

# **Lagrangian Relaxation**

Consider the following mathematical programming problem:

$$(P) \quad \begin{array}{ll} \text{Min} & f(x) \\ \text{s.t} & x \in X_1 \subset R^n \\ & \underline{x \in X_2 \subset R^n} \\ & x \in X \subset R^n \end{array}$$

where  $f : R^n \rightarrow R^1$ .

Suppose that  $X_1$  et  $X_2$  are generated as follows:

$$X_1 = \{x \in R^n : f_i(x) \leq 0, i = 1, \dots, m\}$$
$$X_2 = \{x \in R^n : g_i(x) \leq 0, i = 1, \dots, q\}.$$

Assume that  $x \in X_1$  and  $x \in X$  are constraints **easy** to deal with, then it is interesting to take advantage of putting aside the **complicating** constraints  $x \in X_2$ .

**But** we must account for these in solving then problem.

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$\lambda_i$



Lagrangian relaxation is to replace solving problem (P) by solving a sequence of alternate problems (**lagrangian relaxations**):

$$\begin{aligned}
 (P_\lambda) \quad & G(\lambda) = \text{Min} \left\{ f(x) + \sum_{i=1}^q \lambda_i g_i(x) \right\} \\
 & \text{s.t. } f_i(x) \leq 0 \quad i = 1, \dots, m \\
 & \quad \quad x \in X.
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for different values of the multipliers  $\lambda = [\lambda_1, \dots, \lambda_q]^T \geq 0$ .

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Hence, the **complicating** constraints  $g_1(x), \dots, g_q(x)$  are introduced in the objective function with multipliers  $\lambda_1, \dots, \lambda_q \geq 0$  to simulate their violation, in some sense.

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$$\text{s.t.} \quad x \in X_1 \subset R^n$$

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Referring to lagrangian duality theory, we note that  $G(\lambda)$  is the objective function of the lagrangian dual  $(D)$  of problem  $(P)$  with respect to constraints  $g_1(x), \dots, g_q(x)$ :

$$(D) \quad \text{Max}_{\lambda \geq 0} \{G(\lambda)\} = \text{Max}_{\lambda \geq 0} \left\{ \text{Min}_{\substack{x \in X_1 \\ x \in X}} \left\{ f(x) + \sum_{i=1}^q \lambda_i g_i(x) \right\} \right\}$$

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By the weak duality theorem, the optimal value  $G(\lambda)$  of  $(P_\lambda)$  is a lower bound on the optimal value of  $(P)$ .

This is why  $(P_\lambda)$  is a relaxation of  $(P)$ .

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Furthermore, if we identify a vector of multipliers  $\lambda \geq 0$  such that the optimal solution  $x^\lambda$  of  $(P_\lambda)$  satisfies the conditions:

$$g_i(x^\lambda) \leq 0 \quad i = 1, \dots, q \quad (\text{feasibility})$$

$$\lambda_i g_i(x^\lambda) = 0 \quad i = 1, \dots, q \quad (\text{complementarity})$$

then  $x^\lambda$  is an optimal solution of  $(P)$ .

At each iteration of the **lagrangean relaxation method**, a problem

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$$x \in X.$$

is solved for a fixed value of the multipliers vector  $\lambda = [\lambda_1, \dots, \lambda_q]^T \geq 0$ .

The problem to solve at the next iteration is obtained by modifying the values of the multipliers  $\lambda = [\lambda_1, \dots, \lambda_q]^T \geq 0$ .

One way of doing this modification is by referring to optimizing the dual problem.