

# **1. The Simplex Method**

# Restaurant owner problem

- **Seafoods available:**

30 sea-urchins

24 shrimps

18 oysters

- **Two types of seafood plates to be offered:**

\$8 : including 5 sea-urchins, 2 shrimps et 1 oyster

\$6 : including 3 sea-urchins, 3 shrimps et 3 oysters

- **Problem:** determine the number of each type of plates to be offered by the owner in order to maximize his revenue according to the seafoods available

$$\begin{array}{ll} \max & 8x + 6y \\ \text{Subject to} & \\ & 5x + 3y \leq 30 \\ & 2x + 3y \leq 24 \\ & 1x + 3y \leq 18 \\ & x, y \geq 0 \end{array}$$

# Transforming max problem into min problem

- Consider the following maximisation problem

$$\max f(w)$$

Subject to

$$w \in X \subset R^n$$

where  $f: X \rightarrow R^1$ .

- Let  $w^*$  be a point of  $X$  where the maximum of  $f$  is reached.

- Then  $f(w^*) \geq f(w) \quad \forall w \in X$

or  $-f(w^*) \leq -f(w) \quad \forall w \in X$

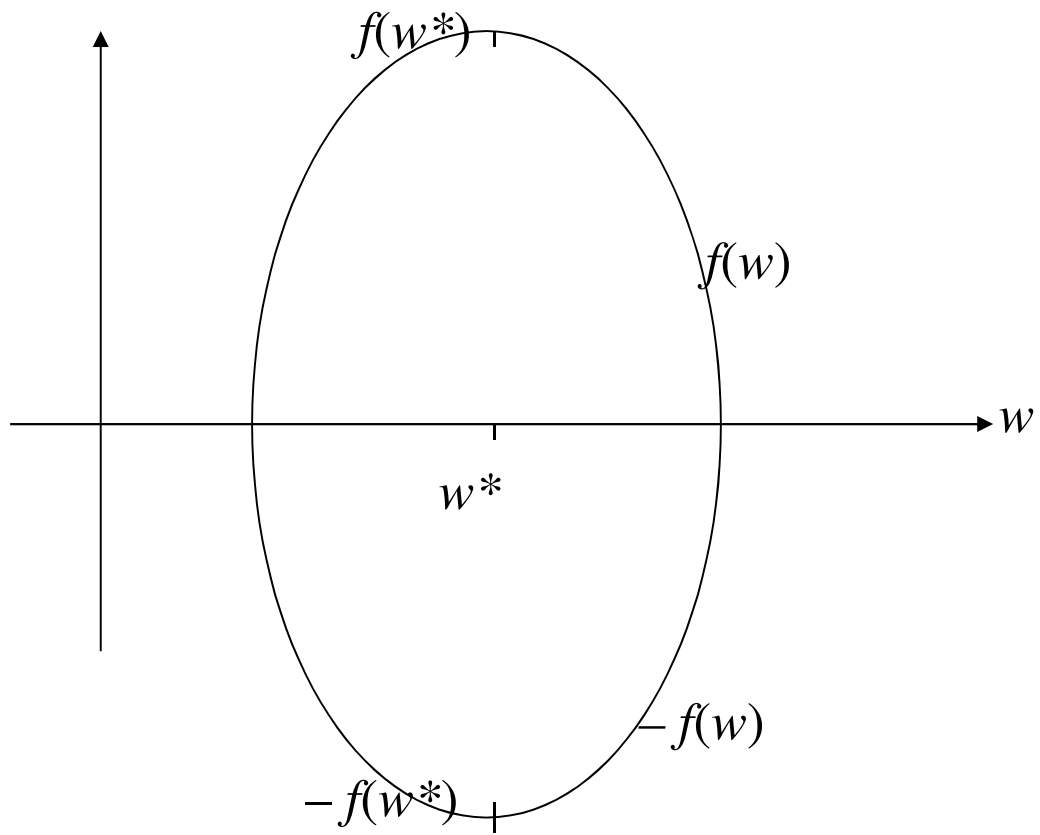
- Consequently

$$-f(w^*) = \min -f(w)$$

Subject to  $w \in X \subset R^n$

and  $w^*$  is a point of  $X$  where the function  $-f(w)$  reaches its minimum.

- Hence if we max  $f(w)$  or if we min  $-f(w)$  we find the same optimal solution  $w^*$ .



# Transforming max problem into min problem

- Furthermore

$$f(w^*) = \max f(w) = - \min -f(w) = - (-f(w^*))$$

- We will always transform max problem into min problem.

- Then  $f(w^*) \geq f(w) \quad \forall w \in X$   
or  $-f(w^*) \leq -f(w) \quad \forall w \in X$

- Consequently

$$-f(w^*) = \min -f(w)$$

$$\text{Subject to } w \in X \subset R^n$$

and  $w^*$  is a point of  $X$  where the function  $-f(w)$  reaches its minimum.

- Hence if we max  $f(w)$  or if we min  $-f(w)$  we find the same optimal solution  $w^*$ .

# Restaurant owner problem

$$\max 8x + 6y$$

Subject to

$$5x + 3y \leq 30$$

$$2x + 3y \leq 24$$

$$1x + 3y \leq 18$$

$$x, y \geq 0$$

$$\min -(8x + 6y)$$

Subject to

$$5x + 3y \leq 30$$

$$2x + 3y \leq 24$$

$$1x + 3y \leq 18$$

$$x, y \geq 0$$

# Solving a problem graphically

- Method to solve problem having only two variables
- Consider the restaurant owner problem transformed into a min problem:

$$\min z = -8x - 6y$$

Subject to

$$5x + 3y \leq 30$$

$$2x + 3y \leq 24$$

$$1x + 3y \leq 18$$

$$x, y \geq 0$$

# Feasible Domain

$$5x + 3y \leq 30$$

$$2x + 3y \leq 24$$

$$1x + 3y \leq 18$$

$$x \geq 0, y \geq 0$$

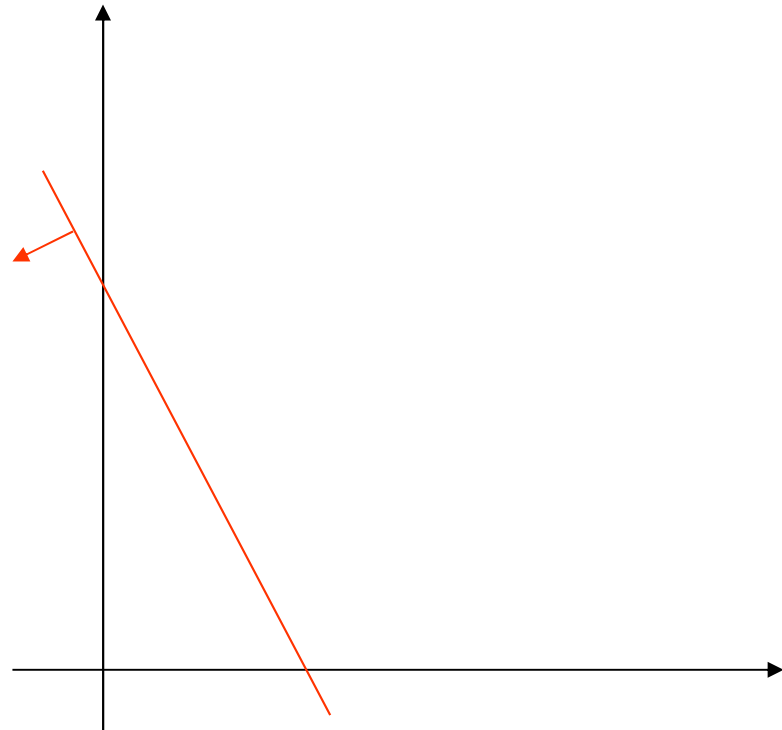
- Draw the line

$$5x + 3y = 30$$

The set of points satisfying  
the constraint

$$5x + 3y \leq 30$$

is under the line since the origin  
satisfies the constraint





# Feasible Domain

$$5x + 3y \leq 30$$

$$2x + 3y \leq 24$$

$$1x + 3y \leq 18$$

$$x \geq 0, y \geq 0$$

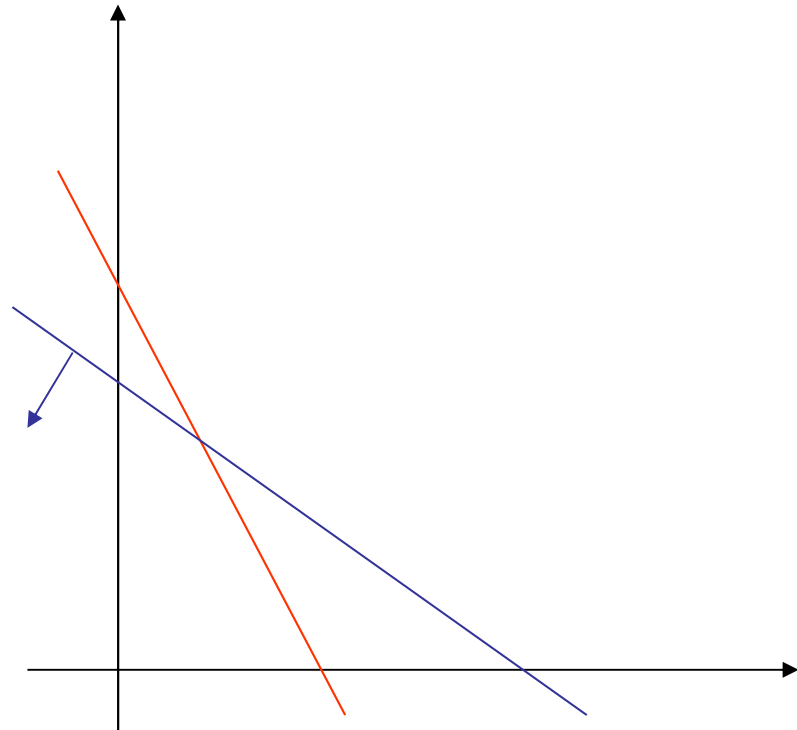
- Draw the line

$$2x + 3y = 24$$

The set of points satisfying  
the constraint

$$2x + 3y \leq 24$$

is under the line since the origin  
satisfies the constraint



# Feasible Domain

$$5x + 3y \leq 30$$

$$2x + 3y \leq 24$$

$$1x + 3y \leq 18$$

$$x \geq 0, y \geq 0$$

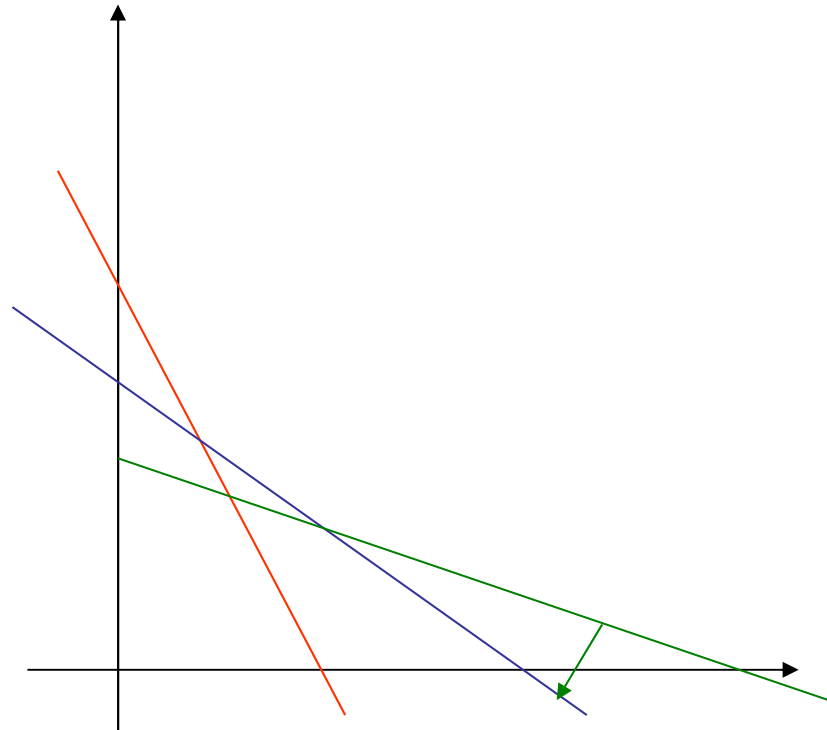
- Draw the line

$$x + 3y = 18$$

The set of points satisfying  
the constraint

$$x + 3y \leq 18$$

is under the line since the origin  
satisfies the constraint



# Feasible Domain

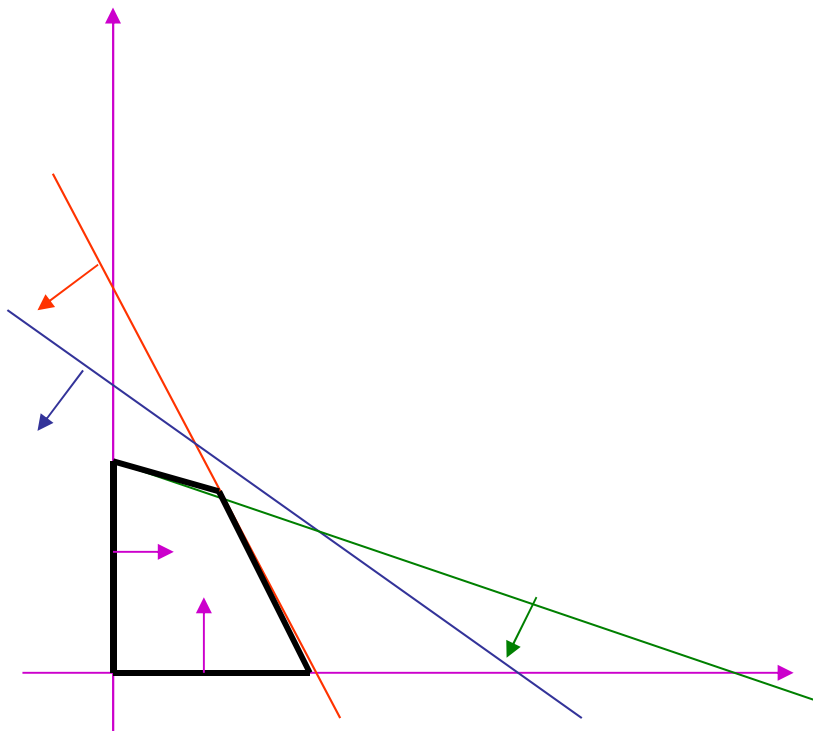
- The set of feasible points for the system

$$5x + 3y \leq 30$$

$$2x + 3y \leq 24$$

$$1x + 3y \leq 18$$

$$x, y \geq 0$$



# Solving the problem

$$5x + 3y \leq 30$$

$$2x + 3y \leq 24$$

$$1x + 3y \leq 18$$

$$x \geq 0, y \geq 0$$

- Consider the economic function :

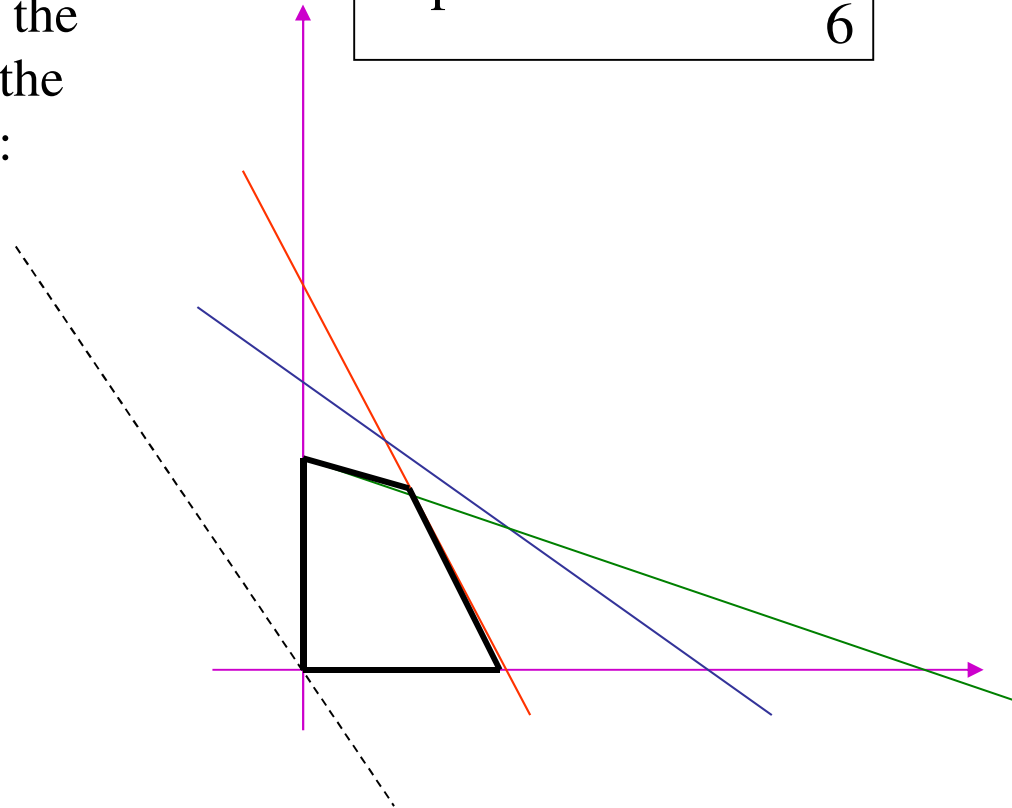
$$z = -8x - 6y.$$

- The more we move away on the right of the origin, the more the objective function decreases:

$$x = 0 \text{ and } y = 0 \Rightarrow z = 0$$

$$y = -\frac{8}{6}x - \frac{z}{6}$$

$$\text{slope of the line: } -\frac{8}{6}$$



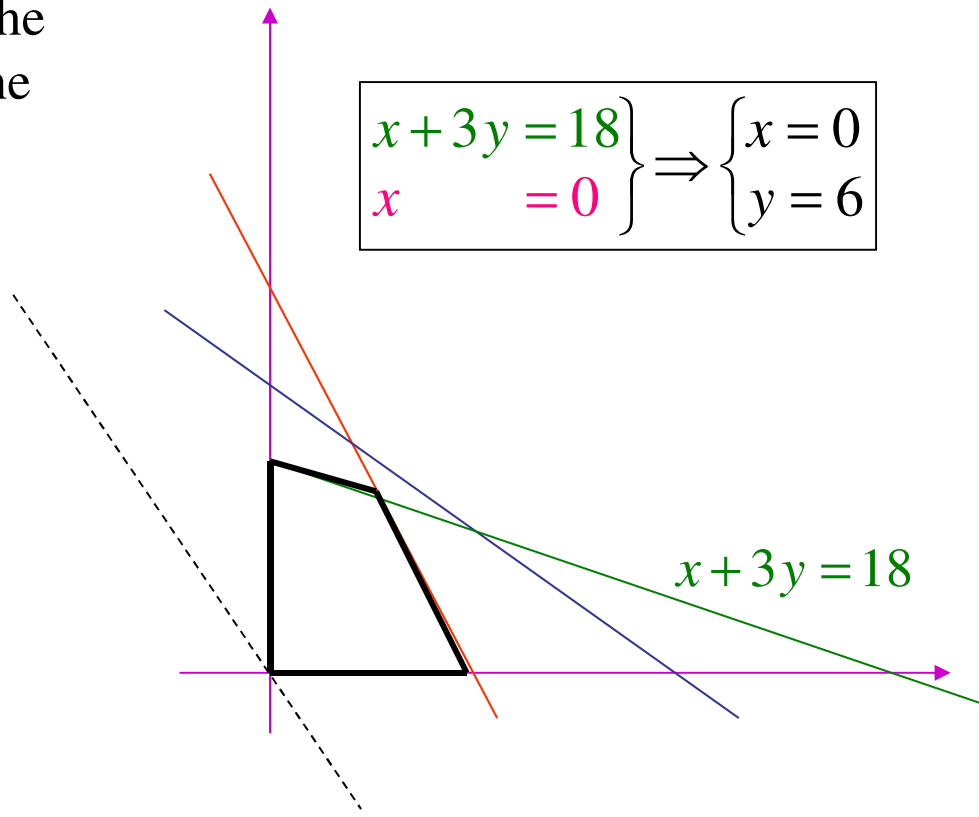
# Solving the problem

$$\begin{aligned} 5x + 3y &\leq 30 \\ 2x + 3y &\leq 24 \\ 1x + 3y &\leq 18 \\ x \geq 0, y &\geq 0 \end{aligned}$$

- Consider the economic function :

$$z = -8x - 6y.$$

- The more we move away on the right of the origin, the more the objective function decreases:
- $x = 0$  and  $y = 0 \Rightarrow z = 0$   
 $x = 0$  and  $y = 6 \Rightarrow z = -36$



# Solving the problem

$$\begin{array}{l} 5x + 3y \leq 30 \\ 2x + 3y \leq 24 \\ 1x + 3y \leq 18 \\ x \geq 0, y \geq 0 \end{array}$$

- Consider the economic function :

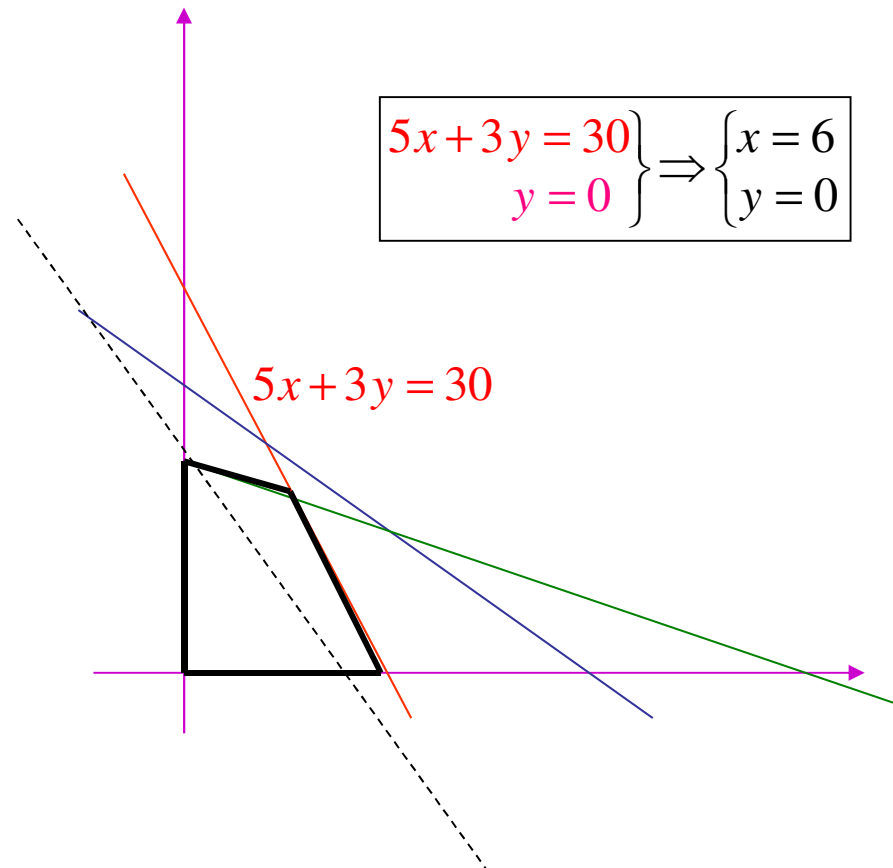
$$z = -8x - 6y.$$

- The more we move away on the right of the origin, the more the objective function decreases:

$$x = 0 \text{ and } y = 0 \Rightarrow z = 0$$

$$x = 0 \text{ and } y = 6 \Rightarrow z = -36$$

$$x = 6 \text{ and } y = 0 \Rightarrow z = -48$$



# Solving the problem

$$\begin{aligned} 5x + 3y &\leq 30 \\ 2x + 3y &\leq 24 \\ 1x + 3y &\leq 18 \\ x \geq 0, y &\geq 0 \end{aligned}$$

- Consider the economic function :

$$z = -8x - 6y.$$

- The more we move away on the right of the origin, the more the objective function decreases:

$$x = 0 \text{ and } y = 0 \Rightarrow z = 0$$

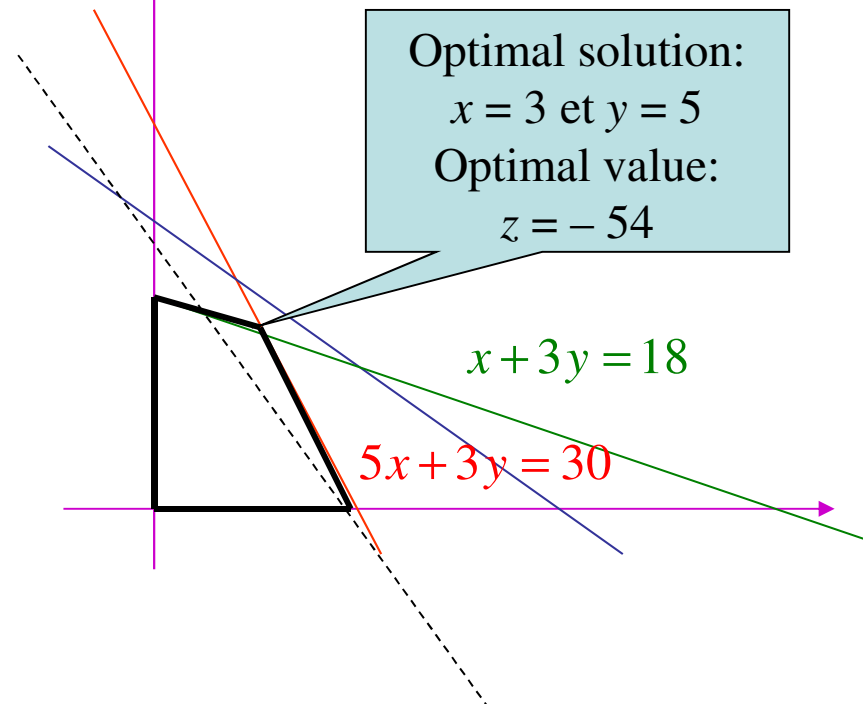
$$x = 0 \text{ and } y = 6 \Rightarrow z = -36$$

$$x = 6 \text{ and } y = 0 \Rightarrow z = -48$$

$$x = 3 \text{ and } y = 5 \Rightarrow z = -54.$$

- Cannot move further on the right without going out of the feasible domain.

$$\left. \begin{aligned} 5x + 3y &= 30 \\ x + 3y &= 18 \\ \hline 4x &= 12 \end{aligned} \right\} \Rightarrow \begin{cases} x = 3 \\ 3 + 3y = 18 \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = 5 \end{cases}$$



# Slack variables

- Modify an inequality constraint into an equality constraint using non negative **slack variables**:

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i \quad \rightarrow \quad a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n + y_i = b_i$$
$$y_i \geq 0$$

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geq b_i \quad \rightarrow \quad a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n - y_i = b_i$$
$$y_i \geq 0$$



## Minimum version of the restaurant owner

- Modify the inequality constraints of the restaurant owner problem into equality constraints using the slack variables  $u$ ,  $p$ , and  $h$ :

$$\min z = -8x - 6y$$

s.t.

$$5x + 3y \leq 30$$

$$2x + 3y \leq 24$$

$$1x + 3y \leq 18$$

$$x, y \geq 0$$

$$\min z = -8x - 6y$$

s.t.

$$5x + 3y + u = 30$$

$$2x + 3y + p = 24$$

$$1x + 3y + h = 18$$

$$x, y, u, p, h \geq 0$$

- The constraints are a linear system including 3 equations and 5 variables.
- 3 of the variables can be evaluated in terms of the other 2 variables

# Dictionary Simplex Method

$$\begin{array}{l} \min z = -8x - 6y \\ \text{s.t.} \quad 5x + 3y + u = 30 \\ \quad \quad 2x + 3y + p = 24 \\ \quad \quad 1x + 3y + h = 18 \\ \quad \quad x, y, u, p, h \geq 0 \end{array}$$

- The constraints are a linear system including 3 equations and 5 variables.
- 3 of the variables can be evaluated in terms of the other 2 variables

$$u = 30 - 5x - 3y$$

$$p = 24 - 2x - 3y$$

$$h = 18 - 1x - 3y$$

$$z = 0 - 8x - 6y$$

- Fixing the values of  $x$  and  $y$  induces the values of the other 3 variables.
- It is sufficient to find non negatives values of  $x$  and  $y$  inducing non negatives values of  $u$ ,  $p$  and  $h$  minimizing the value of  $z$ .
- Infinite number of possibilities. We better have a systematic procedure to find the minimum of  $z$ .

## Find the variable to be increased

- A feasible solution of the linear system

$$u = 30 - 5x - 3y$$

$$p = 24 - 2x - 3y$$

$$h = 18 - 1x - 3y$$

$$z = 0 - 8x - 6y$$

is the following

$$x = y = 0 \Rightarrow u = 30, p = 24, h = 18 \text{ et } z = 0.$$

- We can reduce the value of  $z$  if we increase the value of  $x$ , or that of  $y$ , or both.
- In the Simplex method we increase the value of only one
- To minimize  $z$ , it seems better to increase the value of  $x$  since increasing the value of  $x$  by one unit induces reducing the value of  $z$  by 8 units.

## Limit for increasing the variable

- The non negativity of the variables  $u$ ,  $p$  et  $h$  limits the increase of  $x$

$$u = 30 - 5x - 3y \geq 0$$

$$p = 24 - 2x - 3y \geq 0$$

$$h = 18 - 1x - 3y \geq 0$$

- Since the value of  $y$  is fixed to 0, then

$$u = 30 - 5x \geq 0 \Leftrightarrow x \leq 30 / 5 = 6$$

$$p = 24 - 2x \geq 0 \Leftrightarrow x \leq 24 / 2 = 12$$

$$h = 18 - 1x \geq 0 \Leftrightarrow x \leq 18$$

- The solution remains feasible as long as

$$x \leq \min \{6, 12, 18\} = 6.$$

## New solution

- $$\begin{aligned}u &= 30 - 5x - 3y \\p &= 24 - 2x - 3y \\h &= 18 - 1x - 3y \\z &= 0 - 8x - 6y\end{aligned}$$

The solution remains feasible as long as

$$x \leq \min \{6, 12, 18\} = 6.$$

- In order to minimize  $z$ , we select the largest value of  $x$ :  
i.e.,  $x = 6$ .
- The new solution becomes

$$x = 6, y = 0 \Rightarrow u = 0, p = 12, h = 12 \text{ et } z = -48.$$

## New iteration

- $$\begin{aligned} u &= 30 - 5x - 3y & 5x + 3y + 1u &= 30 \\ p &= 24 - 2x - 3y & 2x + 3y + 1p &= 24 \\ h &= 18 - 1x - 3y & 1x + 3y + 1h &= 18 \\ z &= 0 - 8x - 6y \end{aligned}$$

- The new solution becomes

$$x = 6, y = 0 \Rightarrow u = 0, p = 12, h = 12 \text{ et } z = -48.$$

- This solution is unique for the preceding system when  $y = u = 0$  since the coefficients of the variables  $x, p$  et  $h$  induces a non singular matrix.
- Consequently, to determine another solution, either  $y$  or  $u$  must take a positive value.
- Previously, the analysis was simplified by the fact that the variables  $x$  and  $y$  that could be modified were on the right hand side.

## Obtain an equivalent system

- Modify the system to have  $y$  and  $u$  on the right hand side.
- Use the equation including  $x$  et  $u$  in order to find a relation where  $x$  is a function of  $u$  and  $y$ :
- $$\begin{aligned}u &= 30 - 5x - 3y \Rightarrow 5x = 30 - u - 3y \\p &= 24 - 2x - 3y \\h &= 18 - 1x - 3y \\z &= 0 - 8x - 6y\end{aligned}$$

## Obtain an equivalent system

- Modify the system to have  $y$  and  $u$  on the right hand side.
- Use the equation including  $x$  et  $u$  in order to find a relation where  $x$  is a function of  $u$  and  $y$ :

- $$u = 30 - 5x - 3y \Rightarrow (5x = 30 - u - 3y) \div 5$$
$$\Rightarrow x = 6 - 1/5u - 3/5y$$

$$p = 24 - 2x - 3y$$

$$h = 18 - 1x - 3y$$

$$z = 0 - 8x - 6y$$



## Obtain an equivalent system

- Modify the system to have  $y$  and  $u$  on the right hand side.
- Use the equation including  $x$  et  $u$  in order to find a relation where  $x$  is a function of  $u$  and  $y$ :

$$\bullet \quad u \quad = 30 - 5x - 3y \Rightarrow x = \underline{6 - 1/5u - 3/5y}$$

$$p \quad = 24 - 2x - 3y$$

$$\Rightarrow p = 24 - 2(6 - 1/5u - 3/5y) - 3y$$

$$\Rightarrow p = 12 + 2/5u - 9/5y$$

$$h \quad = 18 - 1x - 3y$$

$$z \quad = 0 \quad - 8x - 6y$$

- Replace  $x$  by its expression in terms of  $u$  and  $y$  in the other equations.

## Obtain an equivalent system

- Modify the system to have  $y$  and  $u$  on the right hand side.
- Use the equation including  $x$  et  $u$  in order to find a relation where  $x$  is a function of  $u$  and  $y$ :

$$\begin{array}{lcl}
 u & = 30 - 5x - 3y & \Rightarrow x = \frac{6 - 1/5u - 3/5y}{p} \\
 p & = 24 - 2x - 3y & \Rightarrow p = 12 + 2/5u - 9/5y \\
 h & = 18 - 1x - 3y & \\
 & & \Rightarrow h = 18 - (6 - 1/5u - 3/5y) - 3y \\
 & & \Rightarrow h = 12 + 1/5u - 12/5y \\
 z & = 0 - 8x - 6y & 
 \end{array}$$

- Replace  $x$  by its expression in terms of  $u$  and  $y$  in the other equations.

## Obtain an equivalent system

- Modify the system to have  $y$  and  $u$  on the right hand side.
- Use the equation including  $x$  et  $u$  in order to find a relation where  $x$  is a function of  $u$  and  $y$ :

- |     |                  |               |                         |                             |                             |
|-----|------------------|---------------|-------------------------|-----------------------------|-----------------------------|
| $u$ | $= 30 - 5x - 3y$ | $\Rightarrow$ | $x$                     | $=$                         | $\frac{6 - 1/5u - 3/5y}{1}$ |
| $p$ | $= 24 - 2x - 3y$ | $\Rightarrow$ | $p$                     | $=$                         | $12 + 2/5u - 9/5y$          |
| $h$ | $= 18 - 1x - 3y$ | $\Rightarrow$ | $h$                     | $=$                         | $12 + 1/5u - 12/5y$         |
| $z$ | $= 0 - 8x - 6y$  |               |                         |                             |                             |
|     |                  | $\Rightarrow$ | $z = 0$                 | $- 8(6 - 1/5u - 3/5y) - 6y$ |                             |
|     |                  | $\Rightarrow$ | $z = -48 + 8/5u - 6/5y$ |                             |                             |

- Replace  $x$  by its expression in terms of  $u$  and  $y$  in the other equations.

# Equivalent system

- We transformed the system

- $$\begin{array}{lcl} u & = 30 - 5x - 3y \Rightarrow & x & = 6 - 1/5u - 3/5y \\ p & = 24 - 2x - 3y \Rightarrow & p & = 12 + 2/5u - 9/5y \\ h & = 18 - 1x - 3y \Rightarrow & h & = 12 + 1/5u - 12/5y \\ z & = 0 - 8x - 6y \Rightarrow & z & = -48 + 8/5u - 6/5y \end{array}$$

## Equivalent system

- We have a new system equivalent to the preceding one (i.e., the two systems have the same set of feasible solutions)
- Note that it is not interesting to increase  $u$  since the value of  $z$  would increase
- We repeat the preceding procedure by increasing the value of  $y$

$$\begin{aligned}x &= 6 - 1/5u - 3/5y \\p &= 12 + 2/5u - 9/5y \\h &= 12 + 1/5u - 12/5y \\z &= -48 + 8/5u - 6/5y\end{aligned}$$

## New iteration

- The non negativity of the variables  $x$ ,  $p$  et  $h$  limits the increase of  $y$  :

$$\begin{aligned}x &= 6 - 1/5u - 3/5y \geq 0 \\p &= 12 + 2/5u - 9/5y \geq 0 \\h &= 12 + 1/5u - 12/5y \geq 0\end{aligned}$$

- Since the value of  $u$  is fixed to 0, then

$$\begin{aligned}x &= 6 - 3/5y \geq 0 \quad \Leftrightarrow \quad y \leq 10 \\p &= 12 - 9/5y \geq 0 \quad \Leftrightarrow \quad y \leq 20/3 \\h &= 12 - 12/5y \geq 0 \quad \Leftrightarrow \quad y \leq 5\end{aligned}$$

- The solution remains feasible as long as

$$y \leq \min \{10, 20/3, 5\} = 5.$$

## New iteration

- $$\begin{aligned}x &= 6 - 1/5u - 3/5y \geq 0 \\p &= 12 + 2/5u - 9/5y \geq 0 \\h &= 12 + 1/5u - 12/5y \geq 0 \\z &= -48 + 8/5u - 6/5y\end{aligned}$$

The solution remains feasible as long as  
 $y \leq \min \{10, 20/3, 5\} = 5.$

- In order to minimize  $z$ , we select the largest value of  $y$ : i.e.,  $y = 5.$
- The new solution is  
 $y = 5, u = 0 \Rightarrow x = 3, p = 3, h = 0 \text{ et } z = -54.$

# Optimal solution

- Modify the system to have  $h$  and  $u$  on the right hand side.
- Use the equation including  $h$  and  $u$  in order to find a relation where  $y$  is a function of  $h$  and  $u$ :

$$h = 12 + 1/5u - 12/5y$$

- Replace  $y$  by its expression in terms of  $u$  and  $h$  in the other equations.
- The system becomes

$$\begin{aligned}x &= 3 - 1/4u + 1/4h \\p &= 3 + 1/4u + 3/4h \\y &= 5 + 1/12u - 5/12h \\z &= -54 + 3/2u + 1/2h\end{aligned}$$

- The solution  $y = 5$ ,  $u = 0$ ,  $x = 3$ ,  $p = 3$ ,  $h = 0$  (where  $z = -54$ ) is then optimal since the coefficients of  $u$  and  $h$  are positive.
- Indeed the value of  $z$  can only increase when the values of  $u$  or  $h$  increase.



It can be shown that the simplex method moves around the feasible domain to identify an optimal solution without ever going inside the feasible domain.

When solving the restaurant owner problem with the simplex method:

The initial solution is

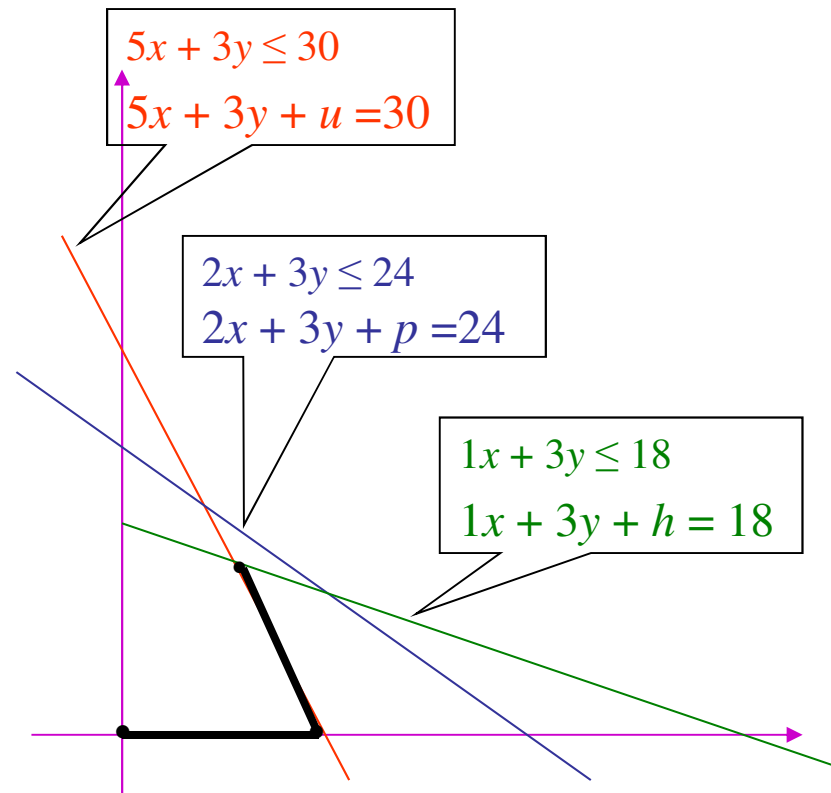
$x = y = 0$  ( $u = 30, p = 24, h = 18$ )  
and the value of  $z = 0$

When increasing the value of  $x$ ,  
the solution becomes

$x = 6, y = 0$  ( $u = 0, p = 12, h = 12$ )  
and the value of  $z = -48$

When increasing the value of  $y$ ,  
the solution becomes

$x = 3, y = 5$  ( $u = 0, p = 3, h = 0$ )  
and the value of  $z = -54$



# Type of solutions encountered in the simplex method

- In all the solutions encountered, only 3 variables are positive!
- Since there are 5 variables, then there exist only  $\binom{5}{3} = \frac{5!}{3!2!} = 10$  different solutions of this type type.
- Can there exists a solution with more than 3 positive variables having a value for  $z$  better than the best solution generated by the simplex.

It can be shown that this is not the case.

# Standard Form

- After modifying the inequality constraints into equality constraints using slack variables, we obtain the **standard form of the problem** where some variables may be slack variables:

$$\begin{aligned} \min \quad & z = c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{Sujet à} \quad & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ & \cdot \quad \quad \cdot \quad \quad \quad \cdot \quad \quad \cdot \\ & \cdot \quad \quad \cdot \quad \quad \quad \cdot \quad \quad \cdot \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

## **Analysis of one iteration**

- To analyse an iteration of the simplex method, suppose that after completing some iterations of the procedure, the variables  $x_1, x_2, \dots, x_m$  are function of the other variables.

# The system

- The system is as follows:

$$\begin{array}{rcl}
 x_1 + & \bar{a}_{1m+1}x_{m+1} + \dots + \bar{a}_{1s}x_s + \dots + \bar{a}_{1n}x_n & = \bar{b}_1 \\
 x_2 + & \bar{a}_{2m+1}x_{m+1} + \dots + \bar{a}_{2s}x_s + \dots + \bar{a}_{2n}x_n & = \bar{b}_2 \\
 \cdot & \cdot & \cdot \\
 x_r + & \bar{a}_{rm+1}x_{m+1} + \dots + \bar{a}_{rs}x_s + \dots + \bar{a}_{rn}x_n & = \bar{b}_r \\
 \cdot & \cdot & \cdot \\
 x_m + & \bar{a}_{mm+1}x_{m+1} + \dots + \bar{a}_{ms}x_s + \dots + \bar{a}_{mn}x_n & = \bar{b}_m \\
 & \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n & = \bar{z} - \bar{z}
 \end{array}$$

- The variables  $x_1, x_2, \dots, x_m$  are **dependent variables** of the values of the other variables that are the **independent variables**.

- The variables  $x_1, x_2, \dots, x_m$  are **dependent variables** of the values of the other variables that are the **independent variables**.
- At each iteration, we transform the system in order to maintain the non negativity of the right hand terms, and hence the **dependent variables** are non negative when the values of the **independent variables** are equal to 0.

$$\begin{array}{rcl}
 x_1 + & & = \bar{b}_1 \\
 & x_2 + & = \bar{b}_2 \\
 & \cdot & \cdot \\
 & & x_r + & = \bar{b}_r \\
 & \cdot & \cdot \\
 & & x_m & = \bar{b}_m
 \end{array}$$

# The system

- The system is as follows

$$\begin{array}{rcl}
 x_1 + & + \bar{a}_{1m+1}x_{m+1} + \dots + \bar{a}_{1s}x_s + \dots + \bar{a}_{1n}x_n & = \bar{b}_1 \\
 x_2 + & + \bar{a}_{2m+1}x_{m+1} + \dots + \bar{a}_{2s}x_s + \dots + \bar{a}_{2n}x_n & = \bar{b}_2 \\
 \cdot & \cdot & \cdot \\
 x_r + & + \bar{a}_{rm+1}x_{m+1} + \dots + \bar{a}_{rs}x_s + \dots + \bar{a}_{rn}x_n & = \bar{b}_r \\
 \cdot & \cdot & \cdot \\
 x_m + & + \bar{a}_{mm+1}x_{m+1} + \dots + \bar{a}_{ms}x_s + \dots + \bar{a}_{mn}x_n & = \bar{b}_m \\
 & + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n & = z - \bar{z}
 \end{array}$$

# The system

- Move the independent variables on the right hand side:

$$\begin{array}{rcl}
 x_1 & = & \bar{b}_1 - \bar{a}_{1m+1}x_{m+1} - \dots - \bar{a}_{1s}x_s - \dots - \bar{a}_{1n}x_n \\
 x_2 & = & \bar{b}_2 - \bar{a}_{2m+1}x_{m+1} - \dots - \bar{a}_{2s}x_s - \dots - \bar{a}_{2n}x_n \\
 \cdot & & \cdot \\
 x_r & = & \bar{b}_r - \bar{a}_{rm+1}x_{m+1} - \dots - \bar{a}_{rs}x_s - \dots - \bar{a}_{rn}x_n \\
 \cdot & & \cdot \\
 x_m & = & \bar{b}_m - \bar{a}_{mm+1}x_{m+1} - \dots - \bar{a}_{ms}x_s - \dots - \bar{a}_{mn}x_n \\
 z & = & \bar{z} + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n
 \end{array}$$



## Step 1: Select the entering variable

- To select the variable to be increased (the **entering variable**), we look at the z equation

$$\begin{array}{rcl}
 x_1 & = & \bar{b}_1 - \bar{a}_{1m+1}x_{m+1} - \dots - \bar{a}_{1s}x_s - \dots - \bar{a}_{1n}x_n \\
 x_2 & = & \bar{b}_2 - \bar{a}_{2m+1}x_{m+1} - \dots - \bar{a}_{2s}x_s - \dots - \bar{a}_{2n}x_n \\
 & \cdot & \cdot \cdot \cdot \\
 x_r & = & \bar{b}_r - \bar{a}_{rm+1}x_{m+1} - \dots - \bar{a}_{rs}x_s - \dots - \bar{a}_{rn}x_n \\
 & \cdot & \cdot \cdot \cdot \\
 x_m & = & \bar{b}_m - \bar{a}_{mm+1}x_{m+1} - \dots - \bar{a}_{ms}x_s - \dots - \bar{a}_{mn}x_n \\
 & & \boxed{z = \bar{z} + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n}
 \end{array}$$

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- To select the variable to be increased (the **entering variable**), we look at the z equation

$$\begin{array}{rcl}
 x_1 & = & \bar{b}_1 - \bar{a}_{1m+1}x_{m+1} - \dots - \bar{a}_{1s}x_s - \dots - \bar{a}_{1n}x_n \\
 x_2 & = & \bar{b}_2 - \bar{a}_{2m+1}x_{m+1} - \dots - \bar{a}_{2s}x_s - \dots - \bar{a}_{2n}x_n \\
 & \cdot & \cdot \qquad \qquad \qquad \cdot \\
 x_r & = & \bar{b}_r - \bar{a}_{rm+1}x_{m+1} - \dots - \bar{a}_{rs}x_s - \dots - \bar{a}_{rn}x_n \\
 & \cdot & \cdot \qquad \qquad \qquad \cdot \\
 x_m & = & \bar{b}_m - \bar{a}_{mm+1}x_{m+1} - \dots - \bar{a}_{ms}x_s - \dots - \bar{a}_{mn}x_n \\
 z & = & \bar{z} + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n
 \end{array}$$

- Denote  $\bar{c}_s = \min_{1 \leq j \leq n} \{ \bar{c}_j \}$

# Step 1: Select the entering variable

- To select the variable to be increased (the **entering variable**), we look at the z equation

$$\begin{aligned}
 x_1 &= \bar{b}_1 - \bar{a}_{1m+1}x_{m+1} - \dots - \bar{a}_{1s}x_s - \dots - \bar{a}_{1n}x_n \\
 x_2 &= \bar{b}_2 - \bar{a}_{2m+1}x_{m+1} - \dots - \bar{a}_{2s}x_s - \dots - \bar{a}_{2n}x_n \\
 &\cdot \\
 &\cdot \\
 x_r &= \bar{b}_r - \bar{a}_{rm+1}x_{m+1} - \dots - \bar{a}_{rs}x_s - \dots - \bar{a}_{rn}x_n \\
 &\cdot \\
 &\cdot \\
 x_m &= \bar{b}_m - \bar{a}_{mm+1}x_{m+1} - \dots - \bar{a}_{ms}x_s - \dots - \bar{a}_{mn}x_n \\
 z &= \bar{z} + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n
 \end{aligned}$$

If  $\bar{c}_s \geq 0$ , then the solution is optimal, and the algorithm stops

- Denote

$$\bar{c}_s = \min_{1 \leq j \leq n} \{ \bar{c}_j \}$$

## Step 1: Select the entering variable

- To select the variable to be increased (the **entering variable**), we look at the z equation

$$\begin{aligned}
 x_1 &= \bar{b}_1 - \bar{a}_{1m+1}x_{m+1} - \dots - \bar{a}_{1s}x_s - \dots - \bar{a}_{1n}x_n \\
 x_2 &= \bar{b}_2 - \bar{a}_{2m+1}x_{m+1} - \dots - \bar{a}_{2s}x_s - \dots - \bar{a}_{2n}x_n \\
 &\vdots \\
 x_{m+1} &= \bar{b}_{m+1} - \bar{a}_{m+1m+1}x_{m+1} - \dots - \bar{a}_{m+1s}x_s - \dots - \bar{a}_{m+1n}x_n \\
 &\vdots \\
 x_n &= \bar{b}_n - \bar{a}_{nm+1}x_{m+1} - \dots - \bar{a}_{ns}x_s - \dots - \bar{a}_{nn}x_n \\
 z &= \bar{z} + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n
 \end{aligned}$$

If  $\bar{c}_s < 0$ , then the variable  $x_s$  becomes the **entering variable**. We move to Step 2.

- Denote  $\bar{c}_s = \min_{1 \leq j \leq n} \{ \bar{c}_j \}$

## Step 2: Select the leaving variable

- We have to identify the largest value of the **entering variable** for the new solution to remain feasible.
- In fact, the increase of the entering variable is limited by the first dependent variable becoming equal to 0. This variable is denoted as the **leaving variable**.
- To identify the largest value of the **entering variable**, we refer to the preceding system of equations:

## Step 2: Select the leaving variable

$$\begin{array}{rcl}
 x_1 & = & \bar{b}_1 - \bar{a}_{1s} x_s \\
 x_2 & = & \bar{b}_2 - \bar{a}_{2s} x_s \\
 \cdot & & \cdot \\
 x_r & = & \bar{b}_r - \bar{a}_{rs} x_s \\
 \cdot & & \cdot \\
 x_m & = & \bar{b}_m - \bar{a}_{ms} x_s \\
 z & = & \bar{z} + \bar{c}_s x_s
 \end{array}$$

- Since the values of the other independent variables remain equal to 0, we can eliminate them from our evaluation

## Step 2: Select the leaving variable

- The conditions insuring that the new solution remains feasible are as follows:

$$x_1 = \bar{b}_1 - \bar{a}_{1s} x_s \geq 0$$

$$x_2 = \bar{b}_2 - \bar{a}_{2s} x_s \geq 0$$

· · ·

$$x_r = \bar{b}_r - \bar{a}_{rs} x_s \geq 0$$

· · ·

$$x_m = \bar{b}_m - \bar{a}_{ms} x_s \geq 0$$

- Two different cases must be analysed.

## Step 2: Select the leaving variable

- The conditions insuring that the new solution remains feasible are as follows:

$$\begin{array}{rcl}
 x_1 & & = \bar{b}_1 - \bar{a}_{1s} x_s \geq 0 \\
 x_2 & & = \bar{b}_2 - \bar{a}_{2s} x_s \geq 0 \\
 \cdot & \cdot & \cdot \\
 x_r & & = \bar{b}_r - \bar{a}_{rs} x_s \geq 0 \\
 \cdot & \cdot & \cdot \\
 x_m & & = \bar{b}_m - \bar{a}_{ms} x_s \geq 0
 \end{array}$$

If  $\bar{a}_{is} \leq 0 \quad \forall 1 \leq i \leq m$  then the value of the entering variable  $x_s$  can be increased to infinity and the values of all dependent variables remain non negative. Indeed, the value of each dependent variable  $x_i$  increases (if  $\bar{a}_{is} < 0$ ) or remains the same (if  $\bar{a}_{is} = 0$ ).

In this case, the algorithm stops indicating that the problem is not bounded below



## Step 2: Select the leaving variable

- The conditions insuring that the new follows:

$$\begin{array}{rcl} x_1 & = & \bar{b}_1 - \bar{a}_{1s} x_s \geq 0 \\ & & \\ x_2 & = & \bar{b}_2 - \bar{a}_{2s} x_s \geq 0 \\ & \cdot & \cdot \\ & & \\ x_r & = & \bar{b}_r - \bar{a}_{rs} x_s \geq 0 \\ & \cdot & \cdot \\ & & \\ x_m & = & \bar{b}_m - \bar{a}_{ms} x_s \geq 0 \end{array}$$

In the second case, if  $\bar{a}_{is} > 0$  for at least one index  $i$ , then the increase of the value of  $x_s$  is limited by the value when a first dependent variable reduces to 0.

Note that only the variables  $x_i$  where  $\bar{a}_{is} > 0$  are relevant. Indeed, if  $\bar{a}_{is} \leq 0$ , then we just observe that the value of  $x_i$  increases or remains the same, and it cannot limit the increase of the **entering variable**  $x_s$ .

## Step 2: Select the leaving variable

- The conditions insuring that the new solution remains feasible are as follows:

$$\begin{array}{rcl} x_1 & = & \bar{b}_1 - \bar{a}_{1s} x_s \geq 0 \\ & & \\ x_2 & = & \bar{b}_2 - \bar{a}_{2s} x_s \geq 0 \\ & \cdot & \cdot \cdot \\ & x_r & = \bar{b}_r - \bar{a}_{rs} x_s \geq 0 \\ & \cdot & \cdot \cdot \\ & x_m & = \bar{b}_m - \bar{a}_{ms} x_s \geq 0 \end{array}$$

The solution remains feasible

$$\Leftrightarrow \forall i \text{ such that } \bar{a}_{is} > 0$$

$$x_i = \bar{b}_i - \bar{a}_{is} x_s \geq 0 \quad \Leftrightarrow \quad x_s \leq \frac{\bar{b}_i}{\bar{a}_{is}}$$

## Step 2: Select the leaving variable

- The conditions insuring that the new solution remains feasible are as follows:

$$\begin{array}{rcl}
 x_1 & = & \bar{b}_1 - \bar{a}_{1s}x_s \geq 0 \\
 x_2 & = & \bar{b}_2 - \bar{a}_{2s}x_s \geq 0 \\
 \cdot & \cdot & \cdot \\
 x_r & = & \bar{b}_r - \bar{a}_{rs}x_s \geq 0 \\
 \cdot & \cdot & \cdot \\
 x_m & = & \bar{b}_m - \bar{a}_{ms}x_s \geq 0
 \end{array}$$

The solution remains feasible

$$\Leftrightarrow \forall i \text{ such that } \bar{a}_{is} > 0$$

$$x_i = \bar{b}_i - \bar{a}_{is}x_s \geq 0 \Leftrightarrow x_s \leq \frac{\bar{b}_i}{\bar{a}_{is}}$$

Consequently, the largest value of the **entering variable**  $x_s$  is

$$x_s = \frac{\bar{b}_r}{\bar{a}_{rs}} = \min_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} : \bar{a}_{is} > 0 \right\}$$

## Step 2: Select the leaving variable

- The conditions insuring that the new solution remains feasible are as follows:

$$\begin{array}{rcl}
 x_1 & & = \bar{b}_1 - \bar{a}_{1s} x_s \geq 0 \\
 & x_2 & = \bar{b}_2 - \bar{a}_{2s} x_s \geq 0 \\
 & \cdot & \cdot \\
 & x_r & = \bar{b}_r - \bar{a}_{rs} x_s \geq 0 \\
 & \cdot & \cdot \\
 & x_m & = \bar{b}_m - \bar{a}_{ms} x_s \geq 0
 \end{array}$$

The independent variable  $x_r$  limiting the increase of the **entering variable**  $x_s$  is the **leaving variable**.

The solution remains feasible

$$\Leftrightarrow \forall i \text{ such that } \bar{a}_{is} > 0$$

$$x_i = \bar{b}_i - \bar{a}_{is} x_s \geq 0 \quad \Leftrightarrow \quad x_s \leq \frac{\bar{b}_i}{\bar{a}_{is}}$$

Consequently, the largest value of the **entering variable**  $x_s$  is

$$x_s = \frac{\bar{b}_r}{\bar{a}_{rs}} = \min_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} : \bar{a}_{is} > 0 \right\}$$

## **Step 3: Pivot to transform the system**

## Step 3: Pivot to transform the system

- Transform the system :

$$\begin{array}{rcl}
 x_1 & = & \bar{b}_1 - \bar{a}_{1m+1}x_{m+1} - \dots - \bar{a}_{1s}x_s - \dots - \bar{a}_{1n}x_n \\
 x_2 & = & \bar{b}_2 - \bar{a}_{2m+1}x_{m+1} - \dots - \bar{a}_{2s}x_s - \dots - \bar{a}_{2n}x_n \\
 \cdot & & \cdot \\
 x_r & = & \bar{b}_r - \bar{a}_{rm+1}x_{m+1} - \dots - \bar{a}_{rs}x_s - \dots - \bar{a}_{rn}x_n \\
 \cdot & & \cdot \\
 x_m & = & \bar{b}_m - \bar{a}_{mm+1}x_{m+1} - \dots - \bar{a}_{ms}x_s - \dots - \bar{a}_{mn}x_n \\
 z & = & \bar{z} + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n
 \end{array}$$

- to take the **entering variable**  $x_s$  on the left replacing the **leaving variable**  $x_r$  and vice-versa.

## Step 3: Pivot to transform the system

- Indeed exchange the role of the variables  $x_s$  et  $x_r$  because
  - the **entering variable**  $x_s$  (being an independent variable with a 0 value) becomes a dependent variable with a non negative value
  - The **leaving variable**  $x_r$  (being a dependent variable with a non negative value) becomes an independent variable with a 0 value
- The set of operations to complete the transformation is referred to as the **pivot**

## Step 3: Pivot to transform the system

$$x_1 = \bar{b}_1 - \bar{a}_{1m+1}x_{m+1} - \dots - \bar{a}_{1s}x_s - \dots - \bar{a}_{1n}x_n$$

$$x_2 = \bar{b}_2 - \bar{a}_{2m+1}x_{m+1} - \dots - \bar{a}_{2s}x_s - \dots - \bar{a}_{2n}x_n$$

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$$x_r = \bar{b}_r - \bar{a}_{rm+1}x_{m+1} - \dots - \bar{a}_{rs}x_s - \dots - \bar{a}_{rn}x_n$$

·

$$x_m = \bar{b}_m - \bar{a}_{mm+1}x_{m+1} - \dots - \bar{a}_{ms}x_s - \dots - \bar{a}_{mn}x_n$$

$$z = \bar{z} + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n$$

$$x_s = \frac{\bar{b}_r}{\bar{a}_{rs}} - \frac{\bar{a}_{rm+1}}{\bar{a}_{rs}}x_{m+1} - \dots - \frac{1}{\bar{a}_{rs}}x_r - \dots - \frac{\bar{a}_{rn}}{\bar{a}_{rs}}x_n$$

Use the  $r^{\text{th}}$  equation to specify  $x_s$  in terms of

$x_{m+1}, \dots, x_{s-1}, x_{s+1}, \dots, x_n,$   
 $x_r$



## Step 3: Pivot to transform the system

$$\begin{aligned}
 x_1 &= \bar{b}_1 - \bar{a}_{1m+1}x_{m+1} - \dots - \bar{a}_{1s}x_s - \dots - \bar{a}_{1n}x_n \\
 x_2 &= \bar{b}_2 - \bar{a}_{2m+1}x_{m+1} - \dots - \bar{a}_{2s}x_s - \dots - \bar{a}_{2n}x_n \\
 &\vdots \\
 x_r &= \bar{b}_r - \bar{a}_{rm+1}x_{m+1} - \dots - \bar{a}_{rs}x_s - \dots - \bar{a}_{rn}x_n \\
 &\vdots \\
 x_m &= \bar{b}_m - \bar{a}_{mm+1}x_{m+1} - \dots - \bar{a}_{ms}x_s - \dots - \bar{a}_{mn}x_n \\
 z &= \bar{z} + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n
 \end{aligned}$$

$$x_s = \frac{\bar{b}_r}{\bar{a}_{rs}} - \frac{\bar{a}_{rm+1}}{\bar{a}_{rs}}x_{m+1} - \dots - \frac{1}{\bar{a}_{rs}}x_r - \dots - \frac{\bar{a}_{rn}}{\bar{a}_{rs}}x_n$$

Replace  $x_s$  specified in terms of  $x_{m+1}, \dots, x_{s-1}, x_{s+1}, \dots, x_n, x_r$ , in each of the other equations

$$x_1 = \bar{b}_1 - \bar{a}_{1m+1}x_{m+1} - \dots - \bar{a}_{1s} \left( \frac{\bar{b}_r}{\bar{a}_{rs}} - \frac{\bar{a}_{rm+1}}{\bar{a}_{rs}}x_{m+1} - \dots - \frac{1}{\bar{a}_{rs}}x_r - \dots - \frac{\bar{a}_{rn}}{\bar{a}_{rs}}x_n \right) - \dots - \bar{a}_{1n}x_n$$

$$x_1 = \left( \bar{b}_1 - \bar{a}_{1s} \frac{\bar{b}_r}{\bar{a}_{rs}} \right) - \left( \bar{a}_{1m+1} - \bar{a}_{1s} \frac{\bar{a}_{rm+1}}{\bar{a}_{rs}} \right) x_{m+1} - \dots - \left( -\bar{a}_{1s} - \frac{1}{\bar{a}_{rs}} \right) x_r - \dots - \left( \bar{a}_{1n} - \bar{a}_{1s} \frac{\bar{a}_{rn}}{\bar{a}_{rs}} \right) x_n$$

## Step 3: Pivot to transform the system

$$\begin{aligned}
 x_1 &= \bar{b}_1 - \bar{a}_{1m+1}x_{m+1} - \dots - \bar{a}_{1s}x_s - \dots - \bar{a}_{1n}x_n \\
 x_2 &= \bar{b}_2 - \bar{a}_{2m+1}x_{m+1} - \dots - \bar{a}_{2s}x_s - \dots - \bar{a}_{2n}x_n \\
 &\cdot \\
 &\cdot \\
 x_r &= \bar{b}_r - \bar{a}_{rm+1}x_{m+1} - \dots - \bar{a}_{rs}x_s - \dots - \bar{a}_{rn}x_n \\
 &\cdot \\
 &\cdot \\
 x_m &= \bar{b}_m - \bar{a}_{mm+1}x_{m+1} - \dots - \bar{a}_{ms}x_s - \dots - \bar{a}_{mn}x_n \\
 z &= \bar{z} + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n
 \end{aligned}$$

$$x_s = \frac{\bar{b}_r}{\bar{a}_{rs}} - \frac{\bar{a}_{rm+1}}{\bar{a}_{rs}}x_{m+1} - \dots - \frac{1}{\bar{a}_{rs}}x_r - \dots - \frac{\bar{a}_{rn}}{\bar{a}_{rs}}x_n$$

Replace  $x_s$  specified in terms of  $x_{m+1}, \dots, x_{s-1}, x_{s+1}, \dots, x_n, x_r$ , in each of the other equations

## Step 3: Pivot to transform the system

$$x_1 = \bar{b}_1 - \bar{a}_{1m+1}x_{m+1} - \dots - \bar{a}_{1s}x_s - \dots - \bar{a}_{1n}x_n$$

$$x_2 = \bar{b}_2 - \bar{a}_{2m+1}x_{m+1} - \dots - \bar{a}_{2s}x_s - \dots - \bar{a}_{2n}x_n$$

⋮

$$x_r = \bar{b}_r - \bar{a}_{rm+1}x_{m+1} - \dots - \bar{a}_{rs}x_s - \dots - \bar{a}_{rn}x_n$$

⋮

$$x_m = \bar{b}_m - \bar{a}_{mm+1}x_{m+1} - \dots - \bar{a}_{ms}x_s - \dots - \bar{a}_{mn}x_n$$

$$z = \bar{z} + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n$$

$$x_s = \frac{\bar{b}_r - \bar{a}_{rm+1}x_{m+1} - \dots - \bar{a}_{rn}x_n}{\bar{a}_{rs}}$$

Replace  $x_s$  specified in terms of  $x_{m+1}, \dots, x_{s-1}, x_{s+1}, \dots, x_n, x_r$ , in each of the other equations

## Step 3: Pivot to transform the system

$$x_1 = \bar{b}_1 - \bar{a}_{1m+1}x_{m+1} - \dots - \bar{a}_{1s}x_s - \dots - \bar{a}_{1n}x_n$$

$$x_2 = \bar{b}_2 - \bar{a}_{2m+1}x_{m+1} - \dots - \bar{a}_{2s}x_s - \dots - \bar{a}_{2n}x_n$$

⋮

$$x_r = \bar{b}_r - \bar{a}_{rm+1}x_{m+1} - \dots - \bar{a}_{rs}x_s - \dots - \bar{a}_{rn}x_n$$

⋮

$$x_m = \bar{b}_m - \bar{a}_{mm+1}x_{m+1} - \dots - \bar{a}_{ms}x_s - \dots - \bar{a}_{mn}x_n$$

$$z = \bar{z} + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n$$

$$x_s = \frac{\bar{b}_r}{\bar{a}_{rs}} - \frac{\bar{a}_{rm+1}}{\bar{a}_{rs}}x_{m+1} - \dots - \frac{1}{\bar{a}_{rs}}x_r - \dots - \frac{\bar{a}_{rn}}{\bar{a}_{rs}}x_n$$

Replace  $x_s$  specified in terms of  $x_{m+1}, \dots, x_{s-1}, x_{s+1}, \dots, x_n, x_r$ , in each of the other equations

## Equivalent system for the next iteration

- The **pivot** generates an equivalent system having the following form

$$\begin{array}{rcl}
 x_1 & & = \tilde{b}_1 - \tilde{a}_{1m+1}x_{m+1} - \dots - \tilde{a}_{1r}x_r - \dots - \tilde{a}_{1n}x_n \\
 x_2 & & = \tilde{b}_2 - \tilde{a}_{2m+1}x_{m+1} - \dots - \tilde{a}_{2r}x_r - \dots - \tilde{a}_{2n}x_n \\
 \cdot & & \cdot \\
 \cdot & & \cdot \\
 x_s & & = \tilde{b}_r - \tilde{a}_{rm+1}x_{m+1} - \dots - \tilde{a}_{rr}x_r - \dots - \tilde{a}_{rn}x_n \\
 \cdot & & \cdot \\
 \cdot & & \cdot \\
 x_m & & = \tilde{b}_m - \tilde{a}_{mm+1}x_{m+1} - \dots - \tilde{a}_{mr}x_r - \dots - \tilde{a}_{mn}x_n \\
 z & = & \tilde{z} + \tilde{c}_{m+1}x_{m+1} + \dots + \tilde{c}_r x_r + \dots + \tilde{c}_n x_n
 \end{array}$$

- Using this new system, we complete a new iteration.

# Tableau format of the simplex method

- We now use the tableau format to complete the iterations of the simplex method.
- We illustrate one iteration of the tableau format for solving the restaurant owner problem.

## Recall the problem

$$\min z = -8x - 6y$$

Sujet à

$$5x + 3y + u = 30$$

$$2x + 3y + p = 24$$

$$1x + 3y + h = 18$$

$$x, y, u, p, h \geq 0$$

## Equivalent tableau format for the system

$$\min z = -8x - 6y$$

Subject to

$$5x + 3y + u = 30$$

$$2x + 3y + p = 24$$

$$1x + 3y + h = 18$$

$$x, y, u, p, h \geq 0$$

$$\min z$$

Subject to

$$5x + 3y + u = 30$$

$$2x + 3y + p = 24$$

$$1x + 3y + h = 18$$

$$-8x - 6y - z = 0$$

$$x, y, u, p, h \geq 0$$

$$u = 30 - 5x - 3y$$

$$p = 24 - 2x - 3y$$

$$h = 18 - 1x - 3y$$

$$z = 0 - 8x - 6y$$

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$u$	5	3	1				30
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0



$$\begin{aligned}
 u &= 30 - 5x - 3y \\
 p &= 24 - 2x - 3y \\
 h &= 18 - 1x - 3y \\
 z &= 0 - 8x - 6y
 \end{aligned}$$

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$u$	5	3	1				30
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

Step 1: Entering criterion

$$\bar{c}_s = \min_{1 \leq j \leq n} \{ \bar{c}_j \}$$

Determine the **entering variable**  
 by selecting the smallest element  
 in the last row of the tableau

$$\min \{-8, -6, 0, 0, 0\} = -8.$$

**$x$  is then the entering variable**

$$u = 30 - 5x - 3y$$

$$p = 24 - 2x - 3y$$

$$h = 18 - 1x - 3y$$

$$z = 0 - 8x - 6y$$

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$u$	5	3	1				30
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

Step 2: leaving criterion

$$x_s = \frac{\bar{b}_r}{a_{rs}} = \min_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i}{a_{is}} : a_{is} > 0 \right\}$$

entering variable

To identify the leaving variable determine the min of the ratio right hand side terms divided by the corresponding elements in the column of the entering variable that are positive:

$$\begin{aligned}
 u &= 30 - 5x - 3y \\
 p &= 24 - 2x - 3y \\
 h &= 18 - 1x - 3y \\
 z &= 0 - 8x - 6y
 \end{aligned}$$

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$u$	5	3	1				30
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

Step 2: leaving criterion

$$x_s = \frac{\bar{b}_r}{a_{rs}} = \min_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i}{a_{is}} : a_{is} > 0 \right\}$$

entering variable

$$\min \{30/5, 24/2, 18\} = 30/5 = 6$$

The corresponding variable  $u$   
becomes the leaving variable

$$u = 30 - 5x - 3y$$

$$p = 24 - 2x - 3y$$

$$h = 18 - 1x - 3y$$

$$z = 0 - 8x - 6y$$

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$u$	5	3	1				30
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

leaving variable

entering variable

Step 3 : Pivot

Transform the system or  
the tableau

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$u$	5	3	1				30
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

← leaving variable

↖ entering variable

RECALL: We use the equation including variable  $x$  and  $u$  to specify  $x$  in terms of  $u$  and  $y$ :

$$\begin{aligned}
 u &= 30 - 5x - 3y \Rightarrow (5x = 30 - u - 3y) / 5 \\
 &\Rightarrow x = 6 - 1/5u - 3/5y
 \end{aligned}$$

This is equivalent to

$$5x + 3y + u = 30$$

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$u$	5	3	1				30
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

← leaving variable

← entering variable

RECALL: We use the equation including variable  $x$  and  $u$  to specify  $x$  in terms of  $u$  and  $y$ :

$$\begin{aligned}
 u &= 30 - 5x - 3y \Rightarrow (5x = 30 - u - 3y) / 5 \\
 &\Rightarrow x = 6 - 1/5u - 3/5y
 \end{aligned}$$

This is equivalent to

$$(5x + 3y + u = 30) / 5$$

dep.var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$u$	5	3	1				30
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

← leaving variable

← entering variable


RECALL: We use the equation including variable  $x$  and  $u$  to specify  $x$  in terms of  $u$  and  $y$ :


$$\begin{aligned}
 u &= 30 - 5x - 3y \Rightarrow (5x = 30 - u - 3y) / 5 \\
 &\Rightarrow x = 6 - 1/5u - 3/5y
 \end{aligned}$$

This is equivalent to

$$(5x + 3y + u = 30) / 5 \Rightarrow x + 3/5y + 1/5u = 6$$

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$u$	5	3	1				30
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0


leaving variable


entering variable



This is equivalent to

$$(5x + 3y + u = 30) / 5 \Rightarrow x + 3/5y + 1/5u = 6$$

In the tableau, this is equivalent to divide the row including the leaving variable by the coefficient of the entering variable in this row



dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$u$	5	3	1				30
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0





 Divide this row by 5  

 leaving variable

This is equivalent to

$$(5x + 3y + u = 30) / 5 \Rightarrow x + 3/5y + 1/5u = 6$$

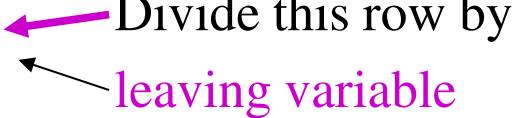
In the tableau, this is equivalent to divide the row including the **leaving variable** by the coefficient of the **entering variable** in this row


dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$u$	5	3	1				30
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

 Divide this row by 5  
 leaving variable  
 entering variable

$$x + \frac{3}{5}y + \frac{1}{5}u = 6$$

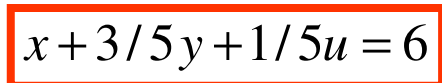
dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$u$	5	3	1				30
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0


 Divide this row by  
 leaving variable


 entering variable

We obtain the following tableau

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$x$	1	$3/5$	$1/5$				6
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0


 $x + 3/5y + 1/5u = 6$

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$x$	1	$3/5$	$1/5$				6
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

- Recall: Replace  $x$  in the other equations

$$p = 24 - 2x - 3y \quad \overset{x}{=} \underline{6 - 1/5u - 3/5y}$$

$$\Rightarrow p = 24 - 2(6 - 1/5u - 3/5y) - 3y$$

This is equivalent to :  $p = 24 - 2(6 - 1/5u - 3/5y) + 2x - 2x - 3y$

$$\Leftrightarrow 2x + 3y + p - 2(x + 3/5y + 1/5u) = 24 - 2(6)$$

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$x$	1	$3/5$	$1/5$				6
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

second row  
minus  
2(the first row)

This is equivalent to :  $p = 24 - 2(6 - 1/5u - 3/5y) + 2x - 2x - 3y$

$$\Leftrightarrow 2x + 3y + p - 2(x + 3/5y + 1/5u) = 24 - 2(6)$$

$$\Leftrightarrow \begin{array}{r} 2x + 3y \qquad \qquad \qquad + p = 24 \\ - 2(x + 3/5y + 1/5u) = 6 \\ \hline 0x + 9/5y - 2/5u + p = 12 \end{array}$$

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$x$	1	$3/5$	$1/5$				6
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

second row  
minus  
2(the first row)

$$0x + 9/5y - 2/5u + p = 12$$

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$x$	1	$3/5$	$1/5$				6
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

second row  
minus  
2(the first row)

The tableau is modified as follows

$$0x + 9/5y - 2/5u + p = 12$$

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$x$	1	$3/5$	$1/5$				6
$p$		$9/5$	$-2/5$	1			12
$h$	1	3			1		18
$-z$	-8	-6				1	0

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	r.h.s.
$x$	1	$3/5$	$1/5$				6
$p$	2	3		1			24
$h$	1	3			1		18
$-z$	-8	-6				1	0

Doing this for the other rows of the tableau

dep. var.	$x$	$y$	$u$	$p$	$h$	$-z$	t	r.h.s.
$x$	1	$3/5$	$1/5$					6
$p$		$9/5$	$-2/5$	1				12
$h$		$12/5$	$-1/5$		1			12
$-z$		$-6/5$	$8/5$			1		48



# Tableau format of the simplex method analysis of one iteration

- Analyse one iteration of the tableau format of the simplex method
- The system

$$\begin{array}{rcl}
 x_1 + & \bar{a}_{1m+1}x_{m+1} + \dots + \bar{a}_{1s}x_s + \dots + \bar{a}_{1n}x_n & = \bar{b}_1 \\
 x_2 + & \bar{a}_{2m+1}x_{m+1} + \dots + \bar{a}_{2s}x_s + \dots + \bar{a}_{2n}x_n & = \bar{b}_2 \\
 \cdot & \cdot & \cdot \\
 x_r + & \bar{a}_{rm+1}x_{m+1} + \dots + \bar{a}_{rs}x_s + \dots + \bar{a}_{rn}x_n & = \bar{b}_r \\
 \cdot & \cdot & \cdot \\
 x_m + & \bar{a}_{mm+1}x_{m+1} + \dots + \bar{a}_{ms}x_s + \dots + \bar{a}_{mn}x_n & = \bar{b}_m \\
 & \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n & = z - \bar{z}
 \end{array}$$

$$\begin{array}{r}
x_1 + \quad \quad \quad + \bar{a}_{1m+1}x_{m+1} + \dots + \bar{a}_{1s}x_s + \dots + \bar{a}_{1n}x_n = \bar{b}_1 \\
x_2 + \quad \quad \quad + \bar{a}_{2m+1}x_{m+1} + \dots + \bar{a}_{2s}x_s + \dots + \bar{a}_{2n}x_n = \bar{b}_2 \\
\vdots \\
x_r + \quad \quad \quad + \bar{a}_{rm+1}x_{m+1} + \dots + \bar{a}_{rs}x_s + \dots + \bar{a}_{rn}x_n = \bar{b}_r \\
\vdots \\
x_m + \bar{a}_{mm+1}x_{m+1} + \dots + \bar{a}_{ms}x_s + \dots + \bar{a}_{mn}x_n = \bar{b}_m \\
\bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_s x_s + \dots + \bar{c}_n x_n = z - \bar{z}
\end{array}$$

can be written in the following tableau

dep. var.	$x_1$	$x_2$	$\dots$	$x_r$	$\dots$	$x_m$	$x_{m+1}$	$\dots$	$x_s$	$\dots$	$x_n$	$-z$	r.h.s.
$x_1$	1						$\bar{a}_{1m+1}$	$\dots$	$\bar{a}_{1s}$	$\dots$	$\bar{a}_{1n}$		$\bar{b}_1$
$x_2$		1					$\bar{a}_{2m+1}$	$\dots$	$\bar{a}_{2s}$	$\dots$	$\bar{a}_{2n}$		$\bar{b}_2$
$\vdots$			$\ddots$						$\ddots$				$\vdots$
$x_r$				1			$\bar{a}_{rm+1}$	$\dots$	$\bar{a}_{rs}$	$\dots$	$\bar{a}_{rn}$		$\bar{b}_r$
$\vdots$					$\ddots$				$\ddots$				$\vdots$
$x_m$						1	$\bar{a}_{mm+1}$	$\dots$	$\bar{a}_{ms}$	$\dots$	$\bar{a}_{mn}$		$\bar{b}_m$
$-z$							$\bar{c}_{m+1}$	$\dots$	$\bar{c}_s$	$\dots$	$\bar{c}_n$	1	$-\bar{z}$

# Step1: Select the entering variable

- Referring to the last row of the tableau, let  $\bar{c}_s = \min_{1 \leq j \leq n} \{\bar{c}_j\}$

<p>If <math>\bar{c}_s \geq 0</math>, then the current solution is optimal, and the algorithm stops</p>						Entering variable		
	$\dots$	$x_{m+1}$	$\dots$	$x_s$	$\dots$	$x_n$	$-z$	r.h.s.
		$\bar{a}_{1m+1}$	$\dots$	$\bar{a}_{1s}$	$\dots$	$\bar{a}_{1n}$		$\bar{b}_1$
		$\bar{a}_{2m+1}$	$\dots$	$\bar{a}_{2s}$	$\dots$	$\bar{a}_{2n}$		$\bar{b}_2$
			$\ddots$		$\ddots$			$\vdots$
		$\bar{a}_{r m+1}$	$\dots$	$\bar{a}_{r s}$	$\dots$	$\bar{a}_{r n}$		$\bar{b}_r$
			$\ddots$		$\ddots$			$\vdots$
	1	$\bar{a}_{m m+1}$	$\dots$	$\bar{a}_{m s}$	$\dots$	$\bar{a}_{m n}$		$\bar{b}_m$
$-z$		$\bar{c}_{m+1}$	$\dots$	$\bar{c}_s$	$\dots$	$\bar{c}_n$	1	$-\bar{z}$

## Step 2: Select the leaving variable

If  $\bar{a}_{is} \leq 0 \quad \forall 1 \leq i \leq m$ ,  
the problem is not  
bounded below,  
and the alg. stops

If  $\exists i$  such that  $\bar{a}_{is} > 0$ ,  
then the sol. remains feasible  $\Leftrightarrow$

$\forall i$  such that  $\bar{a}_{is} > 0$

$$x_i = \bar{b}_i - \bar{a}_{is} x_s \geq 0 \quad \Leftrightarrow \quad x_s \leq \frac{\bar{b}_i}{\bar{a}_{is}}$$

$\dots x_r \dots x_m \quad x_{m+1} \dots x_s \dots x_n \quad -z$	r.h.s.
$\bar{a}_{1m+1} \dots \bar{a}_{1s} \dots \bar{a}_{1n}$	$\bar{b}_1$
$\bar{a}_{2m+1} \dots \bar{a}_{2s} \dots \bar{a}_{2n}$	$\bar{b}_2$
$\vdots$	$\vdots$
$\bar{a}_{rm+1} \dots \bar{a}_{rs} \dots \bar{a}_{rn}$	$\bar{b}_r$
$\vdots$	$\vdots$
$\bar{a}_{mm+1} \dots \bar{a}_{ms} \dots \bar{a}_{mn}$	$\bar{b}_m$

Entering variable

The entering variable  $x_s$  takes the value

$$x_s = \frac{\bar{b}_r}{\bar{a}_{rs}} = \min_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} : \bar{a}_{is} > 0 \right\}$$

## Step 2: Select the leaving variable

dep. var.	$x_1$	$x_2$	$\dots$	$x_r$	$\dots$	$x_m$	$x_{m+1}$	$\dots$	$x_s$	$\dots$	$x_n$	$-z$	r.h.s.
$x_1$	1						$\bar{a}_{1m+1}$	$\dots$	$\bar{a}_{1s}$	$\dots$	$\bar{a}_{1n}$		$\bar{b}_1$
$x_2$		1					$\bar{a}_{2m+1}$	$\dots$	$\bar{a}_{2s}$	$\dots$	$\bar{a}_{2n}$		$\bar{b}_2$
$\vdots$			$\ddots$						$\ddots$				$\vdots$
$x_r$				1			$\bar{a}_{rm+1}$	$\dots$	$\bar{a}_{rs}$	$\dots$	$\bar{a}_{rn}$		$\bar{b}_r$
$\vdots$					$\ddots$				$\ddots$				$\vdots$
$x_m$						1	$\bar{a}_{mm+1}$	$\dots$	$\bar{a}_{ms}$	$\dots$	$\bar{a}_{mn}$		$\bar{b}_m$
$-z$							$\bar{c}_{m+1}$	$\dots$	$\bar{c}_s$	$\dots$	$\bar{c}_n$	1	$-\bar{z}$

Leaving variable points to  $x_r$ .  
Entering variable points to  $x_s$ .

## Step 3: Pivot

The pivot element  $\bar{a}_{rs}$  is located at the intersection of the column including the entering variable  $x_s$  and of the row including the leaving variable  $x_r$

Entering variable

Leaving variable

	$x_1$	$x_2$	$\dots$	$x_r$	$\dots$	$x_m$	$x_{m+1}$	$\dots$	$x_s$	$\dots$	$x_n$	$-\bar{a}_{rs}Z$	r.h.s.
$x_1$	1						$\bar{a}_{1m+1}$		$\bar{a}_{1s}$		$\bar{a}_{1n}$		$\bar{b}_1$
$x_2$		1					$\bar{a}_{2m+1}$		$\bar{a}_{2s}$		$\bar{a}_{2n}$		$\bar{b}_2$
$\vdots$			$\ddots$										$\vdots$
$x_r$				1			$\bar{a}_{rm+1}$		$\bar{a}_{rs}$		$\bar{a}_{rn}$		$\bar{b}_r$
$\vdots$					$\ddots$								$\vdots$
$x_m$						1	$\bar{a}_{mm+1}$		$\bar{a}_{ms}$		$\bar{a}_{mn}$		$\bar{b}_m$
$-Z$							$\bar{c}_{m+1}$		$\bar{c}_s$		$\bar{c}_n$	1	$-\bar{z}$

# Step 3: Pivot

Divide row  $r$  by the pivot element  $\bar{a}_{rs}$  to obtain a new line  $r$ .

dep. var.

Variable d'entrée

Variable de sortie

	$x_1$	$x_2$	$\dots$	$x_r$	$\dots$	$x_m$	$x_{m+1}$	$\dots$	$x_s$	$\dots$	$x_n$	$-\bar{a}_{rs}Z$	r.h.s.
$x_1$	1						$\bar{a}_{1m+1}$	$\dots$	$\bar{a}_{1s}$	$\dots$	$\bar{a}_{1n}$		$\bar{b}_1$
$x_2$		1					$\bar{a}_{2m+1}$	$\dots$	$\bar{a}_{2s}$	$\dots$	$\bar{a}_{2n}$		$\bar{b}_2$
$\vdots$													$\vdots$
$x_r$				1			$\bar{a}_{rm+1}$	$\dots$	$\bar{a}_{rs}$	$\dots$	$\bar{a}_{rn}$		$\bar{b}_r$
$\vdots$													$\vdots$
$x_m$					1		$\bar{a}_{mm+1}$	$\dots$	$\bar{a}_{ms}$	$\dots$	$\bar{a}_{mn}$		$\bar{b}_m$
$-Z$							$\bar{c}_{m+1}$	$\dots$	$\bar{c}_s$	$\dots$	$\bar{c}_n$	1	$-\bar{z}$

$$\frac{1}{\bar{a}_{rs}}$$

# Step 3: Pivot

Divide line  $r$  by the pivot element  $\bar{a}_{rs}$  to obtain a new line  $r$ .

dep. var.

Entering variable

Leaving variable

	$x_1$	$x_2$	$\dots$	$x_r$	$\dots$	$x_m$	$x_{m+1}$	$\dots$	$x_s$	$\dots$	$x_n$	$-\bar{a}_{rs}Z$	r.h.s.
$x_1$	1						$\bar{a}_{1m+1}$	$\dots$	$\bar{a}_{1s}$	$\dots$	$\bar{a}_{1n}$		$\bar{b}_1$
$x_2$		1					$\bar{a}_{2m+1}$	$\dots$	$\bar{a}_{2s}$	$\dots$	$\bar{a}_{2n}$		$\bar{b}_2$
$\vdots$			$\ddots$										$\vdots$
$x_r$							$\frac{1}{\bar{a}_{rs}}$	$\dots$	$\frac{\bar{a}_{rm+1}}{\bar{a}_{rs}}$	$\dots$	$\frac{\bar{a}_{rn}}{\bar{a}_{rs}}$		$\frac{\bar{b}_r}{\bar{a}_{rs}}$
$\vdots$							$\bar{a}_{rs}$	$\dots$	$\bar{a}_{rs}$	$\dots$	$\bar{a}_{rs}$		$\bar{a}_{rs}$
$x_m$					1		$\bar{a}_{mm+1}$	$\dots$	$\bar{a}_{ms}$	$\dots$	$\bar{a}_{mn}$		$\bar{b}_m$
$-Z$							$\bar{c}_{m+1}$	$\dots$	$\bar{c}_s$	$\dots$	$\bar{c}_n$	1	$-\bar{z}$



## Step 3: Pivot

Multiply the new line  $r$  by  $\bar{a}_{is}$ , and subtract this from the line  $i$ . This induces that the coefficient of the **entering variable**  $x_s$  to become equal to 0.

	$x_1$	$x_2$	$\dots$	$x_r$	$\dots$	$x_m$	$\dots$	$x_s$	$\dots$	$x_n$	$-\bar{a}_{rs}Z$	r.h.s.
$x_1$	1					$\bar{a}_{1m+1}$		0		$\bar{a}_{1n}$		$\bar{b}_1$
$x_2$		1				$\bar{a}_{2m+1}$		$\bar{a}_{2s}$		$\bar{a}_{2n}$		$\bar{b}_2$
$\vdots$			$\ddots$									$\vdots$
$x_r$				1		$\frac{\bar{a}_{rm+1}}{\bar{a}_{rs}}$		$\dots 1 \dots$		$\frac{\bar{a}_{rn}}{\bar{a}_{rs}}$		$\frac{\bar{b}_r}{\bar{a}_{rs}}$
$\vdots$						$\frac{\bar{a}_{im+1}}{\bar{a}_{rs}}$		$\frac{\bar{a}_{is}}{\bar{a}_{rs}}$		$\frac{\bar{a}_{in}}{\bar{a}_{rs}}$		$\frac{\bar{b}_i}{\bar{a}_{rs}}$
$x_m$					1	$\bar{a}_{mm+1}$		$\bar{a}_{ms}$		$\bar{a}_{mn}$		$\bar{b}_m$
$-Z$						$\bar{c}_{m+1}$		$\bar{c}_s$		$\bar{c}_n$	1	$-\bar{z}$

Leaving variable

Entering variable

## Step 3: Pivot

Multiply the new line  $r$  by  $\frac{1}{\bar{a}_{rs}}$ , and subtract this from the line  $i$ . This induces that the coefficient of the **entering variable**  $x_s$  to become equal to 0.

dep	$x_1$	$x_2$	$\dots$	$x_r$	$\dots$	$x_m$	$\dots$	$x_s$	$\dots$	$x_n$	$-\bar{a}_{rs}Z$	r.h.s.		
$x_1$	1			$\bar{a}_{1m+1}$				$\bar{a}_{1s}$		$\bar{a}_{1n}$		$\bar{b}_1$		
$x_2$		1		$\bar{a}_{2m+1}$				0		$\bar{a}_{2n}$		$\bar{b}_2$		
$\vdots$			$\ddots$									$\vdots$		
$x_r$				$\frac{1}{\bar{a}_{rs}}$				$\frac{\bar{a}_{rm+1}}{\bar{a}_{rs}}$	$\dots 1 \dots$	$\frac{\bar{a}_{rn}}{\bar{a}_{rs}}$		$\frac{\bar{b}_r}{\bar{a}_{rs}}$		
$\vdots$												$\bar{a}_{rs}$		
$x_m$				1				$\bar{a}_{mm+1}$		$\bar{a}_{ms}$		$\bar{b}_m$		
$-Z$								$\bar{c}_{m+1}$		$\bar{c}_s$		$\bar{c}_n$	1	$-\bar{z}$

Leaving variable

Entering variable

## Step 3: Pivot

Multiply the new line  $r$  by  $\frac{1}{\bar{a}_{rs}}$ , and subtract this from the line  $i$ . This induces that the coefficient of the **entering variable**  $x_s$  to become equal to 0.

dep	$x_1$	$x_2$	$\dots$	$x_r$	$\dots$	$x_{m+1}$	$\dots$	$x_s$	$\dots$	$x_n$	$-\bar{a}_{rs}Z$	r.h.s.
$x_1$	1					$\bar{a}_{r1}$		$\bar{a}_{1s}$		$\bar{a}_{1n}$		$\bar{b}_1$
$x_2$		1				$\bar{a}_{r2}$		$\bar{a}_{2s}$		$\bar{a}_{2n}$		$\bar{b}_2$
$\vdots$			$\ddots$									$\vdots$
$x_r$				1		$\frac{1}{\bar{a}_{rs}}$		$\dots 1 \dots$		$\frac{\bar{a}_{rn}}{\bar{a}_{rs}}$		$\frac{\bar{b}_r}{\bar{a}_{rs}}$
$\vdots$						$\frac{\bar{a}_{ri}}{\bar{a}_{rs}}$				$\frac{\bar{a}_{rn}}{\bar{a}_{rs}}$		$\frac{\bar{a}_{rs}}{\bar{a}_{rs}}$
$x_m$					1	$\bar{a}_{rm+1}$		0		$\bar{a}_{rn}$		$\bar{b}_m$
$-Z$						$\bar{c}_{m+1}$		$\bar{c}_s$		$\bar{c}_n$	1	$-\bar{z}$

Leaving variable

Entering variable

## Step 3: Pivot

Multiply the new line  $r$  by  $\frac{1}{\bar{a}_{rs}}$ , and subtract this from the line  $i$ . This induces that the coefficient of the **entering variable**  $x_s$  to become equal to 0.

dep	$x_1$	$x_2$	$\dots$	$x_r$	$\dots$	$x_{m+1}$	$\dots$	$x_s$	$\dots$	$x_n$	$-\bar{a}_{rs}Z$	r.h.s.
$x_1$	1					$\bar{a}_{1,m+1}$		$\bar{a}_{1s}$		$\bar{a}_{1n}$		$\bar{b}_1$
$x_2$		1				$\bar{a}_{2,m+1}$		$\bar{a}_{2s}$		$\bar{a}_{2n}$		$\bar{b}_2$
$\vdots$			$\ddots$									$\vdots$
$x_r$				1		$\frac{1}{\bar{a}_{rs}}$		$\frac{\bar{a}_{rs}}{\bar{a}_{rs}}$	$\dots$	$\frac{\bar{a}_{rn}}{\bar{a}_{rs}}$		$\frac{\bar{b}_r}{\bar{a}_{rs}}$
$\vdots$												$\bar{a}_{rs}$
$x_m$					1	$\bar{a}_{m,m+1}$		$\bar{a}_{ms}$		$\bar{a}_{mn}$		$\bar{b}_m$
$-Z$						$\bar{c}_{m+1}$		0		$\bar{c}_n$	1	$-\bar{z}$

Leaving variable

Entering variable

## New tableau for the next iteration

dep. var.	$x_1$ $x_2$ $\dots$ $x_r$ $\dots$ $x_m$ $x_{m+1}$ $\dots$ $x_s$ $\dots$ $x_n$ $-z$	r.h.s.
$x_1$	1 $\dots$ $\bar{a}_{1r}$ $\dots$ $\bar{a}_{1m+1}$ $\dots$ 0 $\dots$ $\bar{a}_{1n}$	$\bar{b}_1$
$x_2$	$\dots$ 1 $\dots$ $\bar{a}_{2r}$ $\dots$ $\bar{a}_{2m+1}$ $\dots$ 0 $\dots$ $\bar{a}_{2n}$	$\bar{b}_2$
$\vdots$	$\ddots$	$\vdots$
$x_s$	$\dots$ $\bar{a}_{sr}$ $\dots$ $\bar{a}_{sm+1}$ $\dots$ 1 $\dots$ $\bar{a}_{sn}$	$\bar{b}_r$
$\vdots$	$\ddots$	$\vdots$
$x_m$	$\dots$ $\bar{a}_{mr}$ $\dots$ 1 $\bar{a}_{mm+1}$ $\dots$ 0 $\dots$ $\bar{a}_{mn}$	$\bar{b}_m$
$-z$	$\dots$ $\bar{c}_r$ $\dots$ $\bar{c}_{m+1}$ $\dots$ 0 $\dots$ $\bar{c}_n$ 1	$-\bar{z}$

# Matrix notation

# Matrix notation

- The linear programming problem  
in standard form

$$\begin{array}{ll} \min & z = c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{Subject to} & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ & \cdot \quad \quad \cdot \quad \quad \quad \cdot \quad \quad \cdot \\ & \cdot \quad \quad \cdot \quad \quad \quad \cdot \quad \quad \cdot \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{array}$$

## Restaurant owner problem

$$\min z = [-8, -6, 0, 0, 0] \begin{bmatrix} x \\ y \\ u \\ p \\ h \end{bmatrix}$$

s.t.

$$\begin{pmatrix} 5 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ u \\ p \\ h \end{bmatrix} = \begin{bmatrix} 30 \\ 24 \\ 18 \end{bmatrix}$$

$$x, y, u, p, h \geq 0$$

$$\min z = -8x - 6y$$

$$\text{Subject to } 5x + 3y + u = 30$$

$$2x + 3y + p = 24$$

$$1x + 3y + h = 18$$

$$x, y, u, p, h \geq 0$$



# Matrix notation

- The linear programming problem in standard form

$$\begin{aligned} \min \quad & z = c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{Subject to} \quad & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \end{aligned}$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$x_1, x_2, \dots, x_n \geq 0$$

$\begin{aligned} \min \quad & z = c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$
$\begin{aligned} c, x \in R^n, b \in R^m \\ A \text{ a } m \times n \text{ matrix} \end{aligned}$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$c^T = [c_1, \dots, c_n]$$

# Matrix notation

$$\begin{array}{ll} \min & z \\ \text{Subject to} & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \end{array}$$

$\begin{array}{ll} \min & z \\ \text{Subject to} & Ax = b \\ & c^T x - z = 0 \\ & x \geq 0 \end{array}$ <hr style="width: 80%; margin: 5px auto;"/> $c, x \in R^n, b \in R^m$ $A \text{ a } m \times n \text{ matrix}$
--

$$\begin{array}{ll} & \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\ & c_1x_1 + c_2x_2 + \dots + c_nx_n - z = 0 \\ & x_1, x_2, \dots, x_n \geq 0 \end{array}$$

# Matrix notation

- Consider the matrix formulation of the linear programming problem

$$\begin{aligned} \min \quad & z \\ \text{Subject to} \quad & Ax = b \\ & c^T x - z = 0 \\ & x \geq 0 \end{aligned}$$

- Assume that  $m \leq n$  and that the matrix  $A$  is of full rank (i.e.,  $\text{rank}(A) = m$ , or that the rows of  $A$  are linearly independent)
- A sub matrix  $B$  of  $A$  is a **basis of  $A$**  if it is a  $m \times m$  matrix and non singular (i.e,  $B^{-1}$  exists)

## Restaurant owner problem:

$$A = \begin{matrix} & x & y & u & p & h \\ \begin{pmatrix} 5 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Examples of basis:

$$B_1 = \begin{matrix} & u & p & h \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \quad B_2 = \begin{matrix} & x & p & h \\ \begin{pmatrix} 5 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{matrix} \quad B_3 = \begin{matrix} & x & p & y \\ \begin{pmatrix} 5 & 0 & 3 \\ 2 & 1 & 3 \\ 1 & 0 & 3 \end{pmatrix} \end{matrix}$$

# Matrix notation

- A sub matrix  $B$  of  $A$  is a **basis of  $A$**  if it is a  $m \times m$  matrix and non singular (i.e,  $B^{-1}$  exists)
- To ease the presentation, assume that the basis  $B$  includes the first  $m$  columns of  $A$ , and then

$$A = [B : R]$$

Denote also

$$x = \begin{bmatrix} x_B \\ x_R \end{bmatrix} \quad c = \begin{bmatrix} c_B \\ c_R \end{bmatrix}$$

- The original problem can be written as

$$\begin{array}{ll}
\min & z \\
\text{Subject to} & Ax = b \\
& c^T x - z = 0 \\
& x \geq 0
\end{array}$$

$$\begin{array}{ll}
\min & z \\
\text{Subject to} & [B:R] \begin{bmatrix} x_B \\ x_R \end{bmatrix} = b \\
& \begin{bmatrix} c_B^T \\ c_R^T \end{bmatrix} \begin{bmatrix} x_B \\ x_R \end{bmatrix} - z = 0 \\
& x \geq 0
\end{array}$$

$$\min z$$

$$\text{Subject to } Bx_B + Rx_R = b$$

$$c_B^T x_B + c_R^T x_R - z = 0$$

$$x_B, x_R \geq 0$$

$$\min z$$

$$\text{Subject to } [B:R] \begin{bmatrix} x_B \\ x_R \end{bmatrix} = b$$

$$\begin{bmatrix} c_B^T \\ c_R^T \end{bmatrix} \begin{bmatrix} x_B \\ x_R \end{bmatrix} - z = 0$$

$$x \geq 0$$

- Specify  $x_B$  in terms of  $x_R$  using the constraints of the problem

$$Bx_B + Rx_R = b$$

$$B^{-1}(Bx_B + Rx_R) = B^{-1}b$$

$$B^{-1}Bx_B + B^{-1}Rx_R = B^{-1}b$$

$$Ix_B + B^{-1}Rx_R = B^{-1}b$$

- Then

$$Ix_B = -B^{-1}Rx_R + B^{-1}b$$



$$\begin{array}{ll}
\min z & \\
\text{Subject to} & Bx_B + Rx_R = b \\
& c_B^T x_B + c_R^T x_R - z = 0 \\
& x_B, x_R \geq 0
\end{array}$$

Replacing  $x_B$  by its value in terms of  $x_R$  in the objective function

Note that the two problems are equivalent since the second one is obtained from the first one using elementary operations based on a non singular matrix  $B^{-1}$

$$\begin{array}{ll}
\min z & \\
\text{Subject to} & Ix_B + B^{-1}Rx_R = B^{-1}b
\end{array}$$

$$c_B^T (-B^{-1}Rx_R + B^{-1}b) + c_R^T x_R - z = 0$$

$$x_B, x_R \geq 0$$

$$\begin{aligned}
& \min z \\
\text{Subject to} \quad & Ix_B + B^{-1}Rx_R = B^{-1}b \\
& c_B^T(-B^{-1}Rx_R + B^{-1}b) + c_R^Tx_R - z = 0 \\
& x_B, x_R \geq 0
\end{aligned}$$

Combining the coefficients of  $x_R$

$$\begin{aligned}
& \min z \\
\text{Subject to} \quad & Ix_B + B^{-1}Rx_R = B^{-1}b \\
& 0x_B + (c_R^T - c_B^TB^{-1}R)x_R - z = -c_B^TB^{-1}b \\
& x_B, x_R \geq 0
\end{aligned}$$

$$\min z$$

$$\text{Subject to} \quad Ix_B + B^{-1}Rx_R = B^{-1}b$$

$$0x_B + (c_R^T - c_B^T B^{-1}R)x_R - z = -c_B^T B^{-1}b$$

$$x_B, x_R \geq 0$$

The problem can be specified in the following tableau

dep. var.	$x_B^T$	$x_R^T$	$-z$	r.h.s.
$x_B$	$I$	$B^{-1}R$	$0$	$B^{-1}b$
$-z$	$0$	$c_R^T - c_B^T B^{-1}R$	$1$	$-c_B^T B^{-1}b$

$$\begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \boxed{B^{-1}}$$

	$x$	$y$	$u$	$p$	$h$	$-z$	
$u$	5	3	1	0	0	0	30
$p$	2	3	0	1	0	0	24
$h$	1	3	0	0	1	0	18
$-z$	-8	-6	0	0	0	1	0

Tableau of the second iteration

	$x$	$y$	$u$	$p$	$h$	$-z$	
$x$	1	$\frac{3}{5}$	$\frac{1}{5}$	0	0	0	6
$p$	0	$\frac{9}{5}$	$-\frac{2}{5}$	1	0	0	12
$h$	0	$\frac{12}{5}$	$-\frac{1}{5}$	0	1	0	12
$-z$	0	$-\frac{6}{5}$	$\frac{8}{5}$	0	0	1	48

basic var.	$x_B^T$	$x_R^T$	$-z$	r.h.s.
$x_B$	$I$	$B^{-1}R$	$0$	$B^{-1}b$
$-z$	$0$	$c_R^T - c_B^T B^{-1}R$	$1$	$-c_B^T B^{-1}b$

The variables  $x_B$  (denoted as the dependent variables) associated with the columns of the basis  $B$ , are now denoted **basic variables**

The variables  $x_R$  (denoted independent variables) are now denoted **non basic variables**

basic var.	$x_B^T$	$x_R^T$	$-z$	r.h.s.
$x_B$	$I$	$B^{-1}R$	$0$	$B^{-1}b$
$-z$	$0$	$c_R^T - c_B^T B^{-1}R$	$1$	$-c_B^T B^{-1}b$

To obtain the **basic solution** associated with the basis  $B$

$$\text{let } x_R = 0$$

$$\text{it follows that } x_B = B^{-1}b$$

The basic solution is **feasible** if  $x_B \geq 0$ .

basic var.	$x_B^I$	$x_R^I$	$-z$	r.h.s.
$x_B$	$I$	$B^{-1}R$	$0$	$B^{-1}b$
$-z$	$0$	$c_R^I - c_B^I B^{-1}R$	$1$	$-c_B^I B^{-1}b$

Note that this tableau is identical to the one used to illustrate an iteration of the simplex algorithm.

basic var.	$x_B^T$	$x_R^T$	$-z$	r.h.s.
$x_B$	$I$	$B^{-1}R$	0	$B^{-1}b$
$-z$	0	$c_R^T - c_B^T B^{-1}R$	1	$-c_B^T B^{-1}b$

dep. var.	$x_1$	$x_2$	$\dots$	$x_r$	$\dots$	$x_m$	$x_{m+1}$	$\dots$	$x_s$	$\dots$	$x_n$	$-z$	r.h.s.
$x_1$	1						$\bar{a}_{1m+1}$	$\dots$	$\bar{a}_{1s}$	$\dots$	$\bar{a}_{1n}$		$\bar{b}_1$
$x_2$		1					$\bar{a}_{2m+1}$	$\dots$	$\bar{a}_{2s}$	$\dots$	$\bar{a}_{2n}$		$\bar{b}_2$
$\vdots$			$\ddots$						$\ddots$		$\ddots$		$\vdots$
$x_r$				1			$\bar{a}_{rm+1}$	$\dots$	$\bar{a}_{rs}$	$\dots$	$\bar{a}_{rn}$		$\bar{b}_r$
$\vdots$					$\ddots$				$\ddots$		$\ddots$		$\vdots$
$x_m$						1	$\bar{a}_{mm+1}$	$\dots$	$\bar{a}_{ms}$	$\dots$	$\bar{a}_{mn}$		$\bar{b}_m$
$-z$							$\bar{c}_{m+1}$	$\dots$	$\bar{c}_s$	$\dots$	$\bar{c}_n$	1	$-\bar{z}$



basic var.	$x_B^T$	$x_R^T$	$-z$	r.h.s.
$x_B$	$I$	$B^{-1}R$	$0$	$B^{-1}b$
$-z$	$0$	$c_R^T - c_B^T B^{-1}R$	$1$	$-c_B^T B^{-1}b$

Since all tableaus correspond to basis of  $A$  that are specified by the columns associated with the basic variables (dependent variables), it follows that in the simplex algorithm, we move from one basic feasible solution to a new one where the objective function decreases.

# The simplex multipliers

- Consider the last row of the simplex tableau corresponding to the basis  $B$  associated to the **relative costs** of the variables:

basic var.	$x_B^T$	$x_R^T$	$-z$	r.h.s.
$x_B$	$I$	$B^{-1}R$	$0$	$B^{-1}b$
$-z$	$0$	$c_R^T - c_B^T B^{-1}R$	$1$	$-c_B^T B^{-1}b$

$$\bar{c}_B^T$$

$$\bar{c}_R^T$$

$$\bar{c}_B^T = 0$$

$$\bar{c}_R^T = c_R^T - c_B^T B^{-1}R$$

# The simplex multipliers

- Consider the last row of the simplex tableau corresponding to the basis  $B$  associated to the **relative costs** of the variables:

basic var.	$x_B^T$	$x_R^T$	$-z$	r.h.s.
$x_B$	$I$	$B^{-1}R$	$0$	$B^{-1}b$
$-z$	$0$	$c_R^T - c_B^T B^{-1}R$	$1$	$-c_B^T B^{-1}b$

$$\bar{c}_B^T$$

$$\bar{c}_R^T$$

$$\bar{c}_B^T = 0 = c_B^T - c_B^T = c_B^T - c_B^T B^{-1}B$$

$$\bar{c}_R^T = c_R^T - c_B^T B^{-1}R$$

$$\bar{c}^T = \begin{bmatrix} \bar{c}_B^T & \bar{c}_R^T \end{bmatrix} = \begin{bmatrix} c_B^T & c_R^T \end{bmatrix} - c_B^T B^{-1} [B \ : \ R] = c^T - c_B^T B^{-1}A$$

# The simplex multipliers

$$\bar{c}^T = c^T - c_B^T B^{-1} A$$

Denote the vector  $\pi \in R^m$   
specified by

Then 
$$\pi^T = c_B^T B^{-1}$$

$$\bar{c}^T = c^T - \pi^T A$$

or

$$[\bar{c}_1, \dots, \bar{c}_n] = [c_1, \dots, c_n] - \pi^T [a_{\bullet 1}, \dots, a_{\bullet n}]$$

$$\bar{c}_j = c_j - \pi^T a_{\bullet j}$$

where  $a_{\bullet j}$  denotes the  $j$ th column  
of the constraint matrix  $A$

$\pi$  is the simplex multipliers vector  
associated with the basis  $B$ .

The vector  $\pi$  has one element associated  
with each row (constraint) of the tableau.

	$x$	$y$	$u$	$p$	$h$	$-z$	
$x$	1	$\frac{3}{5}$	$\frac{1}{5}$	0	0	0	6
$p$	0	$\frac{9}{5}$	$-\frac{2}{5}$	1	0	0	12
$h$	0	$\frac{12}{5}$	$-\frac{1}{5}$	0	1	0	12
$-z$	0	$-\frac{6}{5}$	$\frac{8}{5}$	0	0	1	48

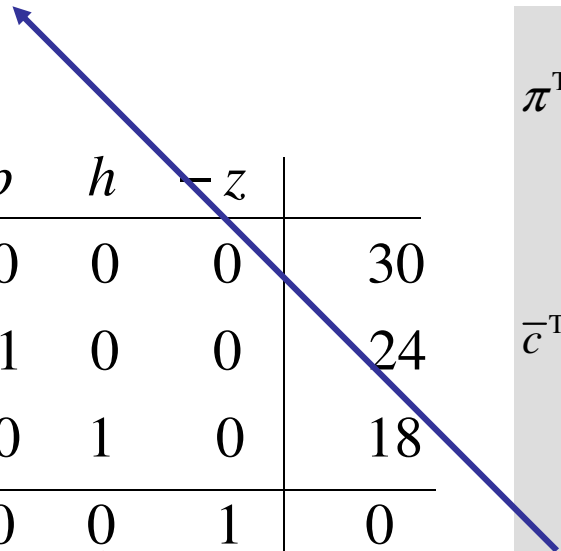
$$\pi^T = c_B^T B^{-1}$$

$$\pi^T = [-8 \ 0 \ 0] \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{2}{5} & 1 & 0 \\ -\frac{1}{5} & 0 & 1 \end{bmatrix} = \left[ -\frac{8}{5} \ 0 \ 0 \right]$$

$$\bar{c}^T = c^T - \pi^T A$$

$$\begin{aligned} \pi^T A &= \begin{bmatrix} -\frac{8}{5} & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -8 & -\frac{24}{5} & -\frac{8}{5} & 0 & 0 \end{bmatrix} \\ \bar{c}^T &= c^T - \pi^T A \\ &= [-8 \ -6 \ 0 \ 0 \ 0] - \begin{bmatrix} -8 & -\frac{24}{5} & -\frac{8}{5} & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\frac{6}{5} & \frac{8}{5} & 0 & 0 \end{bmatrix} \end{aligned}$$

	$x$	$y$	$u$	$p$	$h$	$-z$	
$u$	5	3	1	0	0	0	30
$p$	2	3	0	1	0	0	24
$h$	1	3	0	0	1	0	18
$-z$	-8	-6	0	0	0	1	0



	$x$	$y$	$u$	$p$	$h$	$-z$	
$x$	1	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	3
$p$	0	0	$-\frac{1}{4}$	1	$-\frac{3}{4}$	0	3
$y$	0	1	$-\frac{1}{12}$	0	$\frac{5}{12}$	0	3
$-z$	0	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	54

$$\pi^T = c_B^T B^{-1}$$

$$\pi^T = [-8 \ 0 \ -6] \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{3}{4} \\ -\frac{1}{12} & 0 & \frac{5}{12} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

$$\bar{c}^T = c^T - \pi^T A$$

	$x$	$y$	$u$	$p$	$h$	$-z$	
$u$	5	3	1	0	0	0	30
$p$	2	3	0	1	0	0	24
$h$	1	3	0	0	1	0	18
$-z$	-8	-6	0	0	0	1	0

$$\begin{aligned} \pi^T A &= \begin{bmatrix} -\frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -8 & -6 & -\frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} \\ \bar{c}^T &= c^T - \pi^T A \\ &= [-8 \ -6 \ 0 \ 0 \ 0] - \begin{bmatrix} -8 & -6 & -\frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \frac{3}{2} & 0 & \frac{1}{2} \end{bmatrix} \end{aligned}$$

## The simplex multipliers

$$\bar{c}_j = c_j - \pi^T a_{\bullet j}$$

The simplex multipliers vector  $\pi$  allows to compute the values of the relative costs  $\bar{c}_j$  using the original data of the problem.

$$\bar{c}^T = c^T - \pi^T A$$

The component  $\pi_i$  ( $i = 1, \dots, m$ ) of the simplex multiplier vector can be seen as a weight associated with the row  $i$  of the constraint matrix  $A$ , and then subtracting the linear combination of the rows of the matrix  $A$  using these weights ( $\pi^T A$ ), from the cost vector ( $c^T$ ) of the variables allows to obtain the relative costs of the variables, and to reduce to 0 those of the basic variables.

	$x$	$y$	$u$	$p$	$h$	$-z$	
$x$	1	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	3
$p$	0	0	$-\frac{1}{4}$	1	$-\frac{3}{4}$	0	3
$y$	0	1	$-\frac{1}{12}$	0	$\frac{5}{12}$	0	3
$-z$	0	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	54

$$\pi^T = c_B^T B^{-1}$$

$$\pi^T = [-8 \ 0 \ -6] \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{3}{4} \\ -\frac{1}{12} & 0 & \frac{5}{12} \end{bmatrix} = \left[ -\frac{3}{2} \ 0 \ -\frac{1}{2} \right]$$

$$\bar{c}^T = c^T - \pi^T A$$

	$x$	$y$	$u$	$p$	$h$	$-z$	
$-3/2$	5	3	1	0	0	0	30
0	2	3	0	1	0	0	24
$-1/2$	1	3	0	0	1	0	18
$-z$	-8	-6	0	0	0	1	0

$$\begin{aligned} \pi^T A &= \left[ -\frac{3}{2} \ 0 \ -\frac{1}{2} \right] \begin{bmatrix} 5 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 1 \end{bmatrix} \\ &= \left[ -8 \ -6 \ -\frac{3}{2} \ 0 \ -\frac{1}{2} \right] \\ \bar{c}^T &= c^T - \pi^T A \\ &= [-8 \ -6 \ 0 \ 0 \ 0] - \left[ -8 \ -6 \ -\frac{3}{2} \ 0 \ -\frac{1}{2} \right] \\ &= \left[ 0 \ 0 \ \frac{3}{2} \ 0 \ \frac{1}{2} \right] \end{aligned}$$



## Sensitivity analysis of the optimal value when modifying the right hand side terms

- The simplex multipliers associated with an optimal basic solution allows to analyse the sensitivity of the optimal value when modifying the right hand terms.
- Consider a problem and its version when modifying the right hand terms

$$\begin{array}{ll} \min & z \\ \text{Subject to} & Ax = b \\ & c^T x - z = 0 \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \min & \tilde{z} \\ \text{Subject to} & A\tilde{x} = b + \Delta b \\ & c^T \tilde{x} - \tilde{z} = 0 \\ & \tilde{x} \geq 0 \end{array}$$

## Sensitivity analysis of the optimal value when modifying the right hand side terms

$$\begin{array}{l|l}
 \min z & \min \tilde{z} \\
 \text{Subject to } Ax = b & \text{Subject to } A\tilde{x} = b + \Delta b \\
 c^T x - z = 0 & c^T \tilde{x} - \tilde{z} = 0 \\
 x \geq 0 & \tilde{x} \geq 0
 \end{array}$$

Let  $B^*$  be an optimal basis, and the corresponding basic solution

$$\begin{aligned}
 x_R^* &= 0 \\
 x_{B^*}^* &= B^{*-1}b = \bar{b} \geq 0
 \end{aligned}$$

having the optimal value

$$z^* = c_{B^*}^T x_{B^*}^* + c_R^T x_R^* = c_{B^*}^T B^{*-1}b = c_{B^*}^T \bar{b}$$

# Sensitivity analysis of the optimal value when modifying the right hand side terms

$$\begin{array}{l|l}
 \min z & \min \tilde{z} \\
 \text{Subject to } Ax = b & \text{Subject to } A\tilde{x} = b + \Delta b \\
 c^T x - z = 0 & c^T \tilde{x} - \tilde{z} = 0 \\
 x \geq 0 & \tilde{x} \geq 0
 \end{array}$$

Select a value of the vector  $\Delta b$  to be such that

$$B^{*-1}(b + \Delta b) = B^{*-1}b + B^{*-1}\Delta b \geq 0$$

Hence  $B^*$  remains a feasible basis for the new problem, and the basic feasible solution is as follows

$$\begin{aligned}
 \tilde{x}_R^* &= 0 \\
 \tilde{x}_{B^*}^* &= B^{*-1}(b + \Delta b) \geq 0
 \end{aligned}$$

## Sensitivity analysis of the optimal value when modifying the right hand side terms

Hence  $B^*$  remains a feasible basis for the new problem, and the basic feasible solution is as follows

$$\begin{aligned}\tilde{x}_R^* &= 0 \\ \tilde{x}_{B^*}^* &= B^{*-1}(b + \Delta b) \geq 0\end{aligned}$$

$$\bar{c}^{*\top} = c^\top - \pi^{*\top} A$$

$$\pi^{*\top} = c_{B^*}^\top B^{*-1}$$

Furthermore, since neither the costs  $c_j$  nor the matrix  $A$  have changed, then the simplex multipliers vector  $\pi^*$  remains unchanged. Then the relative costs  $\bar{c}_j$  remains also unchanged and non negative for the new problems.

Then  $B^*$  remains an optimal basis for the new problem.

## Sensitivity analysis of the optimal value when modifying the right hand side terms

An optimal solution of the new problem is then

$$\begin{aligned}\tilde{x}_R^* &= 0 \\ \tilde{x}_{B^*}^* &= B^{*-1}(b + \Delta b) \geq 0\end{aligned}$$

The optimal value of the new problem is as follows:

$$\begin{aligned}\tilde{z}^* &= c_{B^*}^T \tilde{x}_{B^*}^* + c_R^T \tilde{x}_R^* \\ &= c_{B^*}^T B^{*-1}(b + \Delta b) \\ &= c_{B^*}^T B^{*-1}b + c_{B^*}^T B^{*-1}\Delta b \\ &= z^* + \pi^{*T} \Delta b \\ &= z^* + \sum_{i=1}^m \pi_i^* \Delta b_i\end{aligned}$$

$$\pi^{*T} = c_{B^*}^T B^{*-1}$$

$$z^* = c_{B^*}^T B^{*-1}b$$

# Sensitivity analysis of the optimal value when modifying the right hand side terms

An optimal solution of the new problem is then

$$\begin{aligned}\tilde{z}^* &= c_{B^*}^T \tilde{x}_{B^*}^* + c_R^T \tilde{x}_R^* \\ &= c_{B^*}^T B^{*-1} (b + \Delta b) \\ &= c_{B^*}^T B^{*-1} b + c_{B^*}^T B^{*-1} \Delta b \\ &= z^* + \pi^{*T} \Delta b \\ &= z^* + \sum_{i=1}^m \pi_i^* \Delta b_i\end{aligned}$$

Then  $\pi_i^*$  is the marginal rate of variation of the optimal value of the objective function when the right hand term  $b_i$  of the  $i^{th}$  constraint is modified in such a way that the optimal basis remains feasible for the new problem.

	$x$	$y$	$u$	$p$	$h$	$-z$	
$x$	1	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	3
$p$	0	0	$-\frac{1}{4}$	1	$-\frac{3}{4}$	0	3
$y$	0	1	$-\frac{1}{12}$	0	$\frac{5}{12}$	0	3
$-z$	0	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	54

$$\pi^T = c_B^T B^{-1}$$

$$\pi^T = [-8 \ 0 \ -6] \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{3}{4} \\ -\frac{1}{12} & 0 & \frac{5}{12} \end{bmatrix} = \left[ -\frac{3}{2} \ 0 \ -\frac{1}{2} \right]$$

	$x$	$y$	$u$	$p$	$h$	$-z$	
$u$	5	3	1	0	0	0	30
$p$	2	3	0	1	0	0	24
$h$	1	3	0	0	1	0	18
$-z$	-8	-6	0	0	0	1	0

↑   ↑   ↑

	$x$	$y$	$u$	$p$	$h$	$-z$	
$x$	1	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	3
$p$	0	0	$-\frac{1}{4}$	1	$-\frac{3}{4}$	0	3
$y$	0	1	$-\frac{1}{12}$	0	$\frac{5}{12}$	0	3
$-z$	0	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	54

$$\pi^{*T} = c^{*T} B^{*-1}$$

$$\pi^{*T} = [-8 \ 0 \ -6] \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{3}{4} \\ -\frac{1}{12} & 0 & \frac{5}{12} \end{bmatrix} = \left[ -\frac{3}{2} \ 0 \ -\frac{1}{2} \right]$$

$$\tilde{z}^* = z^* + \pi^{*T} \Delta b$$

$$= -54 + \begin{bmatrix} -\frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = -54 - \frac{3}{2} \Delta b_1 + 0 \Delta b_2 - \frac{1}{2} \Delta b_3$$

Let  $\Delta b_2 = \Delta b_3 = 0$ .

$$\Delta b_1 < 0 \Rightarrow -\frac{3}{2} \Delta b_1 > 0 \Rightarrow \tilde{z}^* > z^*$$



# Feasible domain

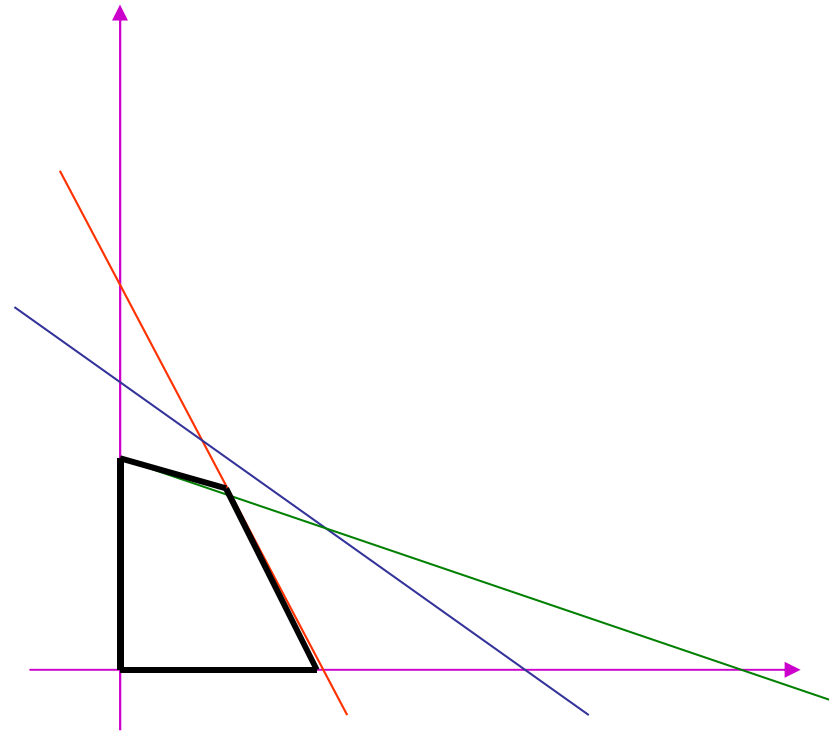
- The feasible domain for the system

$$5x + 3y \leq 30$$

$$2x + 3y \leq 24$$

$$1x + 3y \leq 18$$

$$x, y \geq 0$$



## Solving the problem graphicly

Consider the objective function:

$$z = -8x - 6y \quad .$$

The optimal solution:

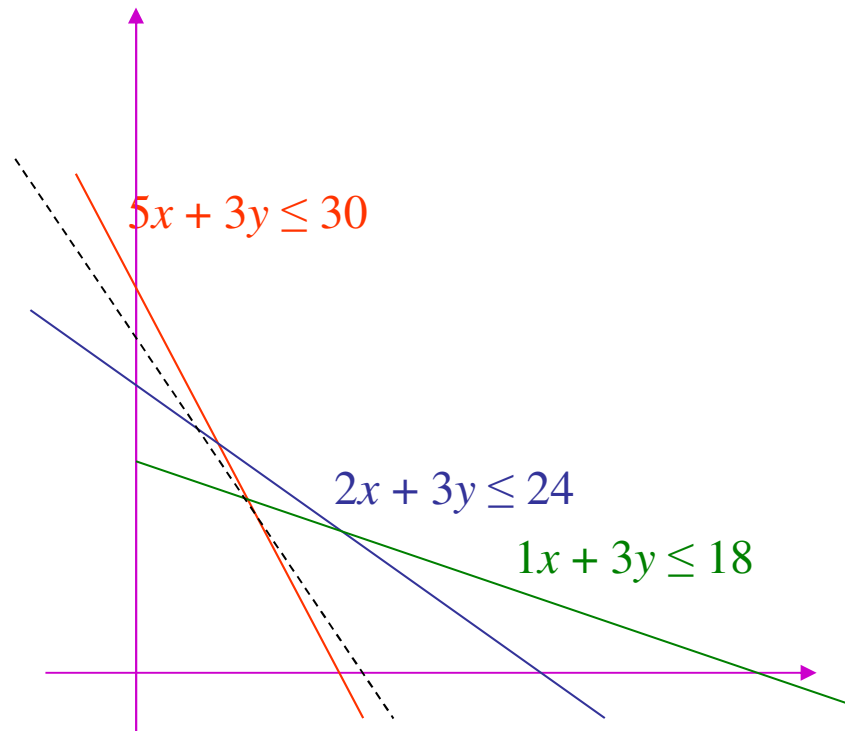
$$x = 3 \text{ and } y = 5 \Rightarrow z = -54.$$

Optimal simplex multipliers vector:

$$\pi^T = [-3/2, 0, -1/2]$$

- If  $b_1 = 30$  becomes  $b_1 + \Delta b_1$  with  $\Delta b_1 < 0$  the size of the feasible domain is reduced

$$5x + 3y = 30 + \Delta b_1 \Leftrightarrow y = -\frac{5}{3}x + \frac{30 + \Delta b_1}{3}$$



	$x$	$y$	$u$	$p$	$h$	$-z$	
$x$	1	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	3
$p$	0	0	$-\frac{1}{4}$	1	$-\frac{3}{4}$	0	3
$y$	0	1	$-\frac{1}{12}$	0	$\frac{5}{12}$	0	3
$-z$	0	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	54

$$\pi^{*T} = c^{*T} B^{*-1}$$

$$\pi^{*T} = [-8 \ 0 \ -6] \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{3}{4} \\ -\frac{1}{12} & 0 & \frac{5}{12} \end{bmatrix} = \left[ -\frac{3}{2} \ 0 \ -\frac{1}{2} \right]$$

$$\tilde{z}^* = z^* + \pi^{*T} \Delta b$$

$$= -54 + \begin{bmatrix} -\frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = -54 - \frac{3}{2} \Delta b_1 + 0 \Delta b_2 - \frac{1}{2} \Delta b_3$$

Let  $\Delta b_2 = \Delta b_3 = 0$ .

$$\Delta b_1 > 0 \Rightarrow -\frac{3}{2} \Delta b_1 < 0 \Rightarrow \tilde{z}^* < z^*$$

# Solving the problem graphicly

Consider the objective function:

$$z = -8x - 6y$$

The optimal solution:

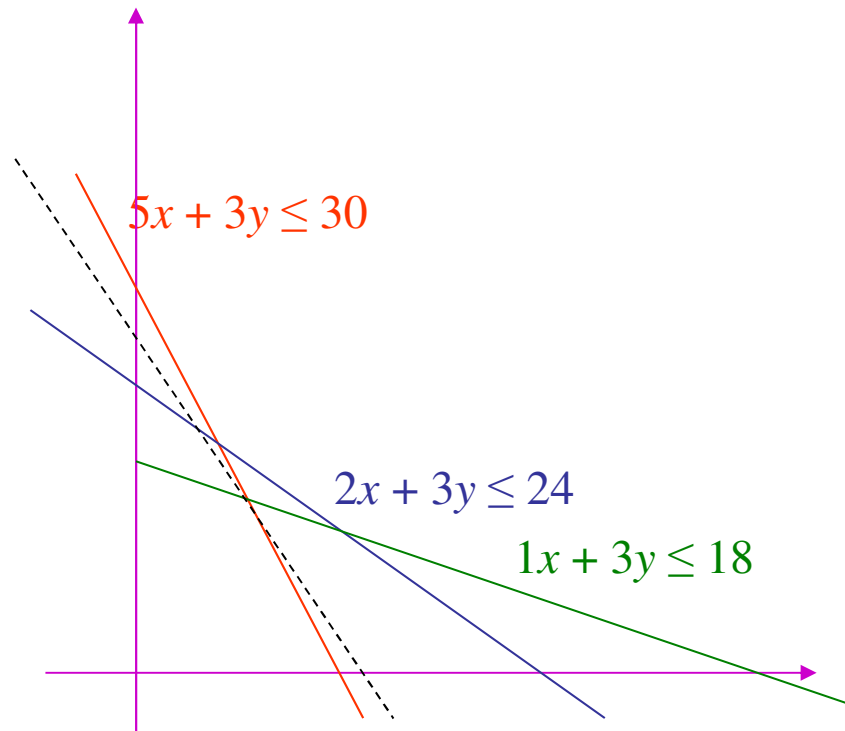
$$x = 3 \text{ and } y = 5 \Rightarrow z = -54.$$

Optimal simplex multipliers vector:

$$\pi^T = [-3/2, 0, -1/2]$$

- If  $b_1 = 30$  becomes  $b_1 + \Delta b_1$  with  $\Delta b_1 > 0$  the size of the feasible domain is increased

$$5x + 3y = 30 + \Delta b_1 \Leftrightarrow y = -\frac{5}{3}x + \frac{30 + \Delta b_1}{3}$$



	$x$	$y$	$u$	$p$	$h$	$-z$	
$x$	1	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	3
$p$	0	0	$-\frac{1}{4}$	1	$-\frac{3}{4}$	0	3
$y$	0	1	$-\frac{1}{12}$	0	$\frac{5}{12}$	0	3
$-z$	0	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	54

$$\pi^{*T} = c^{*T} B^{*-1}$$

$$\pi^{*T} = [-8 \ 0 \ -6] \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{3}{4} \\ -\frac{1}{12} & 0 & \frac{5}{12} \end{bmatrix} = \left[ -\frac{3}{2} \ 0 \ -\frac{1}{2} \right]$$

$$\tilde{z}^* = z^* + \pi^{*T} \Delta b$$

$$= -54 + \begin{bmatrix} -\frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = -54 - \frac{3}{2} \Delta b_1 + 0 \Delta b_2 - \frac{1}{2} \Delta b_3$$

$$\text{Let } \Delta b_1 = \Delta b_2 = 0$$

$$\Delta b_3 < 0 \Rightarrow -\frac{1}{2} \Delta b_3 > 0 \Rightarrow \tilde{z}^* > z^*$$

# Solving the problem graphicly

Consider the objective function:

$$z = -8x - 6y \quad .$$

The optimal solution:

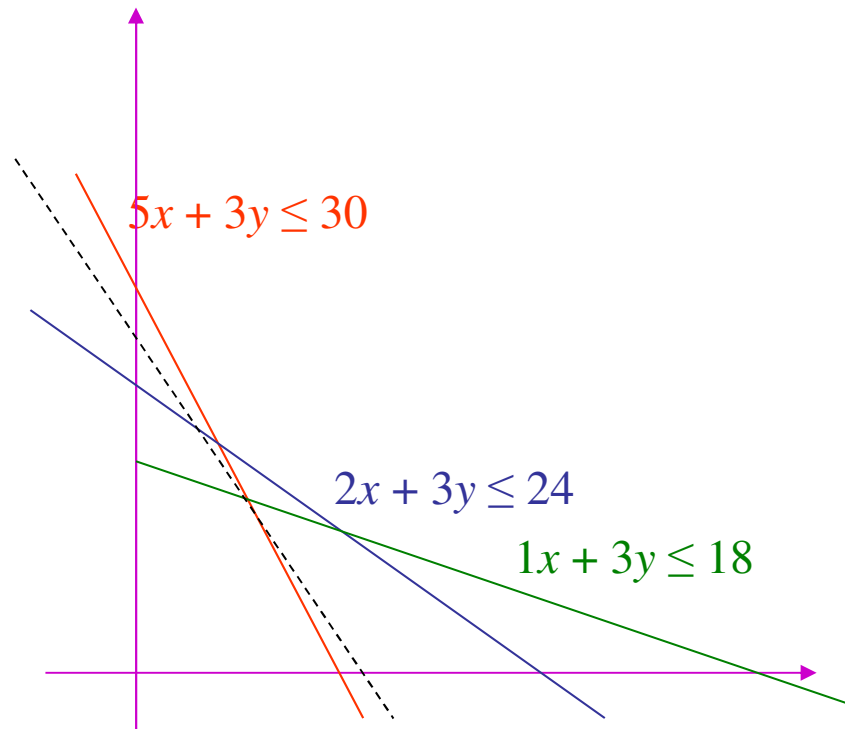
$$x = 3 \text{ and } y = 5 \Rightarrow z = -54.$$

Optimal simplex multipliers vector:

$$\pi^T = [-3/2, 0, -1/2]$$

- If  $b_3 = 18$  becomes  $b_3 + \Delta b_3$  with  $\Delta b_3 < 0$  the size of the feasible domain is reduced

$$x + 3y = 18 + \Delta b_3 \Leftrightarrow y = -\frac{1}{3}x + \frac{18 + \Delta b_3}{3}$$



	$x$	$y$	$u$	$p$	$h$	$-z$	
$x$	1	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	3
$p$	0	0	$-\frac{1}{4}$	1	$-\frac{3}{4}$	0	3
$y$	0	1	$-\frac{1}{12}$	0	$\frac{5}{12}$	0	3
$-z$	0	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	54

$$\pi^{*T} = c^{*T} B^{*-1}$$

$$\pi^{*T} = [-8 \ 0 \ -6] \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{3}{4} \\ -\frac{1}{12} & 0 & \frac{5}{12} \end{bmatrix} = \left[ -\frac{3}{2} \ 0 \ -\frac{1}{2} \right]$$

$$\tilde{z}^* = z^* + \pi^{*T} \Delta b$$

$$= -54 + \begin{bmatrix} -\frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = -54 - \frac{3}{2} \Delta b_1 + 0 \Delta b_2 - \frac{1}{2} \Delta b_3$$

$$\text{Let } \Delta b_1 = \Delta b_3 = 0$$

$$\Delta b_2 < 0 \Rightarrow 0 \Delta b_2 = 0 \Rightarrow \tilde{z}^* = z^*$$

# Solving the problem graphicly

Consider the objective function:

$$z = -8x - 6y$$

The optimal solution:

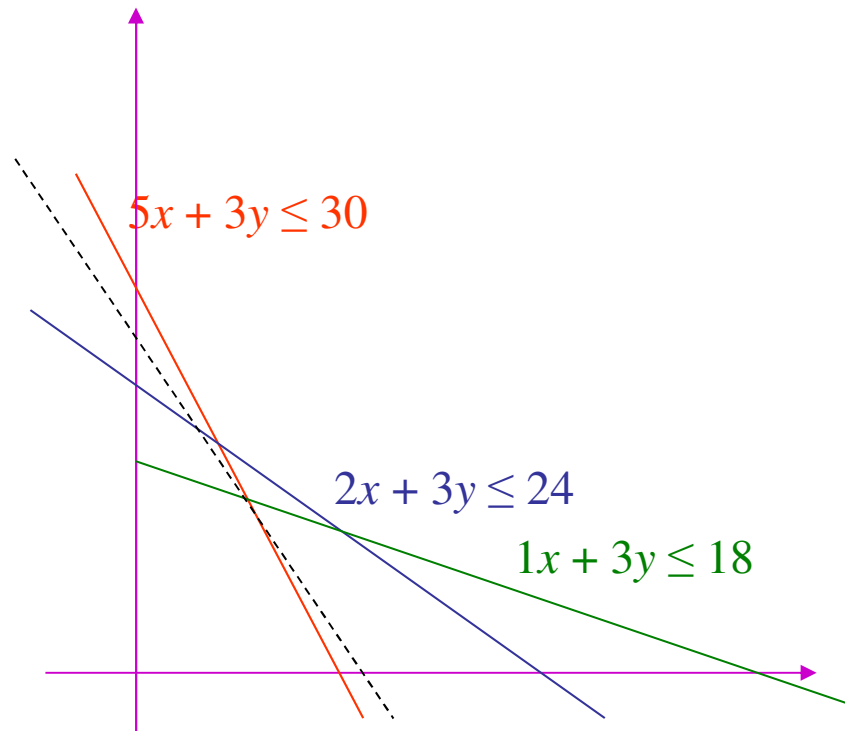
$$x = 3 \text{ and } y = 5 \Rightarrow z = -54.$$

Optimal simplex multipliers vector:

$$\pi^T = [-3/2, 0, -1/2]$$

- If  $b_2 = 24$  becomes  $b_2 + \Delta b_2$  with  $\Delta b_2 < 0$   
the size of the feasible domain  
remains the same

$$2x + 3y = 24 + \Delta b_2 \Leftrightarrow y = -\frac{2}{3}x + \frac{24 + \Delta b_2}{3}$$





# Optimality criterion

- Proposition** In the simplex algorithm, if at some iteration, all the relative costs  $\bar{c}_j \geq 0 \quad \forall j, 1 \leq j \leq n$ , then the current solution is optimal

**Proof:** Without loss of generality, assume that the first  $m$  variables  $x_1, x_2, \dots, x_m$  are basic; i. e.,

basic var.	$x_1$	$x_2$	$\dots$	$x_r$	$\dots$	$x_m$	$x_{m+1}$	$\dots$	$x_s$	$\dots$	$x_n$	$-z$	r.h.s.
$x_1$	1						$\bar{a}_{1m+1}$	$\dots$	$\bar{a}_{1s}$	$\dots$	$\bar{a}_{1n}$		$\bar{b}_1$
$x_2$		1					$\bar{a}_{2m+1}$	$\dots$	$\bar{a}_{2s}$	$\dots$	$\bar{a}_{2n}$		$\bar{b}_2$
$\vdots$			$\ddots$						$\ddots$				$\vdots$
$x_r$				1			$\bar{a}_{rm+1}$	$\dots$	$\bar{a}_{rs}$	$\dots$	$\bar{a}_{rn}$		$\bar{b}_r$
$\vdots$					$\ddots$				$\ddots$				$\vdots$
$x_m$						1	$\bar{a}_{mm+1}$	$\dots$	$\bar{a}_{ms}$	$\dots$	$\bar{a}_{mn}$		$\bar{b}_m$
$-z$							$\bar{c}_{m+1}$	$\dots$	$\bar{c}_s$	$\dots$	$\bar{c}_n$	1	$-c_b^T B^{-1}b$

basic var.	$x_B^T$	$x_R^T$	$-z$	r.h.s.
$x_B$	$I$	$B^{-1}R$	0	$B^{-1}b$
$-z$	0	$c_R^T - c_B^T B^{-1}R$	1	$-c_B^T B^{-1}b$

$$\bar{x}_i = \bar{b}_i \geq 0 \quad i = 1, 2, \dots, m$$

$$\bar{x}_i = 0 \quad i = m + 1, m + 2, \dots, n$$

$$\bar{z} = c_B^T B^{-1}b$$

# Optimality criterion

basic var.	$x_1$ $x_2$ $\dots$ $x_r$ $\dots$ $x_m$ $x_{m+1}$ $\dots$ $x_s$ $\dots$ $x_n$ $-z$	r.h.s.		
$x_1$	1	$\bar{a}_{1m+1}$ $\dots$ $\bar{a}_{1s}$ $\dots$ $\bar{a}_{1n}$	$\bar{b}_1$	
$x_2$		$\bar{a}_{2m+1}$ $\dots$ $\bar{a}_{2s}$ $\dots$ $\bar{a}_{2n}$	$\bar{b}_2$	
$\vdots$	$\ddots$	$\ddots$ $\ddots$	$\vdots$	
$x_r$		$\bar{a}_{rm+1}$ $\dots$ $\bar{a}_{rs}$ $\dots$ $\bar{a}_{rn}$	$\bar{b}_r$	
$\vdots$	$\ddots$	$\ddots$ $\ddots$	$\vdots$	
$x_m$		1	$\bar{a}_{mm+1}$ $\dots$ $\bar{a}_{ms}$ $\dots$ $\bar{a}_{mn}$	$\bar{b}_m$
$-z$		$\bar{c}_{m+1}$ $\dots$ $\bar{c}_s$ $\dots$ $\bar{c}_n$ 1	$-c_B^T B^{-1} b$	

The objective function is as follows

$$z = 0x_1 + \dots + 0x_m + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_n x_n + c_B^T B^{-1} b$$

$$\bar{z} = \dots c_B^T B^{-1} b$$

# Optimality criterion

The objective function is as follows

$$z = 0x_1 + \dots + 0x_m + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_n x_n + c_B^T B^{-1}b$$

Consider another feasible solution  $\bar{x} \geq 0$  with the value

$$\bar{z} = \bar{c}_{m+1} \bar{x}_{m+1} + \bar{c}_{m+2} \bar{x}_{m+2} + \dots + \bar{c}_n \bar{x}_n + c_B^T B^{-1}b$$

But the hypothesis  $\bar{c}_j \geq 0 \quad \forall j, 1 \leq j \leq n$  induces that

$$\bar{z} = \underbrace{\bar{c}_{m+1} \bar{x}_{m+1} + \bar{c}_{m+2} \bar{x}_{m+2} + \dots + \bar{c}_n \bar{x}_n}_{\geq 0} + c_B^T B^{-1}b \geq c_B^T B^{-1}b$$

# Optimality criterion

The objective function is as follows

$$z = 0x_1 + \dots + 0x_m + \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_n x_n + c_B^T B^{-1}b$$

Consider another feasible solution  $\bar{x} \geq 0$  with the value

$$\bar{z} = \bar{c}_{m+1} \bar{x}_{m+1} + \bar{c}_{m+2} \bar{x}_{m+2} + \dots + \bar{c}_n \bar{x}_n + c_B^T B^{-1}b$$

But the hypothesis  $\bar{c}_j \geq 0 \quad \forall j, 1 \leq j \leq n$  induces that

$$\bar{z} = \underbrace{\bar{c}_{m+1} \bar{x}_{m+1} + \bar{c}_{m+2} \bar{x}_{m+2} + \dots + \bar{c}_n \bar{x}_n}_{\geq 0} + c_B^T B^{-1}b \geq c_B^T B^{-1}b = z$$

Then the current solution is optimal. □