

Duality
for
linear programming

Illustration of the notion

- Consider an enterprise producing r items:

$f_k =$ demand for the item $k = 1, \dots, r$

using s components:

$h_l =$ availability of the component $l = 1, \dots, s$

- The enterprise can use any of the n process (activities):

$x_j =$ level for using the process $j = 1, \dots, n$

$c_j =$ the unit cost for using the process $j = 1, \dots, n$

- The process j

produces e_{kj} units of the item $k = 1, \dots, r$

uses g_{lj} units of the component $l = 1, \dots, s$

for each unit of its use

Illustration of the notion

- Consider an enterprise producing r items:
 $f_k =$ demand for the item $k = 1, \dots, r$
 using s components:
 $h_l =$ availability of the component $l = 1, \dots, s$
- The enterprise can use any of the n process (activities):
 $x_j =$ level for using the process $j = 1, \dots, n$
 $c_j =$ the unit cost for using the process $j = 1, \dots, n$
- The process j
 produces e_{kj} units of the item $k = 1, \dots, r$
 uses g_{lj} units of the component $l = 1, \dots, s$
 each time it is used at level 1

- The enterprise problem:** determine the level of each process for satisfying the without exceeding the availabilities in order to minimize the total production cost.

- Model**

$$\begin{aligned} \min z &= \sum_{j=1}^n c_j x_j \\ \text{S. t.} \quad & \sum_{j=1}^n e_{kj} x_j \geq f_k && k = 1, 2, \dots, r \text{ (demands)} \\ & \sum_{j=1}^n g_{lj} x_j \leq h_l && l = 1, 2, \dots, s \text{ (availabilities)} \\ & x_j \geq 0 && j = 1, 2, \dots, n \end{aligned}$$

Illustration of the notion

- A business man makes an offer to buy all the components and to sell the items required by the enterprise to satisfy the demands.
- He must state proper unit prices (to be determined) to make the offer interesting for the enterprise:

v_k item unit price $k = 1, 2, \dots, r$

w_l component unit price $l = 1, 2, \dots, s$.

$$\min z = \sum_{j=1}^n c_j x_j$$

$$\text{S.t.} \quad \sum_{j=1}^n e_{kj} x_j \geq f_k \quad k = 1, 2, \dots, r \text{ (demands)} \quad \boxed{v_k}$$

$$\sum_{j=1}^n g_{lj} x_j \leq h_l \quad l = 1, 2, \dots, s \text{ (availabilities)} \quad \boxed{w_l}$$

$$x_j \geq 0 \quad j = 1, 2, \dots, n$$

Illustration of the notion

The business man must state proper unit prices (to be determined) to make the offer interesting for the enterprise

To complete its analysis, the enterprise must verify that for each process j , the cost of making business with him is smaller or equal than using the process j . But the cost of making business with him is equal to the difference between buying the items required and selling the components unused in order to simulate using one unit of process j (c_j).

$$\begin{aligned}
 \min z &= \sum_{j=1}^n c_j x_j \\
 \text{S. t} \quad & \sum_{j=1}^n e_{kj} x_j \geq f_k \quad k = 1, 2, \dots, r \text{ (demands)} \quad \boxed{v_k} \\
 & \sum_{j=1}^n g_{lj} x_j \leq h_l \quad l = 1, 2, \dots, s \text{ (availabilities)} \quad \boxed{w_l} \\
 & x_j \geq 0 \quad j = 1, 2, \dots, n
 \end{aligned}$$

$$\underbrace{\sum_{k=1}^r e_{kj} v_k}_{\text{buying the items}} - \underbrace{\sum_{l=1}^s g_{lj} w_l}_{\text{selling the components}} \leq c_j$$

Illustration of the notion

$$\underbrace{\sum_{k=1}^r e_{kj} v_k}_{\text{buying the items}} - \underbrace{\sum_{l=1}^s g_{lj} w_l}_{\text{selling the components}} \leq c_j$$

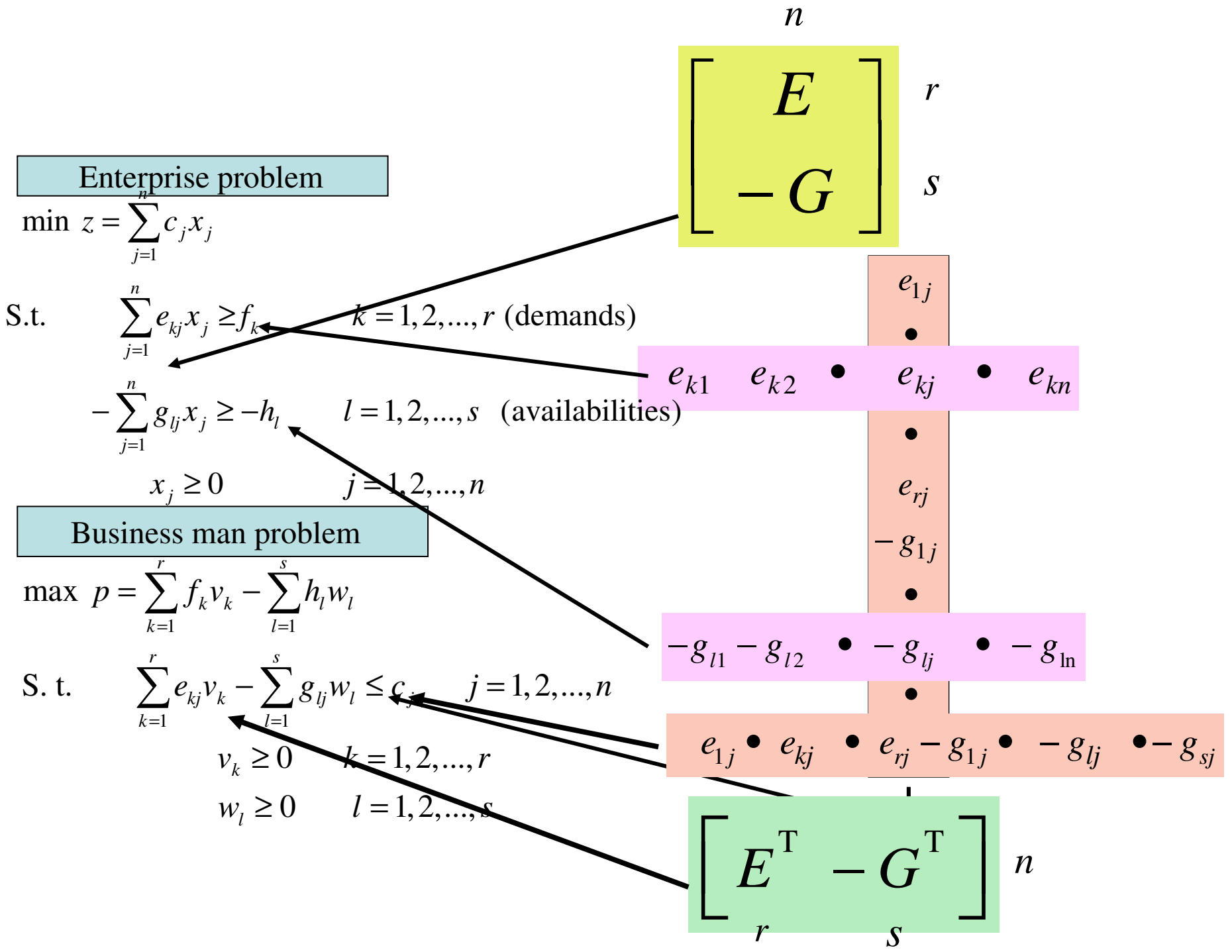
- **The business man problem** is to maximize his profit while maintaining the prices competitive for the enterprise

$$\begin{aligned} \max \quad & p = \sum_{k=1}^r f_k v_k - \sum_{l=1}^s h_l w_l \\ \text{S. t} \quad & \sum_{k=1}^r e_{kj} v_k - \sum_{l=1}^s g_{lj} w_l \leq c_j \quad j = 1, 2, \dots, n \\ & v_k \geq 0 \quad k = 1, 2, \dots, r \\ & w_l \geq 0 \quad l = 1, 2, \dots, s \end{aligned}$$

Illustration of the notion

- The enterprise problem: multiply the availability constraints by -1

$\min z = \sum_{j=1}^n c_j x_j$		$\min z = \sum_{j=1}^n c_j x_j$	
$\text{S. t. } \sum_{j=1}^n e_{kj} x_j \geq f_k \quad k = 1, 2, \dots$		$\text{S. t. } \sum_{j=1}^n e_{kj} x_j \geq f_k \quad k = 1, 2, \dots, r \text{ (demands)}$	
$-1 \times \sum_{j=1}^n g_{lj} x_j \leq h_l \quad l = 1, 2, \dots,$	$j = 1, 2, \dots$	$-\sum_{j=1}^n g_{lj} x_j \geq -h_l \quad l = 1, 2, \dots, s \text{ (availabilities)}$	$j = 1, 2, \dots, n$
$x_j \geq 0$		$x_j \geq 0$	



Primal

$$\min z = \sum_{j=1}^n c_j x_j$$

$$\text{S. t. } \sum_{j=1}^n e_{kj} x_j \geq f_k \quad k = 1, 2, \dots, r \text{ (demands)}$$

$$-\sum_{j=1}^n g_{lj} x_j \geq -h_l \quad l = 1, 2, \dots, s \text{ (availabilities)}$$

$$x_j \geq 0 \quad j = 1, 2, \dots, n$$

Dual

$$\max p = \sum_{k=1}^r f_k v_k - \sum_{l=1}^s h_l w_l$$

$$\text{S. t. } \sum_{k=1}^r e_{kj} v_k - \sum_{l=1}^s g_{lj} w_l \leq c_j \quad j = 1, 2, \dots, n$$

$$v_k \geq 0 \quad k = 1, 2, \dots, r$$

$$w_l \geq 0 \quad l = 1, 2, \dots, s$$

$$\min z = c^T x$$

S. t.

$$\begin{bmatrix} E \\ -G \end{bmatrix} x \geq \begin{bmatrix} f \\ -h \end{bmatrix} \quad \begin{bmatrix} v \\ w \end{bmatrix}$$

$$x \geq 0$$

$$\min c^T x$$

$$\text{S. t. } Ax \geq b$$

$$x \geq 0$$

$$\max p = \begin{bmatrix} f^T & :-h^T \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$$

S. t.

$$\begin{bmatrix} E^T & :-G^T \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \leq c \quad x$$

$$v, w \geq 0$$

$$\max b^T y$$

$$\text{S. t. } A^T y \leq c$$

$$y \geq 0$$

Primal dual problems

Linear programming problem specified with equalities

Primal problem

$$\begin{aligned} \min \quad & c^T x \\ \text{S. t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

y

Dual problem

$$\begin{aligned} \max \quad & b^T y \\ \text{S. t.} \quad & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

x

Linear programming in standard form

Primal problem

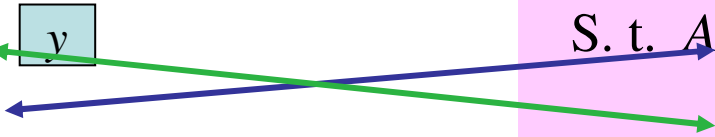
$$\begin{aligned} \min \quad & c^T x \\ \text{S. t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

y

Dual problem

$$\begin{aligned} \max \quad & b^T y \\ \text{S. t.} \quad & A^T y \leq c \end{aligned}$$

x



$$\begin{aligned} \min \quad & c^T x \\ \text{S. t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & c^T x - 0^T s \\ \text{S. t.} \quad & Ax - Is = b \\ & x \geq 0, s \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & b^T y \\ \text{S. t.} \quad & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & b^T y \\ \text{S. t.} \quad & \begin{bmatrix} A^T \\ -I^T \end{bmatrix} y \leq \begin{bmatrix} c \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \max \quad & b^T y \\ \text{S. t.} \quad & A^T y \leq c \\ & -Iy \leq 0 \end{aligned}$$

Duality theorems

- It is easy to show that we can move from one pair of primal-dual problems to the other.
- It is also easy to show that the dual of the dual problem is the primal problem.
- Thus we are showing the duality theorems using the pair where the primal is in the standard form:

primal

$$\begin{aligned} \min \quad & c^T x \\ \text{S. t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \max \quad & b^T y \\ \text{S. t.} \quad & A^T y \leq c \end{aligned}$$

Duality theorems

- **Weak duality theorem**

If $x \in \{x: Ax = b, x \geq 0\}$ (i.e., x is feasible for the primal problem) and if $y \in \{y: A^T y \leq c\}$ (i.e., y is feasible for the dual problem), then

$$b^T y \leq c^T x$$

Proof Indeed, $b^T y = x^T A^T y \leq x^T c$ since $A^T y \leq c$ and $x \geq 0$.

Duality theorems

- **Corollary** If $x^* \in \{x: Ax = b, x \geq 0\}$ and $y^* \in \{y: A^T y \leq c\}$, and if $b^T y^* = c^T x^*$, then x^* and y^* are optimal solutions for the primal and dual problems, respectively..

Proof It follows from the weak duality theorem that for any feasible solution x of the primal problem

$$c^T x \geq b^T y^* = c^T x^*.$$

Consequently x^* is an optimal solution of the primal problem.

We can show the optimality of y^* for the dual problem using a similar proof.

Duality theorems

- **Strong duality theorem** If one of the two primal or dual problem has a finite value optimal solution, then the other problem has the same property, and the optimal values of the two problems are equal. If one of the two problems is unbounded, then the feasible domain of the other problem is empty.

Proof The second part of the theorem follows directly from the weak duality theorem. Indeed, suppose that the primal problem is unbounded below, and thus $c^T x \rightarrow -\infty$. For contradiction, suppose that the dual problem is feasible. Then there would exist a solution $y \in \{y: A^T y \leq c\}$, and from the weak duality theorem, it would follow that $b^T y \leq c^T x$; i.e., $b^T y$ would be a lower bound for the value of the primal objective function $c^T x$, a contradiction.

Recall: The simplex multipliers

$$\bar{c}^T = c^T - c_B^T B^{-1} A$$

Denote the vector $\pi \in R^m$
specified by

$$\pi^T = c_B^T B^{-1}$$

Then

$$\bar{c}^T = c^T - \pi^T A$$

or

$$[\bar{c}_1, \dots, \bar{c}_n] = [c_1, \dots, c_n] - \pi^T [a_{\bullet 1}, \dots, a_{\bullet n}]$$

$$\bar{c}_j = c_j - \pi^T a_{\bullet j}$$

where $a_{\bullet j}$ denotes the j th column
of the constraint matrix A

π is the simplex multipliers vector
associated with the basis B .

The vector π has one element associated
with each row (constraint) of the tableau.

Duality theorems

To prove the first part of the theorem, suppose that x^* is an optimal solution of the primal problem with a value of z^* .

Let $x_{j_1}, x_{j_2}, \dots, x_{j_m}$ be the basic variables.

Let $c_B = [c_{j_1}, c_{j_2}, \dots, c_{j_m}]^T$, and π be the simplex multipliers associated with the optimal basis. Recall that the relative costs of the variables are specified as follows

$$\bar{c}_j = c_j - \pi^T a_{\bullet j} \quad \forall j = 1, 2, \dots, n$$

where $a_{\bullet j}$ denotes the j^{th} column of the matrix A .

Suppose that the basic optimal solution has the following property

$$\bar{c}_j = c_j - \pi^T a_{\bullet j} \geq 0 \quad \forall j = 1, 2, \dots, n$$

Consequently

$$\pi^T a_{\bullet j} \leq c_j \quad \forall j = 1, 2, \dots, n$$

Duality theorems

Suppose that the basic optimal solution has the following property

$$\bar{c}_j = c_j - \pi^T a_{\bullet j} \geq 0 \quad \forall j = 1, 2, \dots, n$$

Consequently

$$\pi^T a_{\bullet j} \leq c_j \quad \forall j = 1, 2, \dots, n$$

or

$$a_{\bullet j}^T \pi \leq c_j \quad \forall j = 1, 2, \dots, n$$

and the matrix format of these relations:

$$A^T \pi \leq c$$

This implies that

$$\pi \in \{y : A^T y \leq c\}$$

i.e., π is a feasible solution of the dual problem.

Duality theorems

Determine the value of the dual objective function for the dual feasible solution π . Recall that

$$\pi = B^{-1\text{T}} c_B.$$

It follows that

$$b^{\text{T}} \pi = b^{\text{T}} B^{-1\text{T}} c_B = (B^{-1} b)^{\text{T}} c_B = x_B^{*\text{T}} c_B = z^*$$

Consequently, it follows from the corollary of the weak duality theorem that π is an optimal solution of the dual problem, and that

$$\pi^{\text{T}} b = z^* .$$

Complementary slackness theory

- We now introduce new necessary and sufficient conditions for a pair of feasible solutions of the primal and of the dual to be optimal for each of these problems.
- Consider first the following pair of primal-dual problems.

primal

$$\begin{array}{ll} \min & c^T x \\ \text{S. t.} & Ax = b \\ & x \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \max & b^T y \\ \text{S. t.} & A^T y \leq c \end{array}$$

x

Complementary slackness theory

- Complementary slackness theorem 1

Let x and y be feasible solution for the primal and the dual, respectively.
Then x and y are optimal solutions for these problems if and only if for all $j = 1, 2, \dots, n$

$$(i) \quad x_j > 0 \quad \Rightarrow \quad a_{\bullet j}^T y = c_j$$

$$(ii) \quad a_{\bullet j}^T y < c_j \quad \Rightarrow \quad x_j = 0$$

$$\begin{array}{l} \min \quad c^T x \\ \text{S. t. } Ax = b \\ \quad \quad x \geq 0 \end{array}$$

$$\begin{array}{l} \max \quad b^T y \\ \text{S. t. } A^T y \leq c \end{array} \quad x$$

Proof (\Leftarrow) First we prove the sufficiency of the conditions. Assume that the conditions (i) et (ii) are satisfied for all $j=1, 2, \dots, n$. Then

$$x_j [a_{\bullet j}^T y - c_j] = 0 \quad \forall j = 1, 2, \dots, n$$

$$\text{Hence } \sum_{j=1}^n x_j [a_{\bullet j}^T y - c_j] = 0$$

Complementary slackness

$$x_j [a_{\bullet j}^T y - c_j] = 0$$

$$\text{Hence } \sum_{j=1}^n x_j [a_{\bullet j}^T y - c_j] = 0$$

$$\text{But } 0 = \sum_{j=1}^n x_j [a_{\bullet j}^T y - c_j] = \sum_{j=1}^n x_j a_{\bullet j}^T y - \sum_{j=1}^n x_j c_j = x^T A^T y - c^T x = b^T y - c^T x$$

Consequently

$$b^T y = c^T x$$

and the corollary of the weak duality theorem implies that x and y are optimal solutions for the primal and the dual problems, respectively.

$$\begin{aligned} \sum_{j=1}^n x_j a_{\bullet j}^T y &= [x_1 a_{\bullet 1}^T + x_2 a_{\bullet 2}^T + \dots + x_n a_{\bullet n}^T] y \\ &= [x_1, x_2, \dots, x_n] \begin{bmatrix} a_{\bullet 1}^T \\ a_{\bullet 2}^T \\ \vdots \\ a_{\bullet n}^T \end{bmatrix} y \\ &= x^T A^T y \end{aligned}$$

Complementary slackness theory

(\Rightarrow) Now we prove the necessity of the conditions. Suppose that the solutions x et y are optimal solutions for the primal and the dual problems, respectively, and

$$b^T y = c^T x.$$

Then referring to the first part of the theorem

$$\sum_{j=1}^n x_j [a_{\bullet j}^T y - c_j] = \sum_{j=1}^n x_j a_{\bullet j}^T y - \sum_{j=1}^n x_j c_j = x^T A^T y - c^T x = b^T y - c^T x = 0$$

Since $x_j \geq 0$ et $a_{\bullet j}^T y \leq c_j \quad \forall j = 1, 2, \dots, n,$

it follows that $x_j [a_{\bullet j}^T y - c_j] = 0 \quad \forall j = 1, 2, \dots, n.$

The proof is completed.

Complementary slackness theory

- Now consider the other pair of primal-dual problems

$$\begin{array}{ll} \min & c^T x \\ \text{S. t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

y

$$\begin{array}{ll} \max & b^T y \\ \text{S. t.} & A^T y \leq c \\ & y \geq 0 \end{array}$$

x

- Complementary slackness theorem 2**

Let x and y be feasible solution for the primal and dual problems, respectively. Then x and y are optimal solutions of these problems if and only if

for all $j = 1, 2, \dots, n$

$$(i) \quad x_j > 0 \quad \Rightarrow \quad a_{\bullet j}^T y = c_j$$

$$(ii) \quad a_{\bullet j}^T y < c_j \quad \Rightarrow \quad x_j = 0$$

for all $i = 1, 2, \dots, m$

$$(iii) \quad a_{i\bullet} \cdot x > b_i \quad \Rightarrow \quad y_i = 0$$

$$(iv) \quad y_i > 0 \quad \Rightarrow \quad a_{i\bullet} \cdot x = b_i$$

Complementary slackness theory

Proof This theorem is in fact a corollary of the complementary slackness theorem 1. Transform the primal problem into the standard form using the slack variables s_i , $i=1,2,\dots,m$:

$$\begin{array}{ll} \min & c^T x \\ \text{Sujet à} & Ax - Is = b \\ & x, s \geq 0 \end{array}$$

$$\begin{array}{ll} \min & c^T x \\ \text{S. t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

The dual of the primal problem in standard form

$$\left. \begin{array}{l} \max \quad b^T y \\ \text{S. t.} \quad A^T y \leq c \\ \quad \quad -I y \leq 0 \end{array} \right\} \equiv \left\{ \begin{array}{l} \max \quad b^T y \\ \text{S. t.} \quad A^T y \leq c \\ \quad \quad \quad I y \geq 0 \end{array} \right.$$

Complementary slackness theory

Use the result in the preceding theorem to this pair of primal-dual problems

$$\begin{array}{ll} \min & c^T x \\ \text{S. t.} & Ax - Is = b \\ & x, s \geq 0 \end{array}$$

y

$$\begin{array}{ll} \max & b^T y \\ \text{S. t.} & A^T y \leq c \\ & -I y \leq 0 \end{array}$$

x

s

For $j=1,2,\dots,n$

$$(i) \quad x_j > 0 \quad \Rightarrow \quad a_{\bullet j}^T y = c_j$$

$$(ii) \quad a_{\bullet j}^T y < c_j \quad \Rightarrow \quad x_j = 0$$

and for $i=1,2,\dots,m$

$$(iii) \quad s_i > 0 \quad \Rightarrow \quad -y_i = 0$$

$$(iv) \quad -y_i < 0 \quad \Rightarrow \quad s_i = 0$$

Complementary slackness theory

For $j=1,2,\dots,n$

$$(i) \quad x_j > 0 \quad \Rightarrow \quad a_{\bullet j}^T y = c_j$$

$$(ii) \quad a_{\bullet j}^T y < c_j \quad \Rightarrow \quad x_j = 0$$

and for $i=1,2,\dots,m$

$$(iii) \quad s_i > 0 \quad \Rightarrow \quad -y_i = 0$$

$$(iv) \quad -y_i < 0 \quad \Rightarrow \quad s_i = 0$$

But $s_i = a_{i\bullet} x - b_i$ and then the conditions become

$$(iii) \quad a_{i\bullet} x > b_i \quad \Rightarrow \quad y_i = 0$$

$$(iv) \quad y_i > 0 \quad \Rightarrow \quad a_{i\bullet} x = b_i$$

$$\begin{array}{ll} \min & c^T x \\ \text{S. t.} & Ax - Is = b \\ & x, s \geq 0 \end{array}$$

Dual simplex algorithm

- The dual simplex method is an iterative procedure to solve a linear programming problem in standard form.

$$\min z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

S. t.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$x_j \geq 0 \quad j = 1, 2, \dots, n$$

Dual simplex algorithm

- At each iteration, a **basic infeasible** solution of problem is available, except at the last iteration, for which the **relative costs of all variables are non negatives**.
- Exemple

$$\begin{array}{l}
 \min \quad z = \quad 3/2u \quad + 1/2h \quad - 27 \\
 \text{S. t.} \quad x \quad + 1/4u \quad - 1/4h = -6/4 \\
 \quad \quad \quad - 1/4u + p - 3/4h = 15/2 \\
 \quad \quad \quad y - 1/12u \quad + 5/12h = 13/2 \\
 \quad \quad \quad x, y, u, p, h \geq 0
 \end{array}$$

'basic var.	x	y	u	p	h	$-z$	r.h.s.
x	1		1/4		-1/4		-6/4
p			-1/4	1	-3/4		15/2
y		1	-1/12		5/12		13/2
$-z$			3/2		1/2	1	27

Dual simplex algorithm

Analyse one iteration of the dual simplex algorithm, and suppose that the current solution is as follows:

basic var.	x_1 x_2 ... x_{j_r} ... x_{j_1} ... x_s ... x_{j_m} ... x_n $-Z$	r.h.s.
x_{j_1}	\bar{a}_{11} \bar{a}_{12} ... 0 ... 1 ... \bar{a}_{1s} ... 0 ... \bar{a}_{1n} 0	\bar{b}_1
.	.	.
x_{j_r}	\bar{a}_{r1} \bar{a}_{r2} ... 1 ... 0 ... \bar{a}_{rs} ... 0 ... \bar{a}_{rn} 0	\bar{b}_r
.	.	.
x_{j_m}	\bar{a}_{m1} \bar{a}_{m2} ... 0 ... 0 ... \bar{a}_{ms} ... 1 ... \bar{a}_{mn} 0	\bar{b}_m
$-Z$	\bar{c}_1 \bar{c}_2 ... 0 ... 0 ... \bar{c}_s ... 0 ... \bar{c}_n 1	$-\bar{z}_0$

$$\bar{c}_j \geq 0 \quad \forall j = 1, 2, \dots, n$$

$$\bar{c}_{j_i} = 0 \quad \forall i = 1, 2, \dots, m$$

Leaving criterion

basic var.	x_1	x_2	...	x_{j_r}	...	x_{j_1}	...	x_s	...	x_{j_m}	...	x_n	$-Z$	r.h.s.
x_{j_1}	\bar{a}_{11}	\bar{a}_{12}	...	0	...	1	...	\bar{a}_{1s}	...	0	...	\bar{a}_{1n}	0	\bar{b}_1
.
x_{j_r}	\bar{a}_{r1}	\bar{a}_{r2}	...	1	...	0	...	\bar{a}_{rs}	...	0	...	\bar{a}_{rn}	0	\bar{b}_r
.
x_{j_m}	\bar{a}_{m1}	\bar{a}_{m2}	...	0	...	0	...	\bar{a}_{ms}	...	1	...	\bar{a}_{mn}	0	\bar{b}_m
$-Z$	\bar{c}_1	\bar{c}_2	...	0	...	0	...	\bar{c}_s	...	0	...	\bar{c}_n	1	$-\bar{z}_0$

$$\bar{c}_j \geq 0 \quad \forall j = 1, 2, \dots, n$$

$$\bar{c}_{j_i} = 0 \quad \forall i = 1, 2, \dots, m$$

If $\bar{b}_i \geq 0 \quad \forall i = 1, 2, \dots, m$, then the solution is feasible and optimal. The algorithm stops.

Leaving criterion

basic var.	x_1 x_2 ... x_{j_r} ... x_{j_1} ... x_s ... x_{j_m} ... x_n $-Z$	r.h.s.
x_{j_1}	\bar{a}_{11} \bar{a}_{12} ... 0 ... 1 ... \bar{a}_{1s} ... 0 ... \bar{a}_{1n} 0	\bar{b}_1
.	.	.
x_{j_r}	\bar{a}_{r1} \bar{a}_{r2} ... 1 ... 0 ... \bar{a}_{rs} ... 0 ... \bar{a}_{rn} 0	\bar{b}_r
.	.	.
x_{j_m}	\bar{a}_{m1} \bar{a}_{m2} ... 0 ... 0 ... \bar{a}_{ms} ... 1 ... \bar{a}_{mn} 0	\bar{b}_m
$-Z$	\bar{c}_1 \bar{c}_2 ... 0 ... 0 ... \bar{c}_s ... 0 ... \bar{c}_n 1	$-\bar{z}_0$

$$\bar{c}_j \geq 0 \quad \forall j = 1, 2, \dots, n$$

$$\bar{c}_{j_i} = 0 \quad \forall i = 1, 2, \dots, m$$

Otherwise, let $\bar{b}_r = \min_{i=1,2,\dots,m} \{\bar{b}_i\}$. If $\bar{a}_{rj} \geq 0 \quad \forall j = 1, 2, \dots, n$, then the feasible domain is empty. Indeed, since

$$\text{for all } x \geq 0, \sum_{j=1}^n \bar{a}_{rj} x_j \geq 0 \quad \text{and} \quad \bar{b}_r < 0$$

it is not possible to find $x \geq 0$ such that $\sum_{j=1}^n \bar{a}_{rj} x_j = \bar{b}_r$.

Leaving criterion

basic var.	x_1 x_2 ... x_{j_r} ... x_{j_1} ... x_s ... x_{j_m} ... x_n $-Z$	r.h.s.
x_{j_1}	\bar{a}_{11} \bar{a}_{12} ... 0 ... 1 ... \bar{a}_{1s} ... 0 ... \bar{a}_{1n} 0	\bar{b}_1
.	.	.
x_{j_r}	\bar{a}_{r1} \bar{a}_{r2} ... 1 ... 0 ... \bar{a}_{rs} ... 0 ... \bar{a}_{rn} 0	\bar{b}_r
.	.	.
x_{j_m}	\bar{a}_{m1} \bar{a}_{m2} ... 0 ... 0 ... \bar{a}_{ms} ... 1 ... \bar{a}_{mn} 0	\bar{b}_m
$-Z$	\bar{c}_1 \bar{c}_2 ... 0 ... 0 ... \bar{c}_s ... 0 ... \bar{c}_n 1	$-\bar{z}_0$

$$\bar{c}_j \geq 0 \quad \forall j = 1, 2, \dots, n$$

$$\bar{c}_{j_i} = 0 \quad \forall i = 1, 2, \dots, m$$

Otherwise, let let $\bar{b}_r = \min_{i=1,2,\dots,m} \{\bar{b}_i\} < 0$. x_{j_r} is the leaving variable.

The pivot will be completed with an element in the r^{th} row.

Leaving criterion

basic var.	x_1 x_2 ... x_{j_r} ... x_{j_1} ... x_s ... x_{j_m} ... x_n $-Z$	r.h.s.
x_{j_1}	\bar{a}_{11} \bar{a}_{12} ... 0 ... 1 ... \bar{a}_{1s} ... 0 ... \bar{a}_{1n} 0	\bar{b}_1
.	.	.
x_{j_r}	\bar{a}_{r1} \bar{a}_{r2} ... 1 ... 0 ... \bar{a}_{rs} ... 0 ... \bar{a}_{rn} 0	\bar{b}_r
.	.	.
x_{j_m}	\bar{a}_{m1} \bar{a}_{m2} ... 0 ... 0 ... \bar{a}_{ms} ... 1 ... \bar{a}_{mn} 0	\bar{b}_m
$-Z$	\bar{c}_1 \bar{c}_2 ... 0 ... 0 ... \bar{c}_s ... 0 ... \bar{c}_n 1	$-\bar{z}_0$

$$\bar{c}_j \geq 0 \quad \forall j=1,2,\dots,n$$

$$\bar{c}_{j_i} = 0 \quad \forall i=1,2,\dots,m$$

$$\begin{array}{l} \bar{b}_1 - \bar{a}_{1s}x_s \\ \vdots \\ \bar{b}_r - \bar{a}_{rs}x_s \\ \vdots \\ \bar{b}_m - \bar{a}_{ms}x_s \end{array}$$

We select the entering variable x_s in such a way that
 1) the value of the leaving variable x_{j_r} increases when
 the value of x_s increases



$$\bar{a}_{rs} < 0$$

Leaving criterion

basic var.	x_1 x_2 ... x_{j_r} ... x_{j_l} ... x_s ... x_{j_m} ... x_n $-Z$	r.h.s.
x_{j_1}	\bar{a}_{11} \bar{a}_{12} ... 0 ... 1 ... \bar{a}_{1s} ... 0 ... \bar{a}_{1n} 0	\bar{b}_1
x_{j_r}	$\left[\begin{array}{cccccccc} \bar{a}_{r1} & \bar{a}_{r2} & \dots & 1 & \dots & 0 & \dots & \bar{a}_{rn} \\ \bar{a}_{rs} & \bar{a}_{rs} & \dots & \bar{a}_{rs} & \dots & \bar{a}_{rs} & \dots & \bar{a}_{rs} \end{array} \right]$	$\left[\begin{array}{c} \bar{b}_r \\ \bar{a}_{rs} \end{array} \right]$
x_{j_m}	\bar{a}_{m1} \bar{a}_{m2} ... 0 ... 0 ... \bar{a}_{ms} ... 1 ... \bar{a}_{mn} 0	\bar{b}_m
$-Z$	\bar{c}_1 \bar{c}_2 ... 0 ... 0 ... \bar{c}_s ... 0 ... \bar{c}_n 1	$-\bar{z}_0$

$$\bar{c}_j \geq 0 \quad \forall j = 1, 2, \dots, n$$

$$\bar{c}_{j_i} = 0 \quad \forall i = 1, 2, \dots, m$$

$$\bar{a}_{rs} < 0 \iff$$

$$\bar{c}_j - \frac{\bar{a}_{rj}}{\bar{a}_{rs}} \bar{c}_s \geq 0, \iff$$

$$\forall j = 1, 2, \dots, n$$

We select the entering variable x_s in such a way that

- 1) the value of the leaving variable x_{j_r} increases when the value of x_s increases
- ii) the relative costs of all the variables remains non negative when the pivot on \bar{a}_{rs} is completed to modify the tableau.

Leaving criterion

basic var.	x_1 x_2 ... x_{j_r} ... x_{j_1} ... x_s ... x_{j_m} ... x_n $-Z$	r.h.s.
x_{j_1}	\bar{a}_{11} \bar{a}_{12} ... 0 ... 1 ... \bar{a}_{1s} ... 0 ... \bar{a}_{1n} 0	\bar{b}_1
x_{j_r}	$\left[\begin{array}{cccccccc} \bar{a}_{r1} & \bar{a}_{r2} & \dots & 1 & \dots & 0 & \dots & \bar{a}_{rn} & 0 \\ \bar{a}_{rs} & \bar{a}_{rs} & \dots & \bar{a}_{rs} & \dots & \dots & \dots & \bar{a}_{rs} & \dots \end{array} \right]$	$\left[\begin{array}{c} \bar{b}_r \\ \bar{a}_{rs} \end{array} \right]$
x_{j_m}	\bar{a}_{m1} \bar{a}_{m2} ... 0 ... 0 ... \bar{a}_{ms} ... 1 ... \bar{a}_{mn} 0	\bar{b}_m
$-Z$	\bar{c}_1 \bar{c}_2 ... 0 ... 0 ... \bar{c}_s ... 0 ... \bar{c}_n 1	$-\bar{z}_0$

$$\bar{c}_j \geq 0 \quad \forall j=1,2,\dots,n$$

$$\bar{c}_{j_i} = 0 \quad \forall i=1,2,\dots,m$$

$$\bar{a}_{rs} < 0 \iff$$

$$\bar{c}_j - \frac{\bar{a}_{rj}}{\bar{a}_{rs}} \bar{c}_s \geq 0, \iff$$

$$\forall j=1,2,\dots,n$$

We select the entering variable x_s in such a way that

If $\bar{a}_{rj} \geq 0$, then the value of \bar{c}_j can only increase since $\bar{c}_s \geq 0$ and $\bar{a}_{rs} < 0$.

ii) the relative costs of all the variables remains non negative when the pivot on \bar{a}_{rs} is completed to modify the tableau.

Leaving criterion

basic var.	x_1 x_2 ... x_{j_r} ... x_{j_l} ... x_s ... x_{j_m} ... x_n $-Z$	r.h.s.
x_{j_1}	\bar{a}_{11} \bar{a}_{12} ... 0 ... 1 ... \bar{a}_{1s} ... 0 ... \bar{a}_{1n} 0	\bar{b}_1
x_{j_r}	$\left[\begin{array}{cccccccc} \bar{a}_{r1} & \bar{a}_{r2} & \dots & 1 & \dots & 0 & \dots & \bar{a}_{rn} \\ \bar{a}_{rs} & \bar{a}_{rs} & \dots & \bar{a}_{rs} & \dots & \bar{a}_{rs} & \dots & \bar{a}_{rs} \end{array} \right]$	$\left[\begin{array}{c} \bar{b}_r \\ \bar{a}_{rs} \end{array} \right]$
x_{j_m}	\bar{a}_{m1} \bar{a}_{m2} ... 0 ... 0 ... \bar{a}_{ms} ... 1 ... \bar{a}_{mn} 0	\bar{b}_m
$-Z$	\bar{c}_1 \bar{c}_2 ... 0 ... 0 ... \bar{c}_s ... 0 ... \bar{c}_n 1	$-\bar{z}_0$

$$\bar{c}_j \geq 0 \quad \forall j=1,2,\dots,n$$

$$\bar{c}_{j_i} = 0 \quad \forall i=1,2,\dots,m$$

$$\bar{a}_{rs} <$$

We select the entering variable x_s in such a way that
 For all j such that $\bar{a}_{rj} < 0$, we have to enforce the non negativity
 of the relative cost by selecting properly the pivot element \bar{a}_{rs} .

$$\bar{c}_j - \frac{\bar{a}_{rj}}{\bar{a}_{rs}} \bar{c}_s \geq 0,$$

⇐

$$\forall j=1,2,\dots,n$$

ii) the relative costs of all the variables remains non
 negative when the pivot on \bar{a}_{rs} is completed to modify
 the tableau.

Entering criterion

For all j such that $\bar{a}_{rj} < 0$, we have to enforce the non negativity of the relative cost; i.e.,

$$\bar{c}_j - \frac{\bar{a}_{rj}}{\bar{a}_{rs}} \bar{c}_s \geq 0, \quad \forall j \text{ such that } \bar{a}_{rj} < 0$$

$$\frac{\bar{c}_j}{\bar{a}_{rj}} - \frac{\bar{c}_s}{\bar{a}_{rs}} \leq 0, \quad \forall j \text{ such that } \bar{a}_{rj} < 0$$

$$\frac{\bar{c}_j}{\bar{a}_{rj}} \leq \frac{\bar{c}_s}{\bar{a}_{rs}}, \quad \forall j \text{ such that } \bar{a}_{rj} < 0$$

Then the index s of the entering variable is such that

$$\frac{\bar{c}_s}{\bar{a}_{rs}} = \max_{j=1,2,\dots,n} \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} : \bar{a}_{rj} < 0 \right\} \quad \text{or} \quad \frac{\bar{c}_s}{-\bar{a}_{rs}} = \min_{j=1,2,\dots,n} \left\{ \frac{\bar{c}_j}{-\bar{a}_{rj}} : \bar{a}_{rj} < 0 \right\}$$

Pivot

- To obtain the simplex tableau associated with the new basis where the entering variable x_s replaces the leaving variable x_r , we complete the pivot on the element $\bar{a}_{rs} < 0$.

Exemple

basic var.	x	y	u	p	h	$-z$	r.h.s.
x	1		1/4		-1/4		-6/4
p			-1/4	1	-3/4		15/2
y		1	-1/12		5/12		13/2
$-z$			3/2		1/2	1	27

- x is the leaving variable, and consequently, the pivot is completed in the first row of the tableau
- h is the entering variable, and consequently, the pivot is completed on the element $-1/4$
- After pivoting, the tableau becomes

basic var.	x	y	u	p	h	$-z$	r.h.s.
h	-4		-1		1		6
p	-3		-1	1			12
y	5/3	1	1/3				4
$-z$			3/2		1/2	1	24

This feasible solution
is optimal

Convergence when the problem is non degenerate

- Non degeneracy assumption:
the relative costs of the non basic variables are positive at each iteration
- **Theorem**: Consider a linear programming problem in standard form.

$$\begin{aligned} \min \quad & z = c^T x \\ \text{Subject to} \quad & Ax = b \\ & x \geq 0 \\ & c, x \in R^n, b \in R^m \\ & A \quad m \times n \text{ matrix} \end{aligned}$$

If the matrix A is of full rank, and if the non degeneracy assumption is verified, then the dual simplex algorithm terminates in a finite number of iterations.

- **Proof:**

Since the rank of matrix A is equal to m , then each basic feasible solution includes m basic variables strictly positive (non degeneracy assumption).

But there is a finite number of ways to select m columns among the n columns of A to specify an $m \times m$ sub matrix of A :

$$\binom{n}{m} = \frac{n!}{m! (n-m)!}$$

But the non feasible basis of A are a subset of these. Then

$$\binom{n}{m} = \frac{n!}{m! (n-m)!}$$

is an upper bound on the number of feasible basis of A .

- The influence of pivoting on the objective function during an iteration of the simplex

basic var.	x_1	x_2	...	x_{j_r}	...	x_{j_1}	...	x_s	...	x_{j_m}	...	x_n	$-Z$
x_{j_1}	\bar{a}_{11}	\bar{a}_{12}	...	0	...	1	...	\bar{a}_{1s}	...	0	...	\bar{a}_{1n}	0
x_{j_r}	\bar{a}_{r1}	\bar{a}_{r2}	...	1	...	0	...	\bar{a}_{rs}	...	0	...	\bar{a}_{rn}	0
x_{j_m}	\bar{a}_{m1}	\bar{a}_{m2}	...	0	...	0	...	\bar{a}_{ms}	...	1	...	\bar{a}_{mn}	0
$-Z$	\bar{c}_1	\bar{c}_2	...	\bar{c}_{j_r}	...	\bar{c}_{j_1}	...	\bar{c}_s	...	\bar{c}_{j_m}	...	\bar{c}_n	1

Deviding row r
by \bar{a}_{rs}

$\bar{b}_r \rightarrow \frac{\bar{b}_r}{\bar{a}_{rs}}$

$\times \bar{c}_s$

Substract from

$$-\bar{z} \rightarrow -\tilde{z} = -\bar{z} - \bar{c}_s \frac{\bar{b}_r}{\bar{a}_{rs}} < -\bar{z}$$

since $\bar{b}_r < 0$, $\bar{a}_{rs} < 0$, and $\bar{c}_s > 0$ under the non degeneracy ass.

$$-\bar{z} \rightarrow -\tilde{z} = -\bar{z} - \bar{c}_s \frac{\bar{b}_r}{a_{rs}} < -\bar{z}$$

since $\bar{b}_r < 0$, $\bar{a}_{rs} < 0$, and $\bar{c}_s > 0$ under the non degeneracy ass.

Then $\tilde{z} > \bar{z}$, and the value of the objective function increases strictly at each iteration.

Consequently, the same basic non feasible solution cannot repeat during the completion of the dual simplex algorithm.

Since the number of basic non feasible solution is bounded, it follows that the dual simplex algorithm must be completed in a finite number of iterations.

Comparing (primal) simplexe alg. and dual simplexe alg.

Simplex alg.

Search in the feasible domain

Search for an entering variable to reduce the value of the objective function

Search for a leaving variable preserving the feasibility of the new solution

Stop when an optimal solution is found or when the problem is not bounded below

Dual simplex alg.

Search out of the feasible domain

Search for a leaving variable to eliminate a negative basic variable

Search for an entering variable preserving the non negativity of the relative costs

Stop when the solution becomes feasible or when the problem is not feasible