Generalizations of the Surrogate Estimation Approach for Sensitivity Analysis

Felisa J. Vázquez-Abad
Department of Computer Science and OR, University of Montreal, and
Department of Electrical and Electronic Engineering, University of Melbourne
e-mail: vazquez@IRO.UMontreal.CA, fva@EE.Mu.oz.AU

Lourdes Zubieta
Department of Business Administration, Bishop's University
e-mail: lzubieta@ubishops.ca

Abstract

Gradient-based optimization of stochastic dynamic systems require "good" estimators of the sensitivity to the control variables. Numerous methods have been proposed for gradient estimation of discrete event systems, usually requiring an impractical amount of information and/or computational effort. Alternative solutions have been proposed in the literature using other estimates to drive the learning algorithm, but up to date such surrogates have been constructed case by case with *ad-hoc* arguments. In this paper we identify three main tools for the systematic construction of surrogate estimators: changes in time scale, changes of variables with the mean flow approach, and changes of variables through uniformization. We introduce each method with a detailed example of application. Surrogate estimation is also generalized with the introduction of correction factors, illustrated in the examples.

1 Introduction

This paper addresses the problem of sensitivity estimation for adaptive control. The model is that of a dynamic stochastic process $\{X_t; t > 0\}$ under a control strategy which is parameterized by a vector $\theta \in \mathbb{R}^c$. Call $\mathcal{F}_t$ the σ-algebra of the process up to time t. To stress the dependence of the dynamics on θ, the probability space is denoted (Ω, P_θ). Such models cover Generalized Semi-Markov Processes (see [5]). The performance function to be optimized is of the form of a finite or infinite horizon average $F(\theta)$. Under some regularity assumptions, [8] show that a stochastic version of the steepest ascent method can achieve asymptotic optimality, provided that "good" estimators of the gradient $\nabla_\theta F(\theta)$ are available. A gradient-based stochastic approximation (SA) algorithm adjusts the control values in the direction of performance improvement, using $\theta_{n+1} = \theta_n + \epsilon_n \widehat{G}_n$, where $\widehat{G}_n \in \mathcal{F}_{\tau(n)}$ is a gradient estimator over a (possibly distributed) estimation interval, and $\tau(n)$ is the epoch of the update (a stopping time). An excellent review on stochastic approximation (SA) is in [8]. Although the last ten years have seen a tremendous growth in the literature on this subject, there are but a few examples of application in large and truly decentralized multidimensional problems, as in [4, 14, 1]. Large complex systems present unsurmountable problems of scalability and decentralization.

If the control problem is expressed in terms of alternative parameters of the model, the gradients w.r.t these other parameters can be used to drive the adaptive control procedure. Expressing sensitivities to control variables in terms of derivatives w.r.t. other parameters of the model is not new [1, 4, 6, 7, 10, 14, 16] but most of the examples are based on heuristic and *ad-hoc* arguments which are problem dependent.

In this paper we identify the useful arguments behind Surrogate Estimation into three basic methodologies: changes in the reference framework (scale and location parameters), changes of variables via the mean flow approach, and changes of variables through uniformization. Section 2 gives a brief background of some of the examples where Surrogate Estimation has been previously used. Section 3 is the core of the paper and presents the methodology via model examples. Section 4 concludes and outlines on-going research.

2 Surrogate Estimation

The surrogate estimation approach was introduced in the routing context in [13, 14], using the so-called mean-flow approach also found in [1] and in [6]. In [13] stochastic approximation is used to optimize the routing parameters in a communications network interconnecting N FIFO queues. At each node j there is an arrival process from the outside assumed to be a renewal process with rate λ_j, independent of service times and other node's arrival processes. Service times at each node are independent of arrivals iid random variables. The nodes apply randomized routing: an item that terminates service at node i is sent to node j with probability θ_{ij}. The goal is to find the optimal routing matrix θ that minimizes $F(\theta)$, a performance function of the the stationary mean queue length at each node j. To this end, sensitivities of end-to-end delays to changes in the routing matrices have to be estimated. The dimension of the control vector θ is quadratic in N and there is one controller at each node, therefore direct estimation would require the amount of information to broadcast to be of order $\mathcal{O}(N^3)$, which is impractically high. Instead, [13, 14] express the global gradients w.r.t. routing parameters in terms of local gradients w.r.t. arrival rates, using the mean flow approach which we extend in Section 3.2.

To our knowledge, the first reference where such surrogates are used in a queueing network is in [6] Chapter 5, section 5.4, where the problem is to estimate sensitivities of overall delay to routing changes in a closed (exponential) network. The flow approximation is used to express the impact of changes to the routing matrices in terms of changes in the visiting ratio at each node. This latter sensitivity is then expressed in terms of the derivatives w.r.t. service rates, using the scaling argument, which we introduce in Section 3.1.

In [3, 4] the control variable is the integer number of "tokens" in a closed network. For a tractable model, they show that there is a unique flow pattern to each network size and transform the problem into finding the optimal flow rates, for which derivative estimation is feasible. In [16] the optimal number of tokens in the initial marking of a Petri Net is sought. Assuming exponential firing times, the sensitivities to the number of tokens is again replaced by IPA derivatives w.r.t. the firing rates. The uniformization argument behind these models is explained with Section 3.3.

3 General Methodology

The section is divided into three parts, each introducing a methodology for building surrogate estimators via a model example.

3.1 Changes in Time Scale

A metro system uses SA to adaptively adjust the frequency between trains at each terminal station. To minimize total waiting time and operating costs we need to estimate the gradient of the total waiting time w.r.t the train frequencies. Consider and arbitrary platform, say the Eastbound Platform on Line L_1 at a transfer station. There are three independent sources of incoming passengers, two from transfers and one from outside. The notation is as follows:

$N_1(\cdot)$: Poisson(λ_1) arrivals from outside.
$M_2^j(\cdot)$: Renewal process of the train arrivals from transferring line L_2, direction $j=1,2$.
$\{T_n^j\}$: inter arrival times of $M_2^j(t)$, with $E[T_n^j]=\mu_2^j$.
$\{Y_n\}\sim G_{\mu_1}$: iid inter departure times on the given platform, with $E[Y_1]=\mu_1$.
$N_2^j(\cdot)$: a Poisson(λ_2^j) process.
P_n^j: number of passengers for transfer in the n-th train from direction j on line L_2. $P_n^j\sim$Poisson$(\lambda_2^j T_n^j)$.

There are two practical problems for the implementation of gradient-search methods for network optimization. The first one is the number of derivatives required for network optimization (the number of different sources of passengers per controller). The second one concerns the sources of randomness of inter departure times. For models where μ_1 is not a scale parameter of the distribution, the derivative w.r.t. μ_1 is not precisely defined mathematically and specific models have to be assumed, each yielding different gradient estimators. In this example we introduce the method of scale changes to express the derivatives w.r.t. $\mu_{2,j}$ in terms of the sensitivities to μ_1, which reduces the number of required estimates to be proportional to the number of platforms in the network and thus becomes independent of the degree of connectivity of the transportation network. This is presented first for the particular case that $\mu_{k,j}$ and $1/\lambda_2^j, j=1,2$ are scale parameters. Later on the same scaling arguments will be used to evaluate a correction factor when the scalability assumption is not satisfied, ths generalizing the method.

Theorem 1 *Let W_1 and $W_2^j, j=1,2$ be the total cumulative waiting time of passengers arriving from outside and from transfer trains, respectively, within $[0,Y)$. $\mu_1=E[Y]$ is a scale parameter and Y is independent of the passenger arrival processes $N_n^j(t)$ (no synchronization). For $t=(T_1^*,T_2^*)$, let E_t be the expectation given that the immediately preceding train arrivals from direction j on Line L_2 happened T_j^* units of time before the start of the current interval. If $E_tW_1, E_tW_2^j$ are continuously differentiable in λ and μ_1,μ_2^j, then:*

$$\mu_1\frac{\partial E_tW_1}{\partial\mu_1}=\lambda_1\frac{\partial E_tW_1}{\partial\lambda_1}+EW_1 \qquad (1)$$

and

$$(T_j^*+\mu_1)\frac{\partial E_tW_2^j}{\partial\mu_1}=-\mu_2^j\frac{\partial E_tW_2^j}{\partial\mu_2^j}+2\,E_tW_2^j. \qquad (2)$$

Proof: Let $\phi(\lambda_1,\mu_1)=W_1$ be the cumulative waiting time of passengers that come from the outside. This is independent of the arrivals of transfer passengers and of the initial residuals t_1,t_2:

$$\phi(\lambda_1,\mu_1)=\sum_{i=1}^{N_1(Y)}(Y-S_i)=\Phi(N_1(t);0\le t\le Y),$$

where S_i are the arrival epochs of $N_1(\cdot)$ and Φ expresses the waiting time as a functional of the process. Then:

$$\phi(\lambda_1, \mu_1 - \Delta) \stackrel{\mathcal{D}}{=} \left(1 - \frac{\Delta}{\mu_1}\right) \phi(\epsilon\lambda_1, \mu_1).$$

To see this, notice that $\phi(\lambda_1, \mu_1 - \Delta)$ is defined over $[0, \epsilon Y]$, with $\epsilon = (1 - \Delta/\mu_1)$, since μ_1 is a scale parameter. Now change the time units to describe the processes to $\tau = t/\epsilon$, $\epsilon = (1 - \Delta/\mu_1)$. In the new time scale the arrival process is a Poisson($\epsilon\lambda_1$). Finally, use the fact that Φ has time units. From this we get:

$$\frac{\phi(\lambda_1, \mu_1) - \phi(\lambda_1, \mu_1 - \Delta)}{\Delta} \stackrel{\mathcal{D}}{=}$$
$$= \left(\frac{\lambda_1}{\mu_1}\right) \frac{\phi(\lambda_1, \mu_1) - \phi(\lambda_1 - \Delta\lambda_1, \mu_1)}{\Delta\lambda_1} + \frac{1}{\mu_1}\phi(\epsilon\lambda_1, \mu_1),$$

where $\Delta\lambda_1 = (\lambda_1/\mu_1)\Delta$. Taking expectations first and then the limits as $\Delta \to 0$ yields the first result (1).

Fix now the direction j from Line L_2 and drop the super index j for ease of notation. This time, represent $W_2 = \phi_{T*}(\mu_2, \mu_1)$ by the functional:

$$\sum_{i=1}^{M_2(Y)} P_i(Y - S_i) = \Phi(N_2(t), M_2(t); -T^* \le t < Y),$$

where S_i are the arrival epochs of the process $M_2(\cdot)$, with $S_0 = -T^*$. Applying the same time scale argument as before, with $\epsilon = (1 - \Delta/\mu_1)$, we obtain $\phi_{T^*}(\lambda_2, \mu_2, \mu_1 - \Delta) \stackrel{\mathcal{D}}{=} \epsilon\phi_{T^*_\epsilon}(\epsilon\lambda_2, \frac{\mu_2}{\epsilon}, \mu_1)$, for $T^*_\epsilon = T^*/\epsilon$). Therefore the finite difference satisfies:

$$\begin{aligned}
&\frac{\phi_{T^*}(\lambda_2, \mu_2, \mu_1) - \phi_{T^*}(\lambda_2, \mu_2, \mu_1 - \Delta)}{\Delta} \\
&\stackrel{\mathcal{D}}{=} \left(\frac{\lambda_2}{\mu_1}\right) \frac{\phi_{T^*}(\lambda_2, \mu_2, \mu_1) - \phi_{T^*}(\lambda_2 - \Delta\lambda_2, \mu_2, \mu_1)}{\Delta\lambda_2} \\
&- \left(\frac{\epsilon\Delta\mu_2}{\Delta}\right) \frac{\phi_{T^*_\epsilon}(\epsilon\lambda_2, \mu_2 + \Delta\mu_2, \mu_1) - \phi_{T^*_\epsilon}(\epsilon\lambda_2, \mu_2, \mu_1)}{\Delta\mu_2} \\
&+ \frac{1}{\mu_1}\phi_{T^*}(\epsilon\lambda_2, \mu_2, \mu_1) + \frac{\Delta\phi^\epsilon_{T^*}(\epsilon\lambda_2, \mu_2, \mu_1)}{\Delta},
\end{aligned} \tag{3}$$

where $\Delta\lambda_2 = (\lambda_2/\mu_1)\Delta$ and $\epsilon\Delta\mu_2 = (\mu_2/\mu_1)\Delta$. The difference process is defined by $\Delta\phi^\epsilon_{T^*}(\epsilon\lambda_2, \mu_2, \mu_1) = \epsilon\phi_{T^*}(\epsilon\lambda_2, \mu_2, \mu_1) - \epsilon\phi_{T^*/\epsilon}(\epsilon\lambda_2, \mu_2, \mu_1)$ and it satisfies:

$$\begin{aligned}
\Delta\phi^\epsilon_{T^*}(\epsilon\lambda_2, \mu_2, \mu_1) &= \Phi(N^\epsilon_2(t), M_2(t); -T^* \le t < Y) \\
&\quad -\Phi(N^\epsilon_2(t), M_2(t); -T^* - \Delta^* \le t < Y) \\
&\stackrel{\mathcal{D}}{=} \Phi(N^\epsilon_2(t), M_2(t); -T^* \le t < Y) \\
&\quad -\Phi(N^\epsilon_2(t), M_2(t); -T^* \le t < Y + \Delta^*).
\end{aligned}$$

Taking expectations and then the limit as $\Delta \to 0$,

$$\lim_{\Delta\to 0} E_t\left[\frac{\Delta\phi^\epsilon_{T^*}(\epsilon\lambda_2, \mu_2, \mu_1)}{\Delta}\right] = -\left(\frac{T^*}{\mu_1}\right)\frac{\partial E_t W}{\partial\mu_1}.$$

Using the assumption of the existence and continuity of the derivatives of the expected waiting times, the limit of the finite difference above gives:

$$\left(1 + \frac{T^*}{\mu_1}\right)\frac{\partial E_t W_2}{\partial\mu_1} = \frac{\lambda_2}{\mu_1}\frac{\partial E_t W_2}{\partial\lambda_2} - \frac{\mu_2}{\mu_1}\frac{\partial E_t W_2}{\partial\mu_2} + \frac{E_t W_2}{\mu_1}.$$

In order to obtain (2), use now the conditional expectation of P_i given $\{T_1, \ldots, T_{M_2(Y)}\}$, with $S_i = T^* + T_1 + \ldots + T_i$, to obtain $E_t W_2 = E_t[\lambda_2 \sum_{i=1}^{M_2(Y)} T_i(Y - S_i)]$, which implies that $\frac{\partial}{\partial\lambda_2}[E_t W_2] = E_t[W_2]/\lambda_2$. ■

The preceding result can be extended to any finite horizon derivative taking m train departures at the given platform instead of just one.

A Numerical Experiment: For a station without transfers, let $\{Y_j\} \sim U[a, b]$, with mean μ and variance σ^2_Y. In the simulations $N_1(\cdot)$ is a Poisson process and $EW_1(\lambda, \mu) = \frac{\lambda}{2}(\mu^2 + \sigma^2_Y)$ is known. Consider the following two models.

Case 1) $Y = \mu + \delta\mu U$, $U \sim U[-1, 1]$. In this case IPA [7] is applicable to both derivatives w.r.t. μ_1 and λ_1. The direct and surrogate IPA estimators in (1) are identical a.s., the latter being defined by:

$$\frac{\partial\widehat{EW}^*_1}{\partial\mu} = \frac{\lambda}{\mu}\left(\frac{\partial\widehat{EW}_1}{\partial\lambda}\right)_{IPA} + \frac{\widehat{EW}_1}{\mu}. \tag{4}$$

Case 2) $Y = \mu + (\mu - a)U$, $U \sim U[-1, 1]$. The IPA direct estimator is unbiased. The surrogate estimator from (1) is again (4). Although μ is not a scale parameter, (4) can give good approximations, as shown in Table 1 with 95% confidence intervals.

Case 2			
ω	Theoretical	Direct	Surrogate
0.5	51.6667	52.958 ± 0.680	50.769 ± 0.140
1.0	53.3333	51.983 ± 0.681	50.527 ± 0.189
1.5	55.0000	54.421 ± 0.756	51.416 ± 0.272
2.0	56.6667	54.395 ± 0.825	51.568 ± 0.346
2.5	58.3333	62.591 ± 0.911	57.140 ± 0.435
3.0	60.0000	54.395 ± 0.825	51.568 ± 0.346
3.5	61.6667	58.978 ± 0.986	56.046 ± 0.577
4.0	63.3333	63.057 ± 1.047	60.527 ± 0.658
4.5	65.0000	62.552 ± 1.118	61.189 ± 0.751
5.0	66.6667	64.377 ± 1.186	64.377 ± 0.842

Table 1: Surrogate estimations for non scalable cases

Table 1 was produced with 10000 consecutive train departures, $\mu = 5$ and $\lambda = 10$. The semi-width is $\omega = \mu - a$. The error from surrogate estimation is relatively small when the scalable part is larger than the non scalable one.

To verify the surrogate equation (2) with transfers we used scalable distributions for Y_i, T_n^j as in Case 1) above, i.e., $Y = \mu_1 + \delta\mu_1 U_1, T^j = \mu_2^j + \delta\mu_1 U_j, j = 1,2$ for several values of δ, where U_1, U_2 and U_3 are iid $U[-1,1]$. Table 2 contains the surrogate estimation results using Theorem 1; it shows the direct and surrogate derivatives of EW with respect to μ_1, where $EW = EW_1 + EW_2 + EW_3$. The simulation uses a fixed value of $T^* = 0.5$, $\lambda_1 = 10$, $\lambda_2 = 7.5$, $\lambda_3 = 10$, $\mu_1 = 5$, $\mu_2^1 = 2$, $\mu_2^2 = 3$, and 10000 consecutive train departures from the given platform.

δ	Direct	Surrogate
0.1	113.055 ± 0.209	113.051 ± 0.206
0.2	122.609 ± 0.700	122.485 ± 0.682
0.3	131.825 ± 0.757	131.435 ± 0.736
0.4	129.266 ± 0.581	128.866 ± 0.564
0.5	133.945 ± 1.351	133.208 ± 1.316

Table 2: Surrogate estimation with transfers.

Robust optimization requires a model-free adaptive control mechanism. Using surrogates for which we may estimate the approximation error may be a valuable alternative, even if the ensuing control is sub-optimal. The following result illustrates how correction terms to (1) may be estimated in the non scalable case.

Theorem 2 *Consider the model for a single station with no transfers. Passengers arrive form the outside according to a renewal process $N(\cdot)$ with rate λ, where $1/\lambda$ is a scale parameter of the distribution of inter arrival times, assumed to have a uniformly bounded density. Inter departure times between consecutive trains are iid of the form $Y_n = \mu X_n + \delta U_n$, where $X_n \sim G$ is independent of μ, $U_n \sim U[0,1]$ is a uniform perturbation and $\delta > 0$ is constant. Call EW the mean cumulative waiting time between trains. Then*

$$\frac{\partial EW}{\partial \mu} = \frac{\lambda}{\mu}\frac{\partial EW}{\partial \lambda} + \frac{1}{\mu} EW + \rho(\delta, \mu), \qquad (5)$$

where the correction term $\rho(\delta,\mu)$ is defined as

$$\frac{1}{\mu}\left[EW - E\left(\frac{E_Y W(\mu X - \delta) + E_Y W(\mu X + \delta)}{2}\right)\right],$$

where $E_Y W(y) = E[W|Y = y]$ is the conditional expectation given the train inter departure time.

Proof: We outline the idea of the proof. As before, write the finite difference with perturbed process at $\mu - \Delta$, and condition on Y to get:

$$\frac{\partial EW}{\partial \mu} = \lim_{\Delta\to 0}\frac{1}{2\Delta}\Big[\int_{-1}^{1} E_Y W(\mu X + \delta u)\, du - \int_{-1}^{1} E_Y W((\mu - \Delta)X + \delta u)\, du\Big].$$

A time scale change with $\epsilon = (1 - \Delta/\mu)$ is performed only for the perturbed process, which has inter departure time $Y = \mu X(1 - \Delta/\mu) + \delta u$. Thus, in the new time scale $Y' = \mu X + \delta V$, where $V \sim U[-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$. In the new time scale passengers arrive according to a Poisson process $N^\epsilon(\cdot)$ with slower rate $\lambda\epsilon$. To complete the proof, the expectation w.r.t. V is calculated integrating in three regions: on $(-1,1)$ the limiting contribution is:

$$\frac{\lambda}{\mu}\frac{\partial EW}{\partial \lambda} - \left(\frac{2}{\mu}\right) EW.$$

A uniformly bounded inter arrival density implies that there is a probability $\mathcal{O}(\Delta)$ that one or more events happen over intervals of size $1/\epsilon - 1 = \Delta/\mu + \mathcal{O}(\Delta^2)$. Thus the integral over $(-\frac{1}{\epsilon}, -1)$ contributes:

$$\frac{\Delta}{2\mu} E[E_Y W(\mu X - \delta)] + \mathcal{O}(\Delta^2),$$

and analogously for the integral over $(1, \frac{1}{\epsilon})$. Putting these results in the finite difference yields (5). ∎

A Numerical Experiment: Continue the previous example for a station witout transfers, where

Case 3) $Y = \mu + \delta U$, $U \sim U[-1,1]$. Both derivatives in (5) can be estimated via IPA, and the correction factor is estimated in parallel using the observations of the system, which adds CPU time. Table 3 shows the results with same parameters as in Table 1, now yielding a theoretical value of 50 for all ω.

ω	Direct	CPU	Surrogate	CPU
0.5	50.325 ± 0.156	0.257	49.557 ± 0.143	0.616
1.0	49.709 ± 0.171	0.349	46.697 ± 0.189	0.854
1.5	49.885 ± 0.217	0.438	48.531 ± 0.275	0.910
2.0	50.643 ± 0.254	0.339	48.905 ± 0.338	0.972
2.5	49.439 ± 0.295	0.320	47.065 ± 0.409	1.539
3.0	50.638 ± 0.361	0.331	50.807 ± 0.513	1.305
3.5	50.360 ± 0.419	0.435	48.998 ± 0.603	1.613
4.0	49.756 ± 0.484	0.323	49.738 ± 0.707	1.806
4.5	55.845 ± 0.470	0.375	61.866 ± 0.727	1.664
5.0	50.732 ± 0.617	0.321	54.657 ± 0.921	1.808

Table 3: Surrogate estimation with correction factor

3.2 Change of Variables: Mean Flow Approach

In [1, 6, 14] the "mean flow" approximation is used to express the dependency between variables. With a simple change of variables, global sensitivities in a queueing network can then be estimated with local derivative estimators w.r.t. arrival or service rates at the individual nodes. The following example illustrates the extension of the method for performance measures that depend both on first and second order moments of the local event rates. Our method computes a "correction" term that the first moment approximation does not forsee. To our knowledge, this is the first example of a change of variables using rates of non-exponential residual event times.

An ATM switch has K buffers where cells coming from different media (real-time voice or video, data, etc) await transmission. Different classes of cells have different service requirements in terms of the tolerated loss rate and delays: the QoS (quality of service). Cell arrivals at the switch are modeled as independent renewal processes for each class. Services use constant time slots corresponding to the required transmission time δ per cell. Let θ_k be the probability that the server will choose queue k for the next service. The process evolves as in [2]: at epochs $t_n = n\delta$, the server chooses a buffer k. If this queue is non-empty, then one cell from buffer k is transmitted over the channel towards its next destination. If a chosen buffer has an empty queue at the start of a transmission slot, then no cell is transmitted (even if an arrival occurs during this time interval). At the end of the each slot, another choice of buffer is again made independently of past history. The process thus described decouples the queueing processes per buffer: for each queue k the server either transmits in slot n, with probability θ_k, or goes on a vacation slot with probability $1-\theta_k$. Consecutive service times (vacation slots plus transmission time) then satisfy $S_n = \delta M_n$, where $\{M_n\}$ are iid Geom(θ_k), thus

$$
\begin{aligned}
ES_n &= \frac{\delta}{\theta_k} \equiv \mu(\theta_k, \delta) \\
ES^2 &= \frac{\delta^2(2-\theta_k)}{\theta_k^2} \equiv \mu^2(\theta_k,\delta)\sigma^2(\theta_k).
\end{aligned}
$$

In [2, 12] the problem is to find the optimal allocation θ^* that will minimize delay or queue length. RPA is used for the derivative w.r.t. θ_k and implemented in a decentralized stochastic approximation procedure to achieve optimal polling. In [1, 4, 13] adaptive routing and flow control are performed using derivatives w.r.t. local arrival rates as surrogates, dramatically decreasing the computation effort. In this example we show how the *same* local derivative estimators w.r.t. arrival rates can be used as surrogates for the polling problem. Each switch could estimate only one derivative per buffer in order for the whole network to reconstruct the optimal routing, flow and polling control concurrently. This problem is thought to be non-scalable in network size, but our results give the first empirical indication that global adaptive control can be implemented with local algorithms that are independent of network size.

Fix now any buffer and drop the index k. The mean flow approach with a first order approximation would assume that a local performance function of interest $\phi(\theta)$ (which does not have time units) depends on θ only through the first moment of the service distribution, $\mu(\theta, \delta)$. We conjecture that the stationary average queue length Q at the chosen buffer depends on the first and second moments of the service distribution. If arrivals are Poisson, then each buffer is approximately a $M/G/1$ model with service times $S_n = \delta M_n$ and $M_n \sim$ Geom(θ). It is an approximation because cells arriving to an empty buffer do not enter service immediately. The Pollaczek-Kintchine formula justifies our conjecture. A change of variables with the second order approximation gives the following:

$$
\begin{aligned}
\frac{\partial Q}{\partial \delta} &= \left(\frac{\partial \mu}{\partial \delta}\right)\frac{\partial Q}{\partial \mu} + \left(\frac{\partial \sigma^2}{\partial \delta}\right)\frac{\partial Q}{\partial \sigma^2} = \left(\frac{1}{\theta}\right)\frac{\partial Q}{\partial \mu} \\
\frac{\partial Q}{\partial \theta} &= \left(\frac{\partial \mu}{\partial \theta}\right)\frac{\partial Q}{\partial \mu} + \left(\frac{\partial \sigma^2}{\partial \theta}\right)\frac{\partial Q}{\partial \sigma^2} = -\left(\frac{\mu}{\theta}\right)\frac{\partial Q}{\partial \mu} - \frac{\partial Q}{\partial \sigma^2} \\
&= -\frac{\delta}{\theta}\frac{\partial Q}{\partial \delta} - \frac{\partial Q}{\partial \sigma^2}.
\end{aligned}
$$

The $M/G/1$ model can be used to approximate the correction term from the second order moment, using $\frac{\partial Q}{\partial \sigma^2} \approx Q/(2-\theta)$. Finally, since δ is a scale parameter of the service distribution, if $1/\lambda$ is a scale parameter of the inter arrival times at buffer k, then the change in time scale of Section 3.1 gives now:

$$
\frac{\partial Q}{\partial \lambda} = \left(\frac{\delta}{\lambda}\right)\frac{\partial Q}{\partial \delta} \approx -\left(\frac{\theta}{\lambda}\right)\left[\frac{\partial Q}{\partial \theta} + \frac{Q}{(2-\theta)}\right]. \tag{6}
$$

In order to assess the precision of (6), we simulated a buffer under random polling with Poisson arrivals at rate λ and $\delta = 1$. The performance measure is the mean average sojourn time per customer W. Based on (6) for the number Q of cells waiting in queue the corresponding surrogate expression is:

$$
\frac{\partial W}{\partial \theta} \approx -\frac{\lambda}{\theta}\frac{\partial W}{\partial \lambda} - \frac{2W}{\theta(2-\theta)} + \frac{1}{\theta(2-\theta)}.
$$

The 95% confidence intervals reported in Table 4 are based on 100 consecutive batches of 2,000 cell arrivals each. The utilization factor is $\rho = \lambda/\theta$. The surrogate estimator from the above relation is indicated by $\widehat{\frac{\partial W^*}{\partial \theta}}$. The derivatives reported in the table correspond to the *approximated* values obtained from the $M/G/1$ closed formulas. Direct estimation of the sensitivity to θ uses the RPA estimator of [2] with the "marked" slot variant. The estimation of the derivatives w.r.t. λ also uses RPA. The experiments show that the estimation using the second order moment approximation gives very similar results than direct estimation. In addition, in all our simulations there is a noticeable reduction in the variance of the estimators, which could be further optimized for efficiency.

3.3 Change of Variables: Uniformization

This example is taken from [10]. A machine processes items at K different speeds. Items arrive for processing according to a renewal process $N(\cdot)$ and either enter service or wait in queue. At the start of each processing

λ	θ	ρ	$\frac{\partial W}{\partial \lambda}$	$\widehat{\frac{\partial W}{\partial \lambda}}$	$-\frac{\partial W}{\partial \theta}$	$-\widehat{\frac{\partial W}{\partial \theta}}$	$-\widehat{\frac{\partial W^*}{\partial \theta}}$
0.20	0.60	0.333	4.375	4.45 $\pm$ 0.115	5.625	5.66 $\pm$ 0.108	6.49 $\pm$ 0.058
0.35	0.75	0.467	3.906	3.91 $\pm$ 0.129	5.156	5.23 $\pm$ 0.147	5.83 $\pm$ 0.083
0.35	0.60	0.583	11.20	11.12 $\pm$ 0.46	13.20	13.14 $\pm$ 0.45	13.95$\pm$ 0.33
0.40	0.65	0.615	10.80	11.05 $\pm$ 0.42	12.80	13.03 $\pm$ 0.44	13.78 $\pm$ 0.31
0.50	0.65	0.769	30.00	29.63 $\pm$ 2.07	33.33	32.93 $\pm$ 2.01	33.77 $\pm$ 1.78
0.35	0.45	0.778	77.50	72.87 $\pm$ 4.85	82.50	78.77 $\pm$ 4.66	79.72 $\pm$ 4.19

Table 4: Comparison of the Estimation

the machine chooses speed k with probability θ_k independently of the history of the process. Higher speeds are more costly. Let $\eta_i = k$ if speed k is chosen for the i-th item. The processing times follow distribution G_k if speed k is chosen, and $E[S_i|\eta_i = k] = \mu_k^{-1}$, with $\mu_k < \mu_j$ if $k < j$. The problem is to minimize a stationary average cost rate $F(\theta)$: the cost rates of operation and storage (in terms of mean waiting time). For such allocation problems, [11] shows that stochastic approximation yields asymptotic optimality if the iterates are driven by the generalized gradient:

$$\mathcal{G}_k[F(\cdot)] = \frac{\partial}{\partial \theta_k} F(\cdot) - \sum_{j=1}^{K} \theta_j \frac{\partial}{\partial \theta_j} F(\cdot),$$

so the problem is that of estimating $\mathcal{G}_k F$. Direct estimation can be performed via RPA (with the swapping phantoms), as explained in detail in [10], but it presents serious drawbacks, notably that it requires $\lambda < \mu_1$, while stability is ensured for $\lambda \sum_k (\theta_k/\mu_k) < 1$. The latter expression defines the feasible set for the control parameter, which may be non-empty even if $\lambda > \mu_1$. In addition, direct estimation via the swapping phantoms requires knowledge of the service distributions, which may not be realistic in an on-line operation where only statistics (empirical distributions) are available.

Under uniformization, the queueing system is a multiclass system where customers of class k require service according to a distribution G_k. Let $\lambda_k = \theta_k \lambda$ be the rate of the corresponding process. Using this alternative model for the same process, everything is described according to the (independent) renewal process per class and the rate vector $\bar{\lambda} = (\lambda_1, \ldots, \lambda_K)$ determines uniquely the dynamics as well as the stationary averages, which are now functions of $\bar{\lambda}$.

Theorem 3 *The stationary average number of items waiting in queue Q satisfies:*

$$\frac{\partial Q}{\partial \lambda_k} = \frac{\partial Q}{\partial \lambda} + \frac{1}{\lambda} \mathcal{G}_k(Q), \tag{7}$$

for all λ, θ within the stability region of the queueing system. If, in addition, $1/\mu_k$ and λ are scale parameters of the service times and inter arrival times respectively, then we also have:

$$\lambda \sum_{k=1}^{K} \theta_k \frac{\partial Q}{\partial \lambda_k} = - \sum_{k=1}^{K} \mu_k \frac{\partial Q}{\partial \mu_k} = \lambda \frac{\partial Q}{\partial \lambda}.$$

Proof: Write $\phi(\bar{\lambda}) = \phi(\lambda, \theta) = Q$. Start with the finite difference:

$$\frac{\partial \phi(\bar{\lambda})}{\partial \lambda_k} = \lim_{\Delta \to 0} \frac{\phi(\bar{\lambda}) - \phi(\bar{\lambda}')}{\Delta},$$

where $\lambda'_i = \lambda_i$ for $i \neq k$ and $\lambda'_k = \lambda_k - \Delta$. Call the system with parameter $\bar{\lambda}'$ the *perturbed* system. It corresponds to parameters $\lambda' = \lambda - \Delta$ and $\theta'_i = \frac{\lambda_i}{\lambda - \Delta} = \theta_i + \delta_i$, for $i \neq k; \theta'_k = \frac{\lambda_k - \Delta}{\lambda - \Delta} = \theta_k - \delta_k$. where: $\delta_i = \frac{\theta_i}{\lambda} \Delta + \mathcal{O}(\Delta^2) = \lambda \theta_i \Delta + \mathcal{O}(\Delta^2)$, $\delta_k = \frac{(1-\theta_k)}{\lambda} \Delta + \mathcal{O}(\Delta^2)$ and then the finite difference satisfies:

$$\frac{\phi(\bar{\lambda}) - \phi(\bar{\lambda}')}{\Delta} = \frac{[\phi(\lambda, \theta) - \phi(\lambda', \theta)] + [\phi(\lambda', \theta) - \phi(\lambda', \theta')]}{\Delta}. \tag{8}$$

The first finite difference in the RHS above will contribute the derivative w.r.t. λ. The second term is expressed as a telescopic sum to isolate the contribution of each component θ_i, θ'_i to the finite difference, as explained in detail in [15] to obtain:

$$\lim_{\Delta \to 0} \frac{\phi(\lambda', \theta) - \phi(\lambda', \theta')}{\Delta} =$$

$$- \sum_{i=1}^{k-1} \frac{\theta_i}{\lambda} \frac{\partial \phi(\lambda, \theta)}{\partial \theta_i} + \frac{(1-\theta_k)}{\lambda} \frac{\partial \phi(\lambda, \theta)}{\partial \theta_k} - \sum_{i=k+1}^{K} \frac{\theta_i}{\lambda} \frac{\partial \phi(\lambda, \theta)}{\partial \theta_i}.$$

Taking now the limit $\Delta \to 0$ in (8) and using the above, (7) is verified. Clearly, this implies

$$\mathcal{G}_k(Q) = \lambda \left(\frac{\partial Q}{\partial \lambda_k} - \sum_{k=1}^{K} \theta_j \frac{\partial Q}{\partial \lambda_j} \right). \tag{9}$$

The second claim is shown using a change in time scale units, as follows. Consider a perturbed system where $\lambda'_1 = \lambda_1 - \Delta$ is used to evaluate a finite difference. Then change the time units using the transformation $\tau = \epsilon t$ with $\epsilon = (1 - \Delta/\lambda_1)$ as in Section 3.1. Since the performance Q does not have time units, the perturbed system under the new time scale behaves exactly as a system with parameters $\lambda_1, \lambda_i + \delta\lambda_1, \mu_i + \delta\mu_i$, where $\delta\lambda_i = (\lambda_i/\lambda_1)\Delta + \mathcal{O}(\Delta^2)$ and similarly, $\mu_i = (\mu_i/\lambda_1)\Delta + \mathcal{O}(\Delta^2)$. Writing the finite difference as a telescopic sum,

$$\frac{\partial Q}{\partial \lambda_1} = -\sum_{i=2}^{K} \frac{\lambda_i}{\lambda_1} \frac{\partial Q}{\partial \lambda_i} - \sum_{i=1}^{K} \frac{\mu_i}{\lambda_1} \frac{\partial Q}{\partial \mu_i},$$

which proves that $\lambda \sum_{k=1}^{K} \theta_k \frac{\partial Q}{\partial \lambda_k} = -\sum_{k=1}^{K} \mu_k \frac{\partial Q}{\partial \mu_k}$. Using now (7) and (9),

$$\frac{\partial Q}{\partial \lambda_k} = \frac{\partial Q}{\partial \lambda} + \frac{\partial Q}{\partial \lambda_k} - \sum_{i=1}^{K} \theta_i \frac{\partial Q}{\partial \lambda_i} \Rightarrow \frac{\partial Q}{\partial \lambda} = \sum_{i=1}^{K} \theta_i \frac{\partial Q}{\partial \lambda_i}$$

■

Equation (9) allows an alternative evaluation of the generalized gradient for adaptive control. In [10] the direct and surrogate estimators are compared, using respectively the swapping and the disappearing phantoms. Besides robustness, the surrogate estimation has less variance. The implementation of such estimates for stochastic optimization is in [11].

The above result can be specialized to recover the estimation described in [7] Chapter 5, section 5.4, where a multiclass queueing system is considered, using $K = 2$. Uniformization generalizes the method and gives insight into the assumptions that the models must satisfy. This method can also be used to mathematically justify the surrogates in [4, 16].

4 Concluding Remarks

Surrogate estimation may prove to be a viable and practical alternative to direct gradient estimates and finite differences. The greatest potential for application lies perhaps with large scale problems where quick on-line measurements have to be available for control purposes. We propose a framework to describe surrogate estimation in a methodological way, explained in more detail in [15]. Our proposed methodology identifies tools by which relationships between variables in the model can be delt with and "Ersätz" estimation can be built. In this paper we have applied such methodology to generalize some known results via correction terms.

Acknowledgements

This work was supported in part by NSERC-Canada grant # WFA0139015 and by a research grant from Bishop's University. Jelena Kapor prepared the computer simulations for the metro problem, and Julie Roy did so for the ATM polling switch.

References

[1] C.G. Cassandras, M.V. Abidi, and D. Towsley. Distributed routing with on-line marginal delay estimation. *IEEE Trans on Comm*, 38:348–359, 1990.

[2] C.G. Cassandras and V. Julka. Scheduling policies using marked/phantom slot algorithms. *QUESTA*, 20:207–254, 1995.

[3] M. Cotton. Contrôle de flux isarithmique adaptatif central dans les réseaux à communication de paquets rapide. Master's thesis, INRS-Telecom, Université du Québec, 1991.

[4] M. Cotton and L.G. Mason. Adaptive isarithmic flow control in fast packet switching networks. *IEEE Trans on Comm*, **vol 42**:000–000, 1994.

[5] P. Glasserman. *Gradient Estimation via Perturbation Analysis*. Kluwer Academic, Boston, 1991.

[6] Y.-C. Ho and X.-R. Cao. Performance sensitivity to routing changes in queueing networks and flexible manufacturing systems using perturbation analysis. *IEEE Trans on Robotics and Atomation*, RA-1:165–172, 1985.

[7] Y.-C. Ho and X.-R. Cao. *Discrete-Event Dynamic Systems and Perturbation Analysis*. Kluwer Academic, Boston, 1991.

[8] H.J. Kushner and G. Yin. *Stochastic Approximation and Applications*. Springer Verlag, New York, 1997.

[9] K. Sigman. The stability of open queueing networks. *Stoch Proc and their Appl*, 35:11–25, 1990.

[10] F.J. Vázquez-Abad. The case of the swapping and of the disappearing phantoms. *Proc of WODES98*, pages 102–107, Great Brittan, 1998.

[11] F.J Vázquez-Abad. Strong points of weak convergence: A study using RPA for automatic learning. *Automatica*, **vol 35**, No 7: 1255–1274, 1999.

[12] F.J. Vázquez-Abad, C.G. Cassandras, and V. Julka. Centralized and decentralized asynchronous optimization of stochastic discrete event systems. *IEEE Trans on Autom Control*, 43:631–655, 1998.

[13] F.J. Vázquez-Abad and H.J. Kushner. A surrogate estimation approach for adaptive routing in communication networks. Technical Report LCDS/CCS 90-2, Brown University, Brown University, 1990.

[14] F.J. Vázquez-Abad and H.J. Kushner. The surrogate estimation approach for sensitivity analysis in queueing networks. *Proc of the 1993 WSC*, pages 347–355, 1993.

[15] F.J Vázquez-Abad and L. Zubieta. Ersätz estimation for sensitivity analysis: a methodology. *DEDS: Theory and Appl*, 1999. (To be submitted).

[16] Y. Wardi, J.M. Proth, N. Sauer, and X.L. Xie. Marking optimization of stochastic timed event graphs by ipa. *DEDS: Theory and Appl*, 6:221–239, 1996.