

Extragradient and Negative Momentum to Optimize GANs

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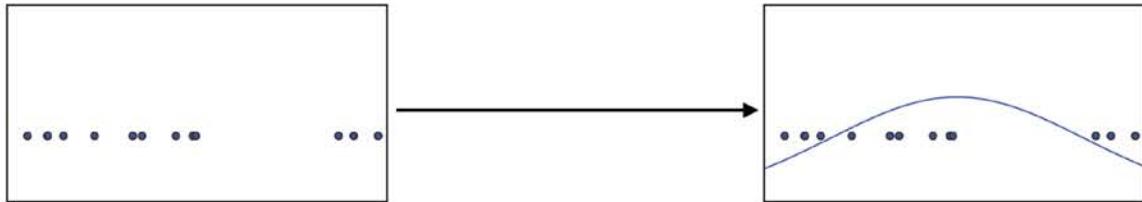
GAN review

(slides courtesy of Aaron Courville)

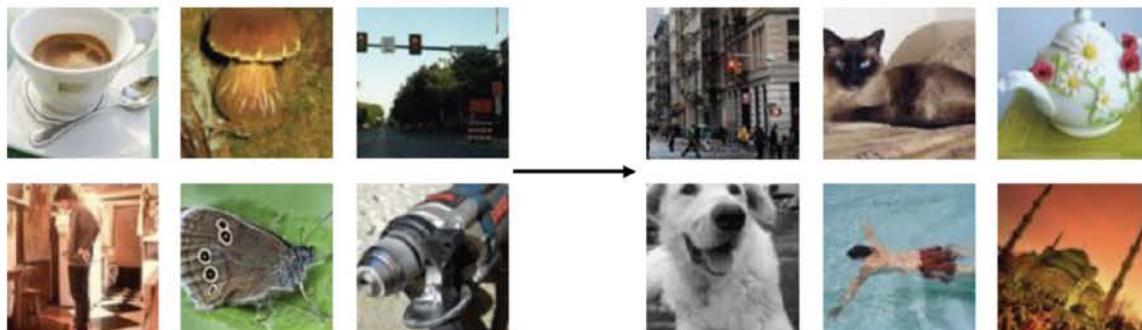
Generative modeling

- Generative models take training samples from some data distribution and learn a model that represents that distribution.

- Density estimation:



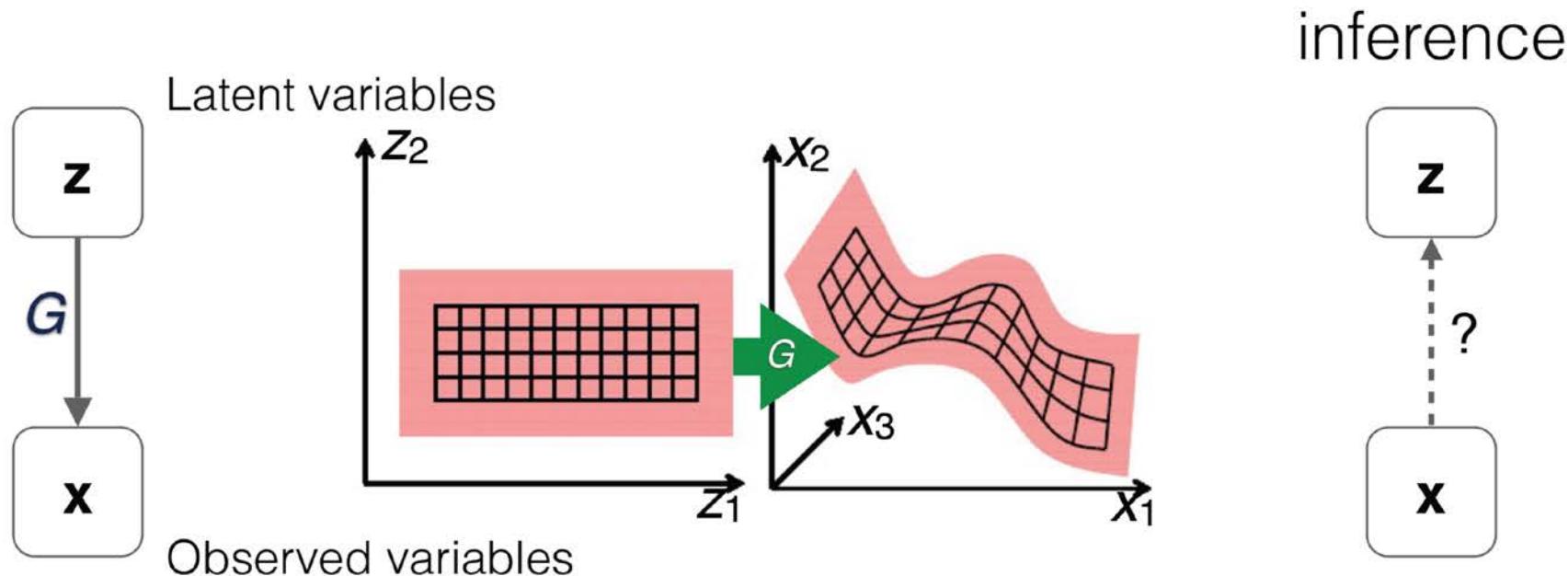
- Sample generation:



Training examples

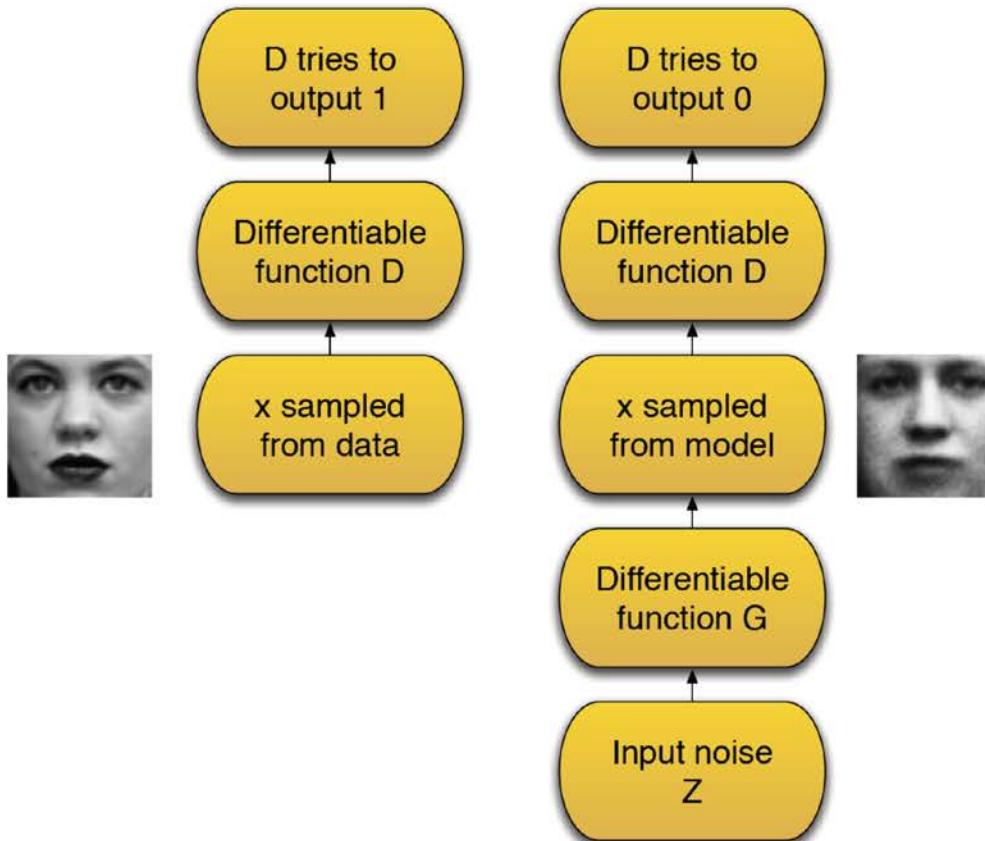
Model samples

How to train a latent variable generative model?



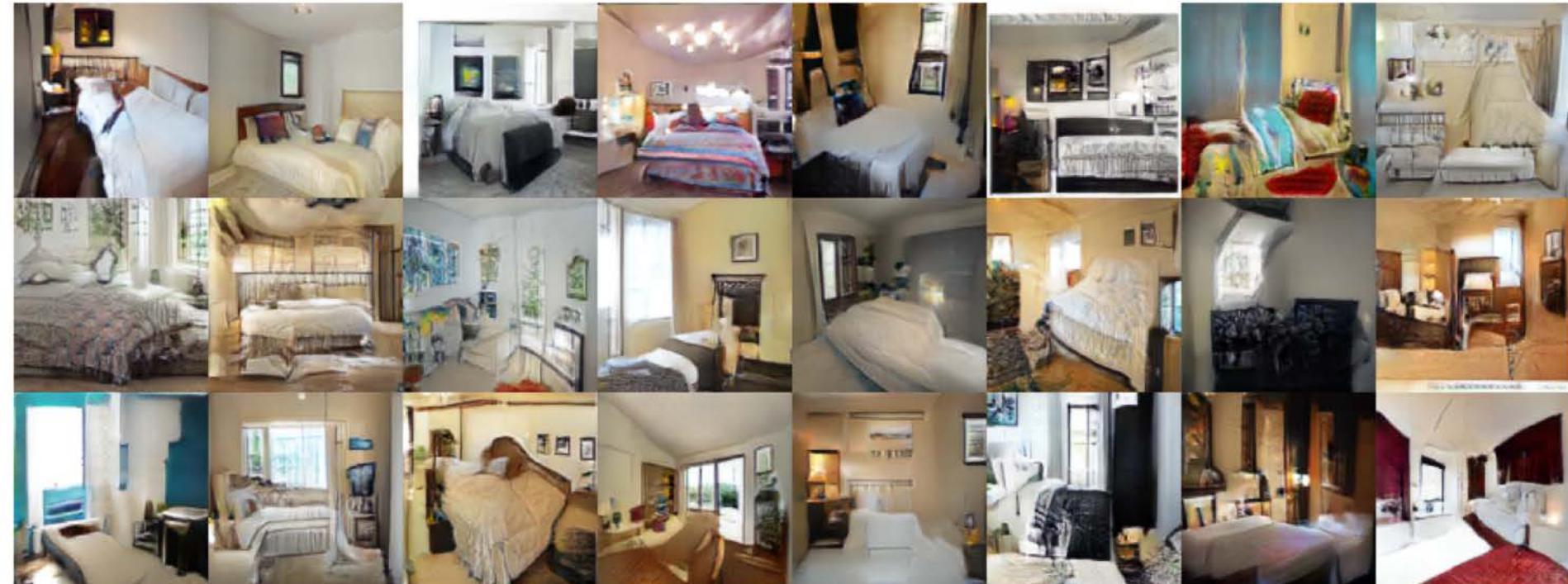
Generative Adversarial Networks

(Goodfellow et al. 2014)



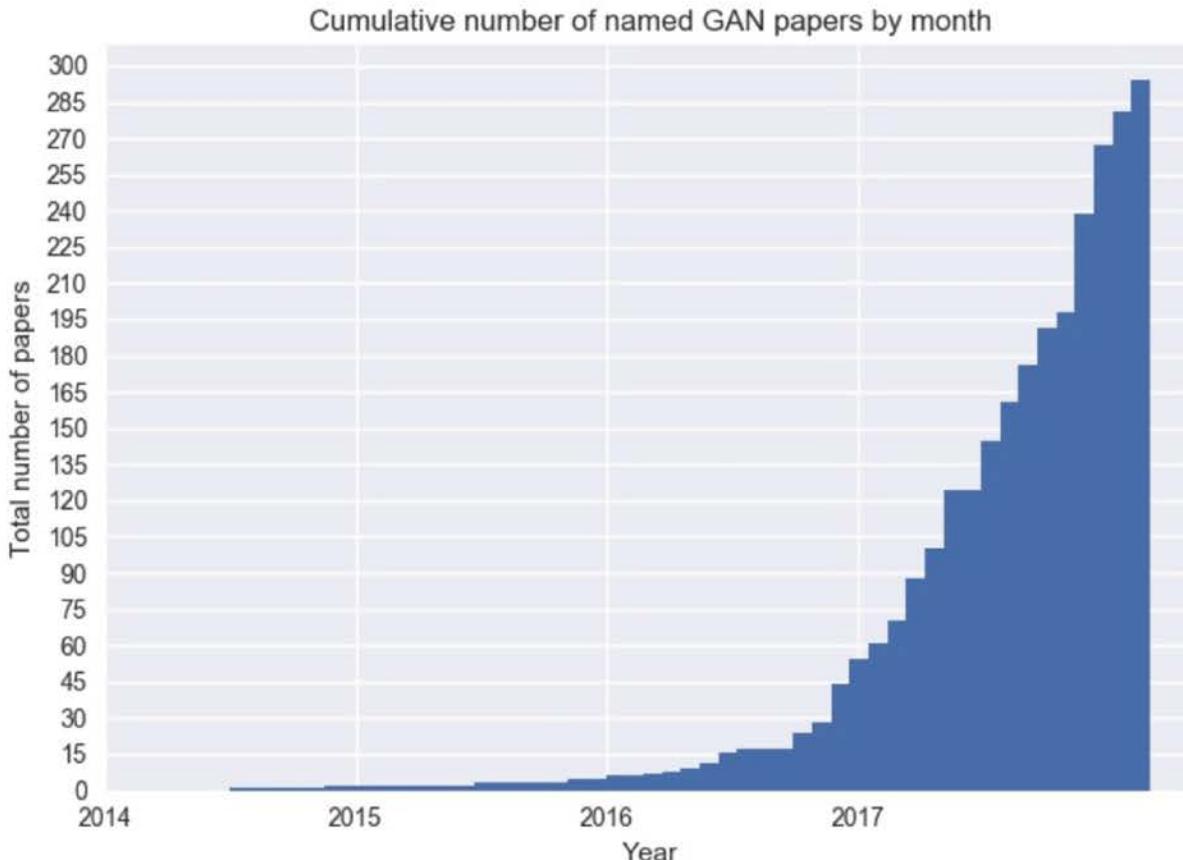
Least-Squares GAN

Xudong Mao, Qing Li[†], Haoran Xie, Raymond Y.K. Lau and Zhen Wang, ArXiv, Feb. 2017



128x128 LSUN bedroom scenes

An explo-GAN of papers



Explosive growth—All the named GAN variants cumulatively since 2014.

Credit: Bruno Gavranović

(from Deep Hunt, blog by Avinash Hindupur)

Optimization for GANs

(my slides!)

GAN and effective task-loss

[Goodfellow et al. NIPS 2014]

- Generative Adversarial Networks (GAN):
 - generator gives $q_\theta \in \mathcal{Q}$
 - discriminator tries to distinguish q_θ from p
 - let $D_\phi(x)$ be belief $\in [0, 1]$ that x came from true distribution p
 - GAN (population) objective:
$$\min_{q_\theta} \max_{D_\phi} \mathbb{E}_{x \sim p} [\log D_\phi(x)] + \mathbb{E}_{x \sim q_\theta} [\log(1 - D_\phi(x))]$$
- if D non-parametric: get $L_p(\theta) \approx \text{JSD}(p||q_\theta)$
 - (KL-like) \Rightarrow no good!
- but D in practice has limited capacity (e.g. NN): $\phi \in \Phi$
 - \Rightarrow induces a meaningful task loss $L_p(\theta)$!
 - “parametric adversarial divergence” [Huang et al. arxiv 2017]

A Variational Inequality Perspective on GANs

[G. Gidel*, H. Berard*, Gaëtan Vignoud, P. Vincent, S. Lacoste-Julien,
arXiv 2018 – to appear at ICLR 2019]

with:



Gauthier
Gidel



Hugo
Berard



Pascal
Vincent



Optimization perspective of GAN

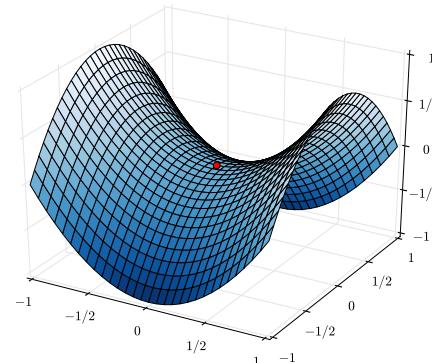
- Min-Max – two player (zero-sum) game:

$$\begin{array}{ccc} \text{generator} & & \text{discriminator params} \\ \text{params} & \searrow & \swarrow \\ & \min_{\theta \in \Theta} \max_{\phi \in \Phi} \bar{\mathcal{L}}(\theta, \phi). & \end{array}$$

- Min-max also called *saddle point* in math. programming

example: WGAN formulation:

$$\min_{\theta \in \Theta} \max_{\phi \in \Phi, \|f_\phi\|_L \leq 1} \mathbb{E}_{\mathbf{x} \sim p} [f_\phi(\mathbf{x})] - \mathbb{E}_{\mathbf{x}' \sim q_\theta} [f_\phi(\mathbf{x}')].$$



More general...

as noted by [Mescheder, Nowocin, Geiger; NIPS 17]...

- Non-zero sum game: **different cost functions**

$$\begin{cases} \theta^* \in \arg \min_{\theta \in \Theta} \mathcal{L}^{(\theta)}(\theta, \phi^*) \\ \phi^* \in \arg \min_{\phi \in \Phi} \mathcal{L}^{(\phi)}(\theta^*, \phi) . \end{cases}$$

$$\mathcal{L}^{(\theta)} = -\mathcal{L}^{(\phi)} \quad , \text{ then } \mathbf{zero-sum} \text{ i.e. min-max)$$

- Example: non-saturating GAN:
(original working version from [Goodfellow et al. NIPS 14])

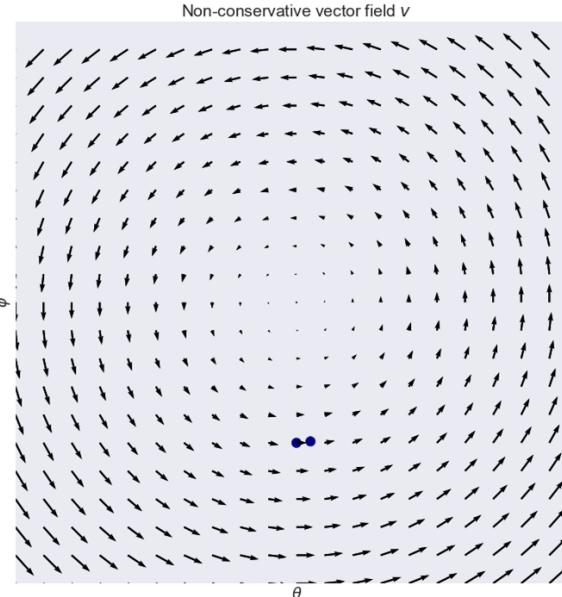
$$\mathcal{L}^{(\theta)}(\theta, \phi) \stackrel{\text{def}}{=} - \mathbb{E}_{\mathbf{x}' \sim q_\theta} \log f_\phi(\mathbf{x}')$$

$$\mathcal{L}^{(\phi)}(\theta, \phi) \stackrel{\text{def}}{=} -\mathbb{E}_{\mathbf{x} \sim p} \log f_\phi(\mathbf{x}) - \mathbb{E}_{\mathbf{x}' \sim q_\theta} \log(1 - f_\phi(\mathbf{x}')) .$$

Pet peeve from GAN literature...

- “Saddle points are hard to optimize”
- example from
Goodfellow NIPS 16 tutorial:
simple bilinear objective
- WGAN with *linear* discriminator
and generator:

$$\min_{\theta \in \Theta} \max_{\phi \in \Phi, \|\phi\| \leq 1} \phi^T \mathbb{E}[\mathbf{X}] - \phi^T \theta \mathbb{E}[\mathbf{Z}] .$$



Standard methods from Math. Prog.

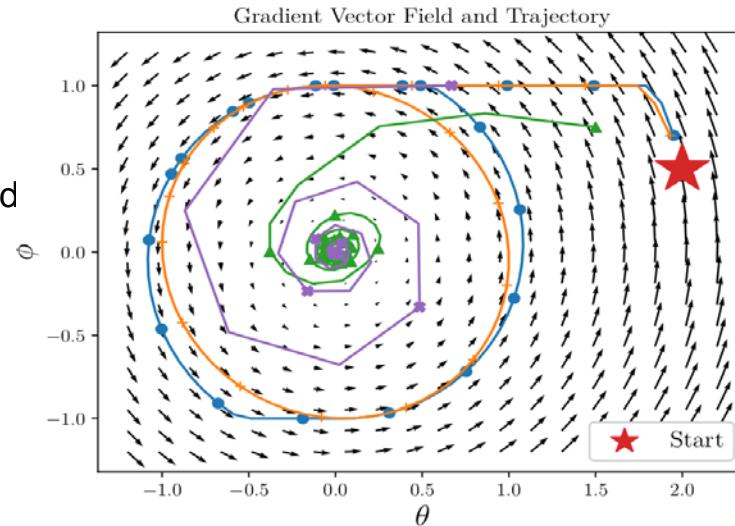
- But saddle points can be optimized!

non-convergent

- Blue: Standard gradient method.
- Orange: Alternating gradient method

convergent

- Green: Gradient method with avg.
- Purple: Extragradient method.



from *variational inequality* literature

GAN as a variational inequality

- new perspective on GANs
 - based on stationary conditions
 - relate to vast literature in math. opt. with standard algorithms

- stationary conditions for game:

$$\begin{cases} \theta^* \in \arg \min_{\theta \in \Theta} \mathcal{L}^{(\theta)}(\theta, \phi^*) \\ \phi^* \in \arg \min_{\phi \in \Phi} \mathcal{L}^{(\phi)}(\theta^*, \phi) . \end{cases}$$

$$\begin{cases} \nabla_{\theta} \mathcal{L}^{(\theta)}(\theta^*, \phi^*)^\top (\theta - \theta^*) \geq 0, \\ \nabla_{\phi} \mathcal{L}^{(\phi)}(\theta^*, \phi^*)^\top (\phi - \phi^*) \geq 0 \end{cases} \quad \forall (\theta, \phi) \in \Theta \times \Phi.$$

- interpretation: each player can't improve cost function

GAN as a variational inequality

- stationary conditions:

$$\begin{cases} \nabla_{\theta}\mathcal{L}^{(\theta)}(\theta^*, \phi^*)^\top(\theta - \theta^*) \geq 0, \\ \nabla_{\phi}\mathcal{L}^{(\phi)}(\theta^*, \phi^*)^\top(\phi - \phi^*) \geq 0 \end{cases} \quad \forall (\theta, \phi) \in \Theta \times \Phi.$$

- rewrite as:

Defining $\mathbf{x} \stackrel{\text{def}}{=} (\theta, \phi)$, $\mathbf{x}^* \stackrel{\text{def}}{=} (\theta^*, \phi^*)$, $\mathcal{X} \stackrel{\text{def}}{=} \Theta \times \Phi$ and,

$$F(\mathbf{x}) \stackrel{\text{def}}{=} \begin{pmatrix} \nabla_{\theta}\mathcal{L}^{(\theta)}(\theta, \phi) \\ \nabla_{\phi}\mathcal{L}^{(\phi)}(\theta, \phi) \end{pmatrix}, \quad (7)$$

the stationary conditions can be compactly formulated as:

$$F(\mathbf{x}^*)^\top(\mathbf{x} - \mathbf{x}^*) \geq 0, \quad \forall \mathbf{x} \in \mathcal{X}. \quad (8)$$

say " \mathbf{x}^* solves the **variational inequality**"

GAN as a variational inequality

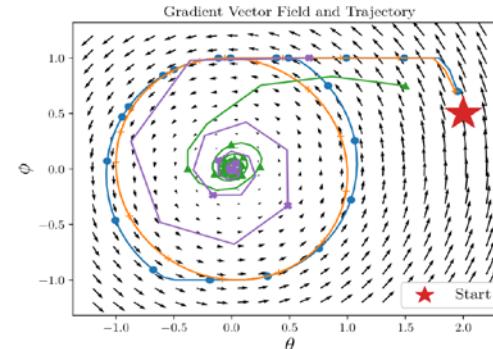
takeaways:

- GAN objective even with different cost functions for the discriminator and the generator,
- Can be formulated as a **variational inequality** .
- Can import to GANs standard variational inequality algorithms (+ make their stochastic extensions)
- With theoretical guarantees (at least for “convex” cost functions)

$$\begin{cases} \theta^* \in \arg \min_{\theta \in \Theta} \mathcal{L}^{(\theta)}(\theta, \phi^*) \\ \phi^* \in \arg \min_{\phi \in \Phi} \mathcal{L}^{(\phi)}(\theta^*, \phi) . \end{cases}$$



$$F(x^*)^\top (x - x^*) \geq 0, \quad \forall x \in \mathcal{X}.$$



Convergent methods for V.I.

gradient method

with averaging

[Bruck 1977]

stochastic: [Nemirovski et al. 09]

extragradient method

[Korpelich 1976]

stochastic: [Juditsky et al. 11]

- smaller variance

Algorithm 1 Stochastic Gradient Method (AvgSGD)

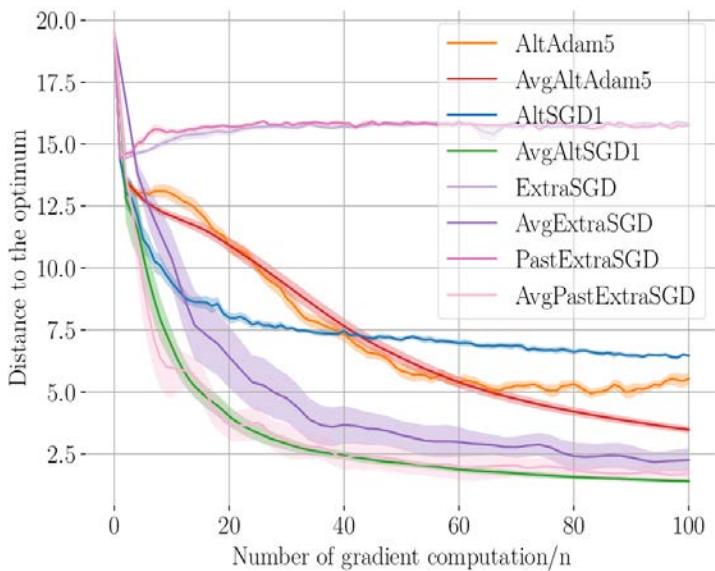
```
1: Let  $x_0 \in \mathcal{X}$ 
2: for  $t = 0 \dots T - 1$  do
3:   Sample  $\xi_t \sim P$ 
4:   Update  $\bar{x}_{t+1} := P_{\mathcal{X}}(\bar{x}_t - \gamma_t F(\bar{x}_t, \xi_t))$ 
5: end for
6: Return  $\bar{x}_T = \sum_{t=0}^{T-1} \gamma_t \bar{x}_t / \sum_{t=0}^{T-1} \gamma_t$ 
```

Algorithm 2 Stochastic Extragradient Method (SEM)

```
1: Let  $x_0 \in \mathcal{X}$ 
2: for  $t = 0 \dots T - 1$  do
3:   Sample  $\xi_t \sim P$   $\triangleright$  Extrapolation step
4:   Compute  $\bar{x}_{t+\frac{1}{2}} := P_{\mathcal{X}}(x_t - \gamma_t F(x_t, \xi_t))$ 
5:   Sample  $\xi_{t+\frac{1}{2}} \sim P$   $\triangleright$  Update step
6:   Update  $x_{t+1} := P_{\mathcal{X}}(\bar{x}_{t+\frac{1}{2}} - \gamma_t F(\bar{x}_{t+\frac{1}{2}}, \xi_{t+\frac{1}{2}}))$ 
7: end for
8: Return  $\bar{y}_T = \sum_{t=0}^{T-1} \gamma_t \bar{x}_{t+1/2} / \sum_{t=0}^{T-1} \gamma_t$ 
```

Bilinear stochastic experiment

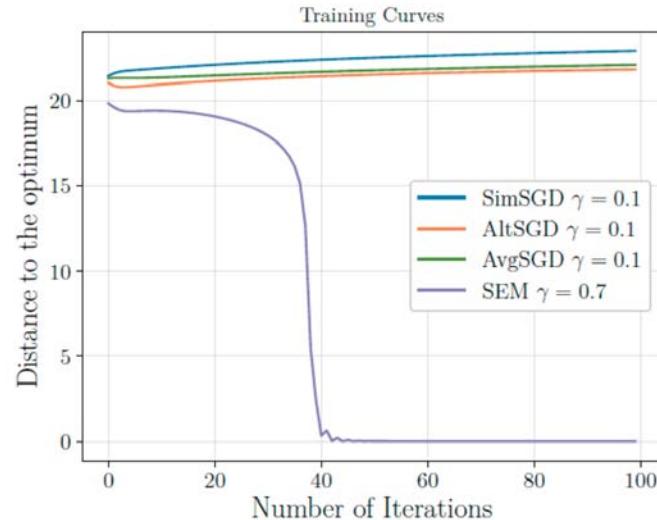
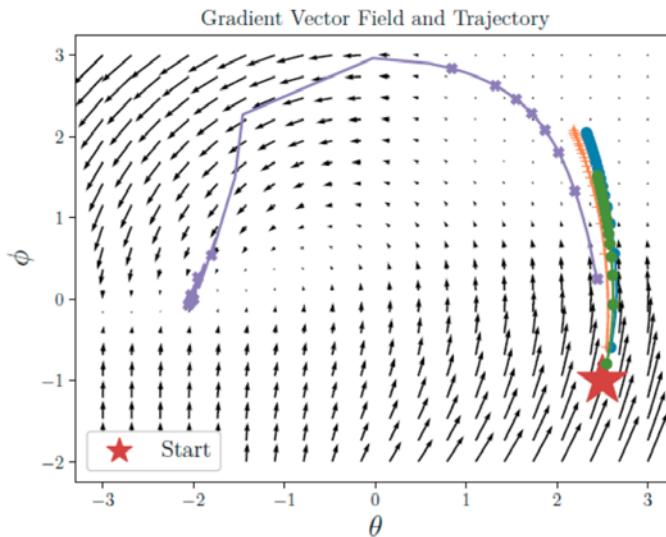
bilinear stochastic objective: $\frac{1}{n} \sum_{i=1}^n \left(\mathbf{x}^\top \mathbf{M}^{(i)} \mathbf{y} + \mathbf{x}^\top \mathbf{a}^{(i)} + \mathbf{y}^\top \mathbf{b}^{(i)} \right).$



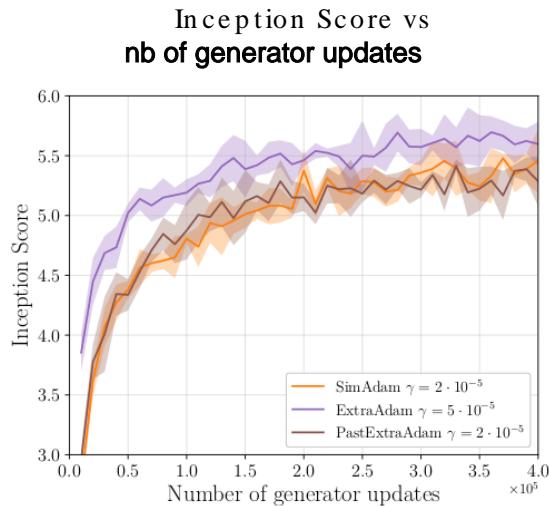
Non-convex toy example

- min-max GAN with:
 - linear discriminator $D_\phi(\mathbf{x}) = \phi^T \mathbf{x}$
 - p & q are diracs $p = \delta_{\mathbf{x}_0} \quad q_\theta = \delta_\theta$

$$\min_{\theta} \max_{\phi} -\log \left(1 + e^{-\phi^T \theta} \right) - \log \left(1 + e^{\phi^T \mathbf{x}_0} \right)$$



Experimental Results: WGAN on CIFAR10



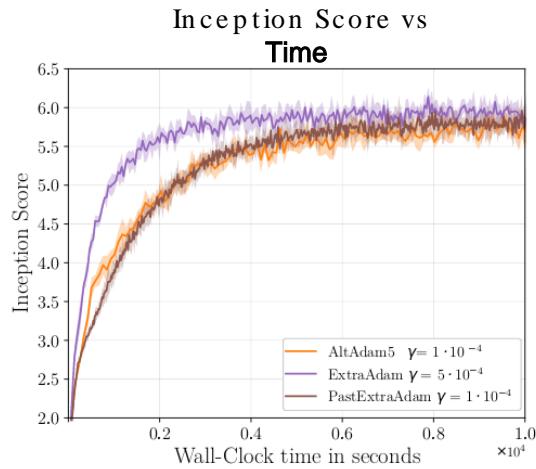
Inception Score on CIFAR10

Model	WGAN			
	Method	no averaging	uniform avg	EMA
SimAdam		$6.05 \pm .12$	$5.83 \pm .16$	$6.08 \pm .10$
AltAdam5		$5.45 \pm .08$	$5.72 \pm .06$	$5.49 \pm .05$
ExtraAdam		$6.38 \pm .09$	$6.38 \pm .20$	$6.37 \pm .08$
PastExtraAdam		5.98 ± 0.15	6.07 ± 0.19	6.01 ± 0.11
OptimAdam		5.74 ± 0.10	5.80 ± 0.08	5.78 ± 0.05

Extragradient Methods

Averaging

Experimental Results: WGAN-GP on CIFAR10



Model	Inception Score on CIFAR10		
	WGAN-GP		
Method	no averaging	no averaging	uniform avg
SimAdam	$6.05 \pm .12$	$6.00 \pm .07$	$6.01 \pm .08$
AltAdam5	$5.45 \pm .08$	$6.25 \pm .05$	$6.51 \pm .05$
ExtraAdam	$6.38 \pm .09$	$6.22 \pm .04$	$6.35 \pm .05$
PastExtraAdam	5.98 ± 0.15	6.27 ± 0.06	6.23 ± 0.13
OptimAdam	5.74 ± 0.10	-	-

Extragradient Methods

Averaging

Latest results

Model	WGAN (DCGAN)			WGAN-GP (ResNet)	
	Method	no averaging	uniform avg	EMA	no averaging
SimAdam	$6.05 \pm .12$	$5.83 \pm .16$	$6.08 \pm .10$	$7.54 \pm .21$	$7.74 \pm .27$
AltAdam5	$5.45 \pm .08$	$5.72 \pm .06$	$5.49 \pm .05$	$7.20 \pm .06$	$7.67 \pm .15$
ExtraAdam	$6.38 \pm .09$	$6.38 \pm .20$	$6.37 \pm .08$	$7.79 \pm .09$	$8.26 \pm .12$
PastExtraAdam	$5.98 \pm .15$	$6.07 \pm .19$	$6.01 \pm .11$	$7.71 \pm .12$	$7.84 \pm .18$
OptimAdam	$5.74 \pm .10$	$5.80 \pm .08$	$5.78 \pm .05$	$7.80 \pm .07$	$7.99 \pm .12$

Discussion

- New framework called variational inequality to consider GAN objectives.
- Tapped into the optimization literature to bring new techniques to train GANs
- With caveats:
 - Averaging seems to consistently improve the inception score at convergence,
 - Extragradient looks to be more stable for step size tuning.
- Just considered two standard algorithms.
- Could design algorithm specific to GANs inspired from variational inequalities.

Negative Momentum for Improved Game Dynamics

Gauthier Gidel, Reyhane Askari Hemmat, Mohammad Pezeshki, Gabriel Huang, Remi Lepriol, Simon Lacoste-Julien, Ioannis Mitliagkas

<https://arxiv.org/abs/1807.04740>
to appear at AISTATS 2019

Game dynamics are

~~weird~~

fascinating

Start with optimization
dynamics

Optimization

$$\theta^* \in \arg \min_{\theta \in \Theta} \mathcal{L}^{(\theta)}(\theta)$$

Smooth, differentiable cost function, L

→ Looking for stationary (fixed) points
(gradient is 0)

→ Gradient descent

Optimization

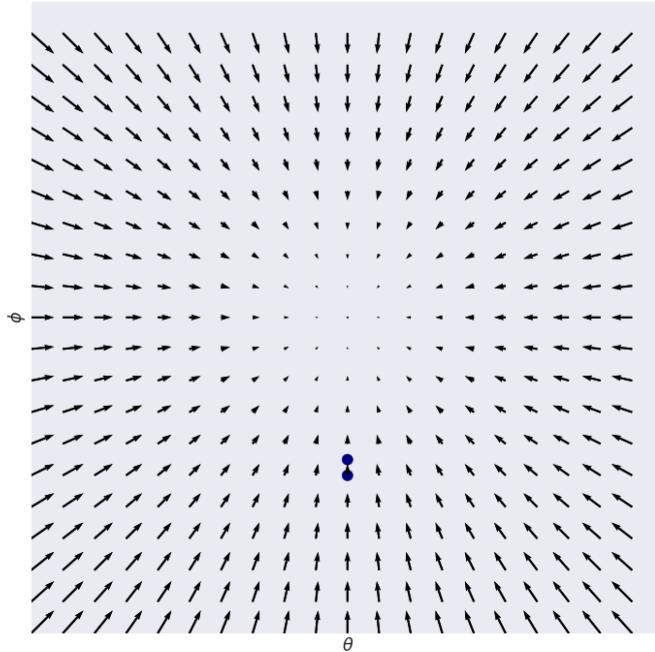
$$\mathbf{v}(\boldsymbol{\theta}) = \nabla \mathcal{L}^{(\boldsymbol{\theta})}(\boldsymbol{\theta})$$

Conservative vector field



Straightforward dynamics

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \eta \mathbf{v}(\boldsymbol{\theta}_t)$$



Gradient descent

$$\mathbf{v}(\boldsymbol{\theta}) = \nabla \mathcal{L}^{(\boldsymbol{\theta})}(\boldsymbol{\theta})$$

Conservative vector field

→

Straightforward dynamics

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \eta \mathbf{v}(\boldsymbol{\theta}_t)$$

Fixed-point analysis

$$F_\eta(\boldsymbol{\theta}) = \boldsymbol{\theta} - \eta \mathbf{v}(\boldsymbol{\theta})$$

Jacobian of operator

$$\nabla F_\eta(\boldsymbol{\theta}) = I - \eta \underline{\nabla \mathbf{v}(\boldsymbol{\theta})}$$

Hessian of objective, L

Local convergence

Theorem 1 (Prop. 4.4.1 Bertsekas [1999]). *If the spectral radius $\rho_{\max} \stackrel{\text{def}}{=} \rho(\nabla F_\eta(\omega^*)) < 1$, then, for ω_0 in a neighborhood of ω^* , the distance of ω_t to the stationary point ω^* converges at a linear rate of $\mathcal{O}((\rho_{\max} + \epsilon)^t)$, $\forall \epsilon > 0$.*

Eigenvalues of op. Jacobian

$$\lambda(\nabla F_\eta(\theta)) = 1 - \eta \lambda(\nabla v(\theta))$$

If $\rho(\theta^*) = \max |\lambda(\theta^*)| < 1$, then
fast local convergence

Jacobian of operator

$$\nabla F_\eta(\theta) = I - \eta \underline{\nabla v(\theta)}$$

Hessian of objective, L
Symmetric, real-eigenvalues

Momentum (heavy ball, Polyak 1964)

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \eta \boldsymbol{v}(\boldsymbol{\theta}_t) + \beta (\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})$$

Jacobian of momentum operator

$$\nabla F_{\eta, \beta}(\boldsymbol{\theta}_t, \boldsymbol{\theta}_{t-1}) = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ \mathbf{I}_n & \mathbf{0}_n \end{bmatrix} - \eta \begin{bmatrix} \nabla \boldsymbol{v}(\boldsymbol{\theta}_t) & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{bmatrix} + \beta \begin{bmatrix} \mathbf{I}_n & -\mathbf{I}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{bmatrix}$$

**Non-symmetric, with complex eigenvalues
→ Rotations in augmented state-space**

Games

Games

Nash Equilibrium

$$\theta^* \in \arg \min_{\theta \in \Theta} \mathcal{L}^{(\theta)}(\theta, \varphi^*)$$

$$\varphi^* \in \arg \min_{\varphi \in \Phi} \mathcal{L}^{(\varphi)}(\theta^*, \varphi)$$

Smooth, differentiable L
→ Looking for local Nash eq

→ Gradient descent

→ **Simultaneous**

→ **Alternating**

Game dynamics

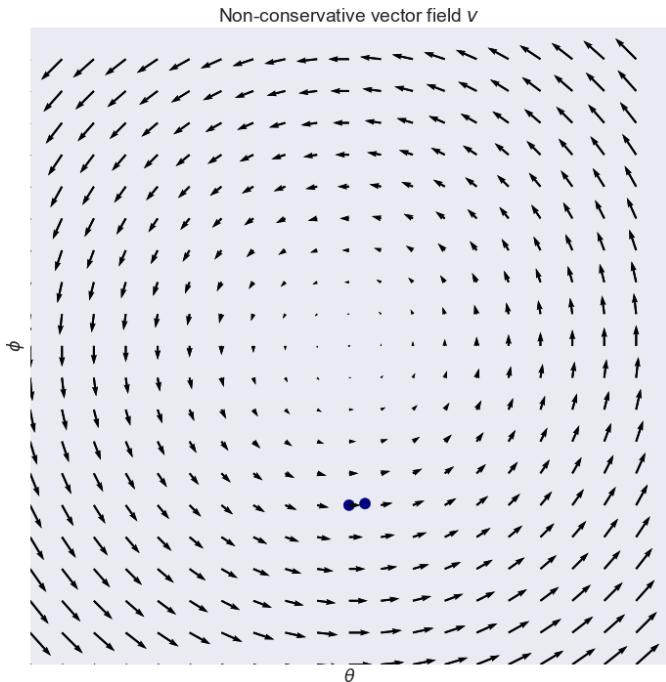
$$v(\varphi, \theta) := \begin{bmatrix} \nabla_{\varphi} \mathcal{L}^{(\varphi)}(\varphi, \theta) \\ \nabla_{\theta} \mathcal{L}^{(\theta)}(\varphi, \theta) \end{bmatrix}$$

Non-conservative vector field



Rotational dynamics

$$F_{\eta}(\varphi, \theta) \stackrel{\text{def}}{=} [\varphi \quad \theta]^{\top} - \eta v(\varphi, \theta)$$



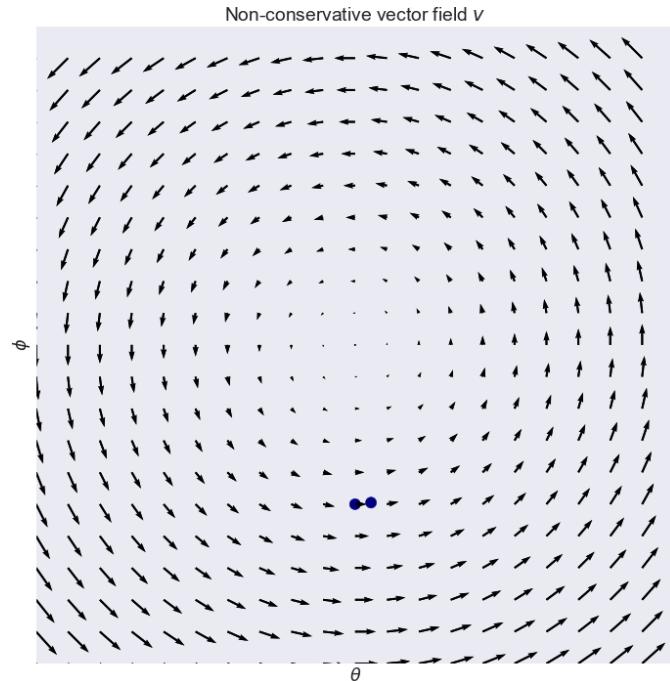
Game dynamics under gradient descent

$$F_\eta(\varphi, \theta) \stackrel{\text{def}}{=} [\varphi \quad \theta]^\top - \eta v(\varphi, \theta)$$

Jacobian is non-symmetric, with complex eigenvalues → Rotations in decision space

Games already demonstrate rotational dynamics.

Can momentum help/hurt??



Spoiler

Positive momentum can be bad for adversarial games

Practice that was very common when GANs were first invented:

- Recent work reduced the momentum .
- Not an accident

Our results

Negative momentum is “optimal” on simple bilinear game

Negative momentum values are locally preferable near 0 on a more general class of games

Negative momentum is empirically best for certain zero sum games like “saturating GANs”

Momentum on games

Recall Polyak's momentum (on top of simultaneous grad. desc.):

$$\boldsymbol{x}_{t+1} = \boldsymbol{x}_t - \eta \boldsymbol{v}(\boldsymbol{x}_t) + \beta(\boldsymbol{x}_t - \boldsymbol{x}_{t-1}), \quad \boldsymbol{x}_t = (\boldsymbol{\theta}_t, \boldsymbol{\phi}_t)$$

Fixed point operator requires a **state augmentation** :
(because we need previous iterate)

$$F_{\eta, \beta}(\boldsymbol{x}_t, \boldsymbol{x}_{t-1}) := \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ \mathbf{I}_n & \mathbf{0}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{x}_{t-1} \end{bmatrix} - \eta \begin{bmatrix} \boldsymbol{v}(\boldsymbol{x}_t) \\ \mathbf{0}_n \end{bmatrix} + \beta \begin{bmatrix} \mathbf{I}_n & -\mathbf{I}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{x}_{t-1} \end{bmatrix}$$

Bilinear game

$$\min_{\theta} \max_{\varphi} \theta^\top A \varphi$$

Method	β	Bounded	Converges
Simultaneous	$\beta \in \mathbb{R}$	✗	✗
Alternated	>0	✗	✗
	0	✓	✗
	<0	✓	✓

“Proof by picture”

Gradient descent



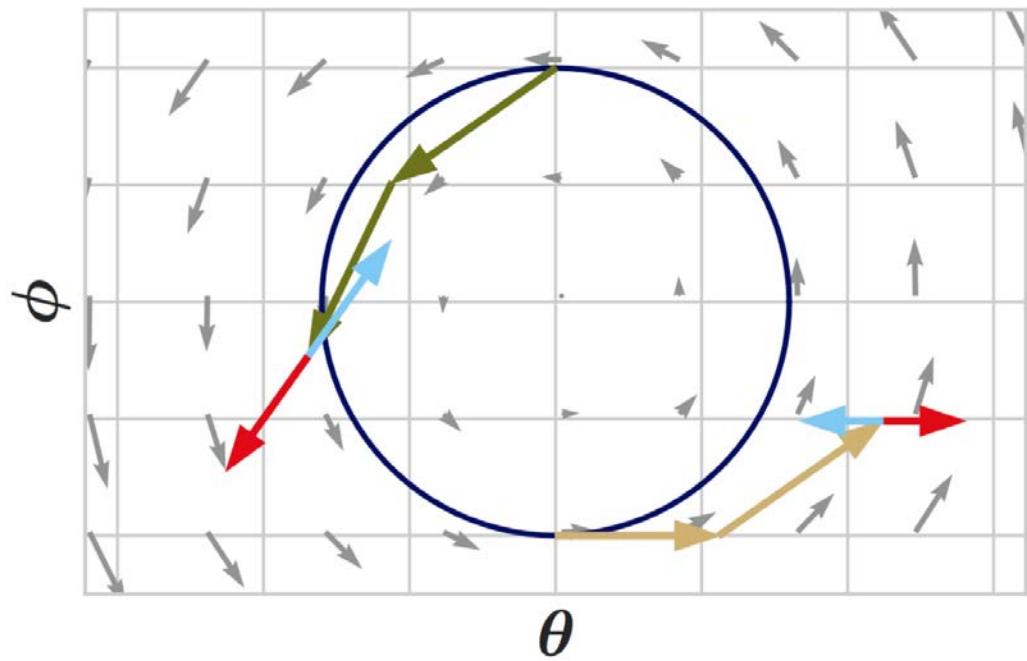
Simultaneous

→ Alternating

Momentum

→ Positive

→ Negative



General games

Without momentum

$$F_\eta(\varphi, \theta) \stackrel{\text{def}}{=} [\varphi \quad \theta]^\top - \eta \mathbf{v}(\varphi, \theta)$$

$$\nabla F_\eta(\theta) = I - \eta \nabla \mathbf{v}(\theta)$$

$$\text{Sp}(\nabla F_\eta(\omega^*)) = \{1 - \eta\lambda | \lambda \in \text{Sp}(\nabla \mathbf{v}(\omega^*))\}$$

Theorem 4: Given a large class of games, a small negative momentum can improve the magnitude of the limiting eigenvalues

With momentum

$$\begin{aligned} F_{\eta, \beta}(\omega_t, \omega_{t-1}) &= (\omega_{t+1}, \omega_t) \\ \omega_{t+1} &:= \omega_t - \eta \mathbf{v}(\omega_t) + \beta(\omega_t - \omega_{t-1}) \end{aligned}$$

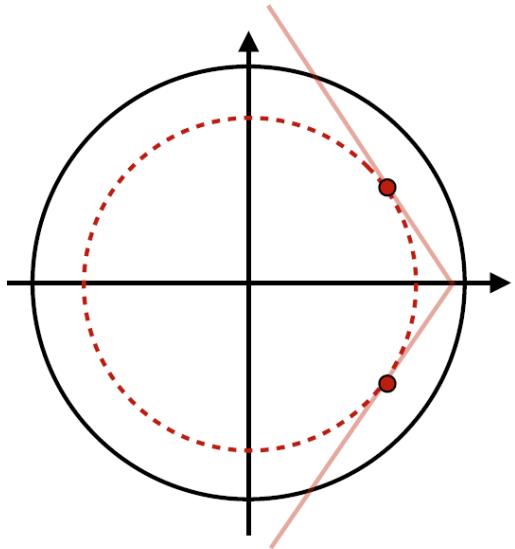
$$\begin{bmatrix} I_n & \mathbf{0}_n \\ I_n & \mathbf{0}_n \end{bmatrix} - \eta \begin{bmatrix} \nabla \mathbf{v}(\omega_t) & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{bmatrix} + \beta \begin{bmatrix} I_n & -I_n \\ \mathbf{0}_n & \mathbf{0}_n \end{bmatrix}$$

$$\mu_{\pm}(\beta, \eta, \lambda) := (1 - \eta\lambda + \beta) \frac{1 \pm \Delta^{\frac{1}{2}}}{2},$$

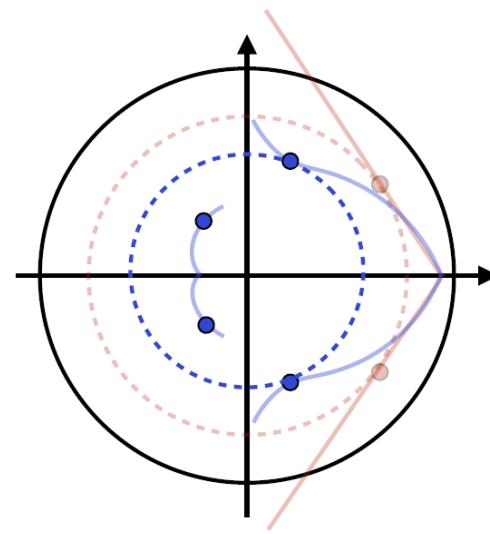
Shifting the eigen-values




Without momentum



With negative momentum



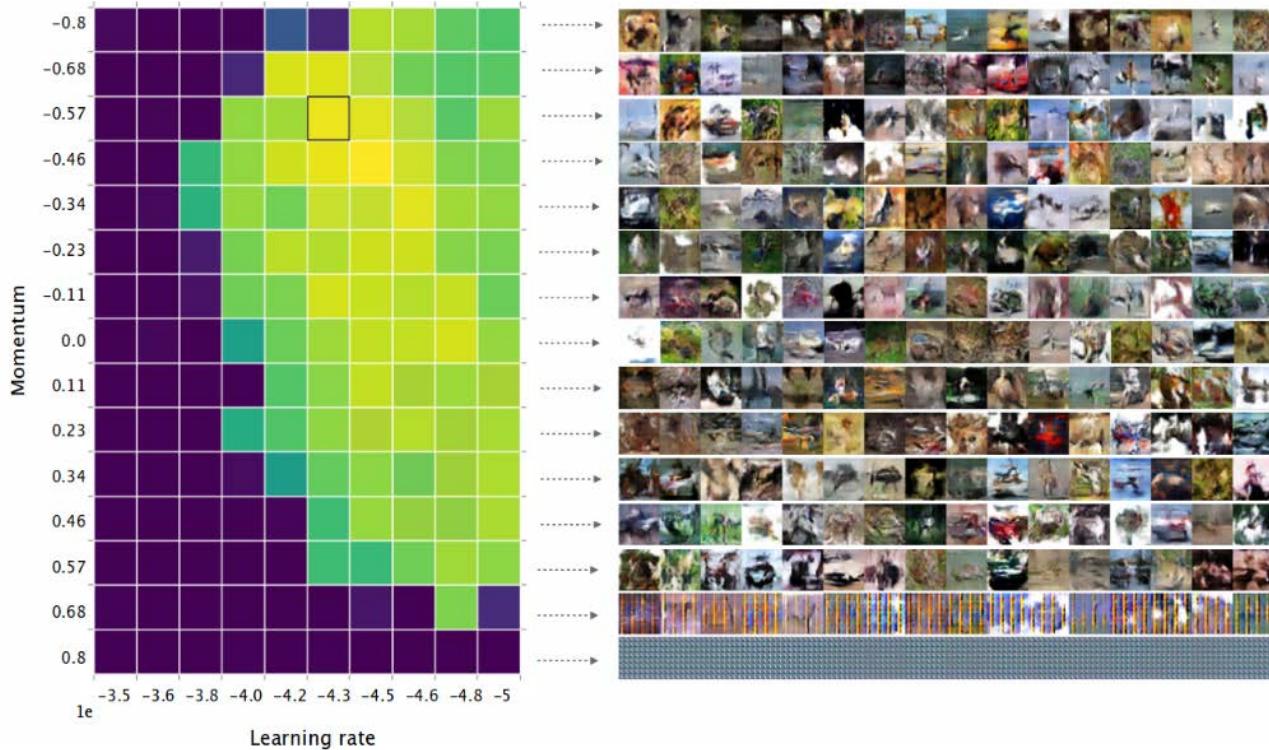
$$\text{Sp}(\nabla F_\eta(\omega^*)) = \{1 - \eta\lambda | \lambda \in \text{Sp}(\nabla v(\omega^*))\}$$

$$\mu_{\pm}(\beta, \eta, \lambda) := (1 - \eta\lambda + \beta) \frac{1 \pm \Delta^{\frac{1}{2}}}{2},$$

Empirical results

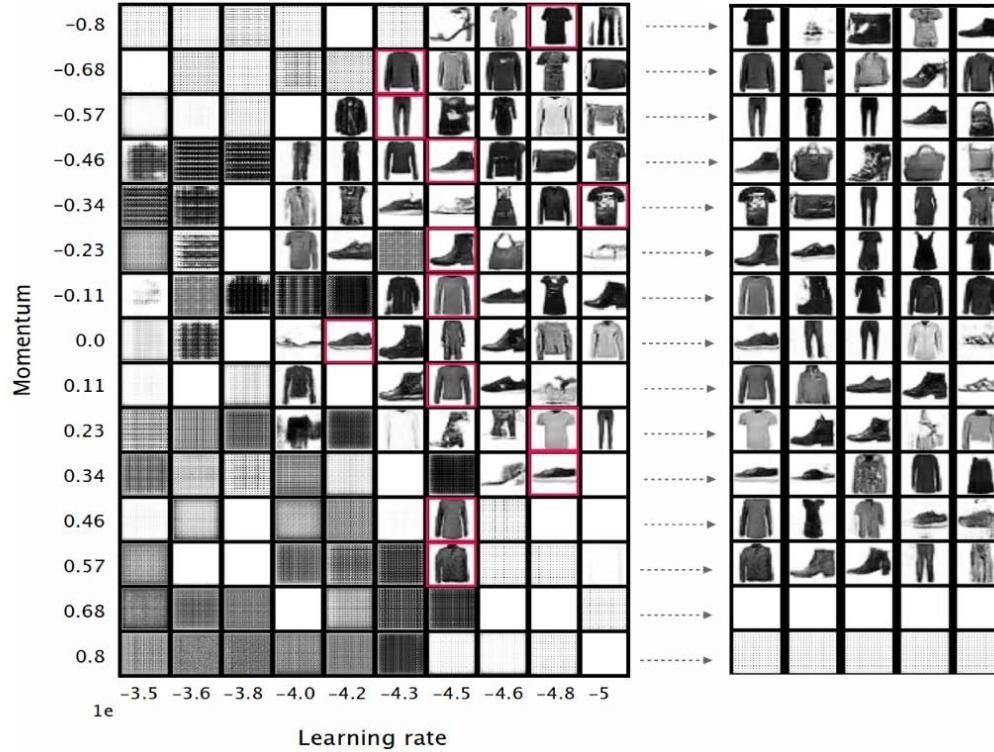
What happens in practice ?

CIFAR-10 :



What happens in practice ?

Fashion MNIST:



Negative Momentum

To sum up:

- Negative momentum seems to improve the behaviour due to “bad” eigenvalues.
- Optimal for a class of games
- Empirically optimal on “saturating” GANs

SMOOTH GAMES OPTIMIZATION AND MACHINE LEARNING WORKSHOP

Room 512 ABEF, Friday Dec 7th,
NeurIPS2018 , Montreal.

VIDEOS OF THE WORKSHOP

<https://sgo-workshop.github.io/>

Thank you !