Gomory Cutting Plane Algorithm

Let *A* be an $m \times n$ matrix, and consider the following integer program:

IP: min
$$\mathbf{c'x}$$

subject to $A\mathbf{x} = \mathbf{b}$
 $\mathbf{x} \ge \mathbf{0}$
 \mathbf{x} integer

Let LP be the linear programming relaxation, and suppose that LP is solved to optimality, obtaining an optimal basis B and a corresponding optimal tableau T. Let \mathbf{x}^* denote the optimal solution obtained for the LP relaxation.

Let v_{ij} denote entry of the optimal tableau in the *i*th row and the *j*th column. From the *i*th row, where i = 1, ..., m, we have the valid equality

$$x_{B(i)} + \sum_{j \notin B} v_{ij} x_j = v_{i0}.$$
 (1)

Since all of the x_i are nonnegative, we have that

$$x_{B(i)} + \sum_{j \notin B} \left[v_{ij} \right] x_j \le v_{i0},$$

is a valid inequality for IP. Additionally, since all of the x_j are integer in IP, the right side of the inequality is an integer. Hence

$$x_{B(i)} + \sum_{j \notin B} \left\lfloor v_{ij} \right\rfloor x_j \le \left\lfloor v_{i0} \right\rfloor \tag{2}$$

is also a valid inequality for IP. Subtracting (1) from (2), we obtain the valid inequality

$$\sum_{j \notin B} \left(\left\lfloor v_{ij} \right\rfloor - v_{ij} \right) x_j \le \left\lfloor v_{i0} \right\rfloor - v_{i0}.$$

Adding the slack variable x_{n+1} , we add the constraint

$$\sum_{j \notin B} \left(\left\lfloor v_{ij} \right\rfloor - v_{ij} \right) x_j + x_{n+1} = \left\lfloor v_{i0} \right\rfloor - v_{i0}$$

to the optimal tableau, and add A_{n+1} to the basis *B*. Note that the tableau is prepared to proceed. Since $\lfloor v_{i0} \rfloor - v_{i0} < 0$ but the reduced costs remain nonnegative, we proceed with the dual simplex algorithm.

Observe that \mathbf{x}^* does not satisfy (2), and so the feasible region of the linear programming relaxation is reduced by adding the inequality. The Gomory Cutting Plane Algorithm (also known as the Fractional Dual Algorithm) is to iteratively add these cuts until an integral solution is obtained. Note that every feasible integral solution to IP satisfies (2), and hence no integral solutions are lost.

To show that the Gomory Cutting Plane terminates after a finite number of steps, we require that the dual simplex algorithm utilizes lexicographic anticycling rules. We thus summarize an iteration of the Gomory Cutting Plane algorithm:

1. Choose the first (*i.e.*, highest) row i where v_{i0} is not integral. (Note that this includes the zeroth row.)

2. Add the cut

$$\sum_{j \notin B} \left(\left\lfloor v_{ij} \right\rfloor - v_{ij} \right) x_j + x_{n+1} = \left\lfloor v_{i0} \right\rfloor - v_{i0}$$

to the bottom of the optimal tableau, and add A_{n+1} to the basis *B*.

- 3. Proceed with the dual simplex algorithm using the lexicographic anticycling rules:
 - (a) Start with all columns j = 1, ..., n + 1 lex-positive.
 - (b) Choose any row ℓ where $v_{\ell 0} < 0$ for the pivot row. Choose the pivot column j as the index that maximizes

$$\frac{\text{lex-max}}{j:v_{\ell j}<0}\left[\frac{1}{v_{\ell j}}B^{-1}A_j\right].$$

Note that the lexicographic anticycling rule maintians that all columns j = 1, ..., n remain lexpositive.

Theorem 1. The Gomory Cutting Plane Algorithm terminates in a finite number of steps.

Proof. Let T^r be the optimal tableau obtained after the r^{th} cut has been added, with entries v_{ij}^r and columns T_j^r . Since the optimal cost is nondecreasing, v_{00}^r is nonincreasing. Additionally, cuts are always added to the bottom of the tableau. Hence, $T_0^1 >_L T_0^2 >_L \dots$

Assume that v_{00}^r is bounded below. Then v_{00} converges to some w_{00} . Suppose that after k iterations, $v_{00}^k < \lfloor w_{00} \rfloor + 1$. Write $v_{00}^k = \lfloor w_{00} \rfloor + f_{00}^k$, where $0 \le f_{00}^k < 1$. In the next iteration, the row i = 0 will be chosen and the cut

$$\sum_{j \notin B} \left(\left\lfloor v_{0j}^k \right\rfloor - v_{0j}^k \right) x_j + x_{n^k + 1} = -f_{00}^k$$

is added. When running dual simplex, let p be the first column of the first pivot. This has the effect that

$$v_{00}^{k+1} = v_{00}^{k} + \frac{v_{0p}^{k}}{\left\lfloor v_{0p}^{k} \right\rfloor - v_{0p}^{k}} f_{00}^{k}$$

Note that $v_{0p}^k \ge 0$ (since is a reduced cost in the k^{th} optimal tableau), and hence $0 \le \left[v_{0p}^k\right] \le v_{0p}^k$. Thus $\frac{v_{0p}^k}{\left[v_{0p}^k\right] - v_{0p}^k} \le -1$, and so

$$v_{00}^{k+1} \le v_{00}^k - f_{00}^k = \lfloor w_{00} \rfloor.$$

Since v_{00}^r is nonincreasing and converging to w_{00} , we have that w_{00} is an integer and $v_{00}^{k+1} = w_{00}$. Moreover, $v_{00}^r = w_{00}$ for all iterations $r \ge k+1$.

We use the same method on v_{10}^r , v_{20}^r , and so on. Since the 0th column of T^r is lex-decreasing, we have that v_{10}^r is converging to some w_{10} (if bounded from below). However, to use the method, we need that $v_{1p}^k \ge 0$. Note that v_{00}^k is fixed at w_{00} so $v_{0p}^k = 0$. But the lexicographic anticycling rule for dual simplex ensures that A_p^k remains lex-positive, which implies that $v_{1p}^k \ge 0$.

We continue in this way for all rows that correspond to variables of the original integer program, obtaining an integral solution. $\hfill \Box$