Calculus of Constructions to \( \lambda_H \)
A simple functional language, where CIC is used for the type language:

\[(\text{exp}) \ e ::= \ x \ | \ n \ | \ e \ e \ | \ e[A] \ | \ \lambda x : A. e \ | \ \Lambda X : A. f\]

The type of types is an inductive definition:

Inductive \( \Omega : \ast : = \text{Snat} : \text{Nat} \to \Omega \)

\[\begin{align*}
& | \to : \Omega \to \Omega \to \Omega \\
& | \forall : \Pi k : \ast. (k \to \Omega) \to \Omega
\end{align*}\]

CIC types are \( \lambda_H \) kinds; CIC kinds are \( \lambda_H \) kind schemas
Generalize to functions

Applying previous formula \( C[e] : S[\tau] e \):

\[
(+) : \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat} \quad \Rightarrow \quad C[(+)] : S[\text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat}] (+)
\]

\[
sort : \text{List} \rightarrow \text{List} \quad \Rightarrow \quad C[\text{sort}] : S[\text{List} \rightarrow \text{List}] \text{ sort}
\]

And since we want:

\[
C[(+)] : \forall n, m : \text{Nat}. \text{Snat } n \rightarrow \text{Snat } m \rightarrow \text{Snat } (n + m)
\]

\[
C[\text{sort}] : \forall l : \text{List}. \text{Slist } l \rightarrow \text{Slist } (\text{sort } l)
\]

We get:

\[
S[\tau_1 \rightarrow \tau_2] f \equiv \forall x : \tau_1. S[\tau_1] x \rightarrow S[\tau_2] (f x)
\]

\[
C[\lambda x : \tau.e] \quad \equiv \quad \Lambda x : \tau. \lambda x' : (S[\tau] x).C[e]
\]
What about variables

How to compute $C[x]$?

intensional type analysis or dictionary passing…
What about variables

How to compute $C[x]$?

*intensional type analysis or dictionary passing...*

Refine $C[\lambda x:\tau . e] \equiv \Lambda x : \tau . \lambda x' : (S[\tau][x]).C[e]$ into

\[
C\  c\  [\lambda x : \tau . e] \equiv \Lambda x : \tau . \lambda x' : (S[\tau][x]).C\  \{c,\ x \mapsto \ x'\} \ [e]
\]

\[
C\  c\  [x] \equiv c(x)
\]
What about type variables

How to compute $S[t]$?

$$\text{head} : \Pi t : \ast . \text{List } t \rightarrow t$$

$C[\text{head}] : \forall t : \ast . \forall l : \text{List } t . \text{Slist } t \ l \rightarrow S[t] \ (\text{head } t \ l)$

Becomes

$C[\text{head}] : \forall t : \ast . \forall \text{St : } t \rightarrow \Omega . \forall l : \text{List } t . \text{Slist } t \ l \rightarrow \text{St } (\text{head } t \ l)$

So

$$S \ s [t] \equiv s(t)$$

$$S \ s [\Pi t : \kappa . \tau] \equiv \forall t : \kappa . \forall \text{St : } ??? . S \ {s, t \mapsto \text{St}} \ [\tau]$$
Really, $S[\tau]$ is similar to $C[e]$

Just like $C[e] : S[\tau]e$ we have $S[\tau] : S'[\kappa]\tau$

In a sense, Snat is treated as “the single value” of $S'[\star]\text{Nat}$

$$S'[\star]\tau \equiv \tau \rightarrow \Omega$$

We also have to define $S'[\kappa]$ for other kinds

But at least, CC has no kind variables, so the tower stops here
Back to array access

We had

\[ \text{aref} : \Pi t : *, n : \text{Nat}, m : \text{Nat}. (n < m) \rightarrow \text{Array } t m \rightarrow t \]

which leads to

\[ C[\text{aref}] : \forall t : *, St : t \rightarrow \Omega, n : \text{Nat}, m : \text{Nat}, P : n < m, A : \text{Array } t m. \]

\[ \text{Snat } n \rightarrow \text{Snat } m \rightarrow \text{Slt } n \ m \ P \rightarrow \text{Sarray } t \ m \ A \]

\[ \rightarrow St (\text{aref } t \ n \ m \ P \ A) \]

Requires the usual extra info to make efficient
Conclusion

\[ \lambda_H \] is a good target for ML, OO, and dependent types

\[ S[\tau]e \] is injective, so all type information is preserved

Traditional compilation techniques can be used, e.g. CPS and closure conversion become straightforward

We have generalized it to pure type systems

A surface language for \[ \lambda_H \] could avoid duplication

Need to extend it to inductive definitions
The type farm has declared independence