RPA Pathwise Derivative Estimation of Ruin Probabilities

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Abstract

In the context of classical Ruin Theory, we adress the problem of estimating derivatives of ruin probabilities with respect to the claim arrival intensity. We study two approaches, one via a regenerative storage process and the other, importance sampling. In both cases we can apply the Rare Perturbation Analysis (RPA) method for sensitivity estimation. Computer simulation results are provided.

1 Introduction

The surplus process of an insurance portfolio is defined as the wealth obtained by the premium payments minus the reimboursements made at the times of claims,

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \qquad t \geq 0,$$

where $N(t)$ is a Poisson process with intensity λ that models the epochs when claims are made and the corresponding amounts $\{Y_i\}$ are assumed to be i.i.d. random variables with distribution G and mean β. Premiums are received at constant rate c. Set $T = \min\{t : U(t) < 0\}$, the ruin probability is

$$\psi(u, \lambda) = P\{T < \infty\}.$$

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If $c \le \lambda\beta$, then $\psi(u,\lambda) = 1$ for all initial endowment u (Gerber 1979). As a consequence of this result, it is common to assume that premiums satisfy $c > \lambda\beta$, which we will do.

The variant of the model that we consider is given in Gerber (1979), Asmussen (1985), Asmussen and Nielsen (1995). It is assumed that the wealth available is invested at some continuously compounded interest rate δ. The resulting surplus process is of the form:

$$dU(t) = (c + \delta U(t))dt - dS(t); \quad U(0) = u \tag{1}$$

where $S(t) = \sum_{i=1}^{N(t)} Y_i$ is the compound Poisson process of cumulative claims. Use the notation $\{T_n, n \ge 0\}$ as the event epochs of the process $N(t)$, with $T_0 \equiv 0$, and denote by $W_n = T_n - T_{n-1}$ the interarrival times.

Since for general distributions G there is no analytical expression for the ruin probability available, one often resorts to simulation. However, estimating $\psi(u,\lambda)$ via direct simulations of the process $U(t)$ requires estimating $P(A)$, where A is a rare event. For paths where ruin is not achieved, there is no infallible stopping rule for the simulation.

Section 2 presents two methods for estimating ψ consistently by simulation. The first method uses indirect estimation via the Surplus Process, a queueing related process for which an overflow probability equals ψ, as will be summarized in Section 2.1 (see Asmussen (1985), Gerber (1979), Michaud (1993)). The second method was introduced by Asmussen and Nielsen (1995) for a general framework and it uses Importance Sampling, a common method for estimation of rare event probabilities, as explained in Bratley, Fox, and Schrage (1987), and will be presented in detail in Section 2.2. For the model with interest this method prescribes simulating another process where claims follow a non-homogeneous Poisson process with rates that are adapted to the natural filtration of the process (also known as Cox processes). We restate the results of Asmussen and Nielsen (1995) with an independent formulation for a particular claim amount distribution.

Three parameters of the model are of importance: the premium rate, the claim arrival rate and the mean claim amount. Lower premiums are more competitive, but might increase the probability of ruin. The sensitivities of the ruin probability to these parameters may therefore provide important information in establishing premiums and evaluating sensitivities to risk. In Vázquez-Abad and Zubieta (1999) the relationships between the derivatives of the ruin probability to each of the three parameters are developped. It is further shown that they may all be expressed in terms of the sensitivity to the accident rate λ, but this may be even more difficult to compute than evaluating the ruin probability itself.

This work focuses on the estimation of the sensitivities of $\psi(u,\lambda)$ to the arrival rate λ using the RPA method. Section 3 is devoted to the application of RPA to both the Storage Process and the Importance Sampling Method of Section 2. The Phantom RPA method is applied directly to the Storage Process in Section 2.2.1, using the same techniques as in Brémaud and Vázquez-Abad (1992), but a continuity correction term must be included for the Importance Sampling Mehtod, as discussed in Section 2.2.2. Furthermore, the phantom version of RPA requires extra simulations in this model, which may hinder efficiency. The Virtual RPA method of Section 2.2.3 avoids this problem. To our knowledge, this is the first application of the method of Baccelli and Brémaud (1993), which we implement for the non-homogeneous Poisson arrivals model.

Section 4 describes the Phantom and Virtual Estimators and provides the recursion formulas to program these methods efficiently. The Simulation results are in Section 5, where Functional Estimation via the Storage Process is discussed. Our simulations compare the two estimation methods for the exponential claim distribution, where formulas for ψ and its derivatives are available, thus allowing for a better assessement of the behaviour of the estimators.

2 Estimating the Ruin Probability

2.1 The Storage Process

Consider the general model for $\delta \geq 0$. Gerber (1979) shows that the complement of the probability of ruin $\bar{\psi}(u) = 1 - \psi(u)$ satisfies a renewal equation when the accidents occur according to a Poisson process, namely:

$$c\frac{\partial}{\partial u}\bar{\psi}(u) = \lambda\bar{\psi}(u) - \lambda \int_{-\infty}^{u} \bar{\psi}(u)\, dG(y)$$

with boundary condition $\lim_{u\to\infty} \bar{\psi}(u) = 1$.

Using this, one can establish the duality between the surplus process (1) and a storage or queueing process, as shown in Dufresne and Gerber (1989). In this case, for general distribution G, the storage process is:

$$X(t) = dS(t) - (c + \delta X(t))^{+} dt; \quad X(0) = x_0 \tag{2}$$

where $(x)^{+} \equiv \max(0, x)$. This process regenerates when $c > \lambda\beta$ even if $\delta = 0$, the stationary measure exists and is unique. In Vázquez-Abad and LeQuoc (1999) the equivalence between $X(\cdot)$ and a queueing process is discussed. Asmussen (1985) and Gerber (1979) show that the complement of the overflow probability of $X(\cdot)$ satisfies the same renewal equation as $\bar{\psi}(u)$ with the same boundary condition, which implies:

$$\psi(u, \lambda) = \lim_{t\to\infty} P\{X(t) > u\} = \lim_{t\to\infty} \frac{1}{t} \sum_{n=1}^{N(t)} D_n \tag{3}$$

where $N(t)$ is the number of claims received within $[0, t)$ and D_n is the total amount of time that the process $X(\cdot)$ spends above level u within the interval $[T_{n-1}, T_n)$. As shown in Figure 1, for $t \in [T_{n-1}, T_n)$ the process $X(t)$ follows an ODE and therefore is given by:

$$\begin{aligned}
X(T_{n-1} + s) &= \left(e^{-\delta s}[X(T_{n-1}) - \frac{c}{\delta}(e^{\delta s} - 1)]\right)^{+}, \quad s \in [0, W_n) \\
X(T_n) &= \left(e^{-\delta W_n}[X(T_{n-1}) - \frac{c}{\delta}(e^{\delta W_n} - 1)]\right)^{+} + Y_n
\end{aligned}$$

The embedded process $\{X(T_n); n = 0, 1, \ldots\}$ can be used for discrete event simulation in order to estimate the ruin probability as well as its sensitivities.

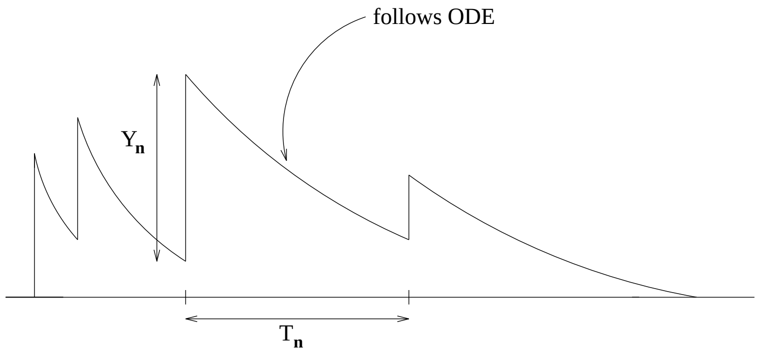

Figure 1: A typical cycle in a trajectory of $X(t)$.

2.2 Importance Sampling

2.2.1 The Linear Model

Consider the linear model ($\delta = 0$) for the surplus process $U(t)$, and let

$$\tau = \min \left\{ n : u + cT_n \leq \sum_{i=1}^{n} Y_i \right\}. \tag{4}$$

It is a stopping time and, if $h[(W_i, Y_i); i = 1, 2, \ldots] = \mathbf{1}_{\{\tau < \infty\}}$ then $E[h] = \psi(u, \lambda)$, where E denotes expectation with respect to the measure of the process when the accidents occur according to a Poisson process with rate λ and the claims are i.i.d.$\sim G$.

The method of importance sampling presented in Siegmund (1976), Asmussen and Nielsen (1995), Bratley, Fox, and Schrage (1987), and Ross (1997) is based on a change of measure under which ruin is certain. Let $\lambda_0 > \lambda$ and

$$\mu = (\lambda_0 - \lambda)/c > 0.$$

Let $g(y)$ denote the density of Y_1 (or $P[Y_1 = y]$ in the case of discrete claim distributions) and assume that $Y_i > 0$ a.s.

Suppose that a process $U(t)$ is simulated, with accidents occurring at a rate λ_0 and claim amounts following the exponentially tilted distribution: $\tilde{g}(y) = Ke^{\mu y} g(y)$ and $K^{-1} = E[e^{\mu Y}]$, also known as the Esscher transform of Y. Naturally, this can only be well defined if the moment generating function of Y exists on an open interval containing μ, that is, if $E[e^{\mu Y}] < \infty$. Otherwise, the method can still be used under a different change of measure and different stability assumptions, as presented in Vázquez-Abad and LeQuoc (1999). Set λ_0 at a value such that ruin is certain. By independence of the random variables $\{W_i, Y_i\}$,

$$\begin{aligned} P\{\tau = n\} &= \tilde{E}_{\lambda_0} \left\{ \prod_{i=1}^{n} \frac{\lambda e^{-\lambda W_i} g(Y_i)}{\lambda_0 e^{-\lambda_0 W_i} K e^{\mu Y_i} g(Y_i)} \mathbf{1}_{\{\tau = n\}} \right\} \\ &= \tilde{E}_{\lambda_0} \left\{ \mathbf{1}_{\{\tau = n\}} \left(\frac{\lambda}{K\lambda_0} \right)^n \exp\left\{ (\lambda_0 - \lambda) T_n - \frac{(\lambda_0 - \lambda)}{c} \sum_{i=1}^{n} Y_i \right\} \right\} \end{aligned}$$

where $\tilde{E}_{\lambda_0}$ denotes expectation under the new measure. Since $\psi(u, \lambda) = \sum_{n \geq 0} P\{\tau = n\}$ and $\tau < \infty$ $\tilde{P}_{\lambda_0}$-a.s., then, using $c\mu = (\lambda_0 - \lambda)$:

$$\psi(u, \lambda) = \tilde{E}_{\lambda_0}\left[\left(\frac{\lambda}{K\lambda_0}\right)^{\tau} e^{\mu[cT_\tau - S_\tau]}\right] \tag{5}$$

where $S_n = \sum_{i=1}^{n} Y_i$. The term appearing inside the expectation, called the likelihood ratio, is the Radon-Nikodym derivative of the original measure (restricted to the set $\{\tau < \infty\}$) w.r.t. the new measure. Notice that, by (4),

$$(\lambda_0 - \lambda)T_\tau - \frac{(\lambda_0 - \lambda)}{c}\sum_{i=1}^{\tau} Y_i \leq -\frac{(\lambda_0 - \lambda)u}{c} < 0, \quad a.s.$$

Let R be the *adjustment coefficient* of the original process Gerber (1979), defined by the implicit equation

$$\lambda E[e^{RY}] = \lambda + Rc. \tag{6}$$

The condition for certain ruin under the new measure is $\lambda_0 > c\tilde{E}_{\lambda_0}[Y]$. It follows from Asmussen and Nielsen (1995) that under some conditions, $\lambda + Rc > c\tilde{E}_{\lambda_0}[Y]$. Choosing $\lambda_0 = \lambda + Rc$, we have $\mu = R$ and $\lambda/K = \lambda E[e^{RY}] = \lambda_0$. Then the likelihood ratio is less than one on the trajectories where $\tau < \infty$, which bounds the variance of the estimator. Otherwise, if $\lambda + Rc < c\tilde{E}_{\lambda_0}[Y]$, then in order to have certainty of ruin we must choose $\lambda_0 > \lambda + Rc$, in which case it is necessary to assume that the new claim distribution satisfies $\tilde{E}_{\lambda_0}[e^{\alpha\tau}] < \infty$, for $\alpha = \ln\lambda - \ln K\lambda_0$.

Remark: There is an implicit equation relating the moment generating function of τ, $\varphi_\tau(z) = E[e^{z\tau}]$ and that of the claim distribution $\varphi_Y(z)$, given by: $\varphi_\tau(z) = \varphi_Y[z/c + \lambda/c(1 - \varphi_\tau(z))]$. This formula follows from Tackács argument (see Kleinrock (1976), pp. 206-216) for the busy periods of an $M/G/1$ queue with accelerated service at rate c. It is therefore possible to verify the existence of $\varphi_\tau(z)$ for some claim distributions.

2.2.2 Model with Interest

When $\delta > 0$ in (1), there isn't a *fixed* value λ_0 that ensures certainty of ruin under a similar change of measure as in the linear case. However, a change of measure under which ruin is certain is possible by adjusting the values of the new rate and tilt parameter depending on the current state. That is, at claim number n, generate the next arrival epoch W_{n+1} as an exponential with a different rate λ_n and generate the claim size Y_{n+1} under the Esscher transform with tilt μ_n, where both λ_n, μ_n depend on the state $U(T_n)$.

In Asmussen and Nielsen (1995) a general model for the nonlinear dynamics that includes (1) is studied and the measure is changed with (λ_n, μ_n) an adapted process itself. In order to illustrate the method in a simple manner, we present it for the particular case when the claims are exponentially distributed. Other claim distributions are treated similarly.

Consider the embedded discrete time process $U_n = U(T_n)$ (with the obvious abuse of notation), shown in Figure 2. From (1), $U(\cdot)$ follows the dynamics of

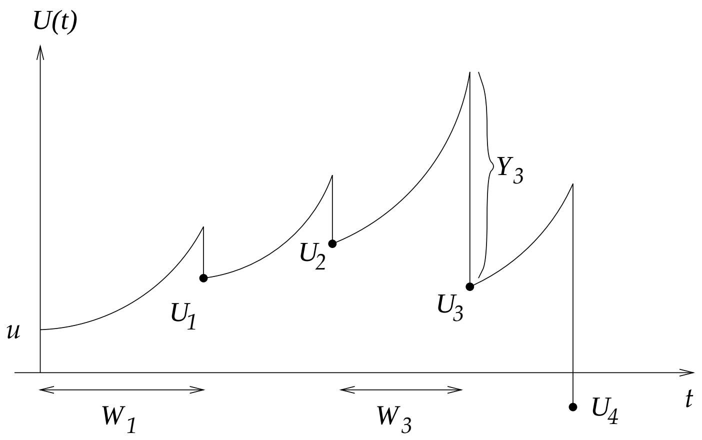

Figure 2: Continuous process $U(t)$ and embedded process $\{U_n\}$ (dots)

an ODE between claims, and then jumps according to the amount of the claim, yileding:

$$U_{n+1} = U_n e^{\delta W_{n+1}} + c\left(\frac{e^{\delta W_{n+1}} - 1}{\delta}\right) - Y_{n+1} \tag{7}$$

Define R_n as the local adjustment coefficient that satisfies:

$$\lambda + (c + \delta U_n)R_n = \lambda E[e^{R_n Y}],$$

which for the exponential case reduces to $R_n = \beta^{-1} - \lambda/c_n$, with $c_n = c + \delta U_n$. Set $\lambda_n = \lambda + c_n R_n = c_n/\beta$ and $\mu_n = R_n = (\lambda_n - \lambda)/c_n$. Consider now the process (7) where

$$W_{n+1} \sim \exp\left(\frac{c_n}{\beta}\right), \text{ independent of } \quad Y_{n+1} \sim \exp\left(\frac{\lambda}{c_n}\right). \tag{8}$$

The symbol "$\sim \exp(\alpha)$" means that the random variable has exponential distribution with rate α. These relations define the new measure $\tilde{P}$ under which the surplus process is simulated. Asmussen and Nielsen (1995) assume that $\tilde{E}[e^{\lambda_0 Y}] < \infty$, where

$$\lambda_0 = \sup_{n \le \tau} \lambda_n.$$

Clearly under this measure, $\{U_n\}$ is a Markov Process. The time of ruin is now defined accordingly, as $\tau = \min\{n : U_n < 0\}$.

The two results that we now show for the case of exponential claims are (i) that under the new measure ruin is certain, and (ii) that the corresponding Importance Sampling estimator has a variance bounded by one.

Writing $U_{n+1} = U_n + X_{n+1} - Y_{n+1}$ and using (8) we have, for $z \ge 0$:

$$\begin{aligned} \tilde{P}[\tau = n+1 | U_n = z] &= P\{Y_{n+1} > X_{n+1} + z | U_n = z\} \\ &= \tilde{E}\left\{\exp\left[-\frac{\lambda z}{c + \delta z} - \frac{\lambda}{c + \delta z}(e^{\delta W_{n+1}} - 1)\left(z + \frac{c}{\delta}\right)\right]\right\} \\ &\ge \tilde{E}\left\{\exp\left[-\frac{\lambda}{\delta} - \left(\frac{\lambda}{\delta}\right)(e^{\delta W_{n+1}} - 1)\right]\right\} \ge e^{-\lambda/\delta} = p_0 > 0 \end{aligned}$$

where we have used $\lambda z/(c+\delta z) \leq \lambda/\delta$ for all $z \geq 0$ and (8). Therefore, $\tilde{P}[\tau = n+1|U_n] \geq p_0 > 0$ and the bound p_0 is independent of $U_n \geq 0$, which implies that τ is stochastically dominated by a geometric random variable with parameter p_0. Since a geometric random variable is finite a.s., then $\tilde{P}[\tau < \infty] = 1$ under the new measure.

By definition of λ_n and R_n, it follows that $(\lambda E[e^{R_n Y}]/\lambda_n) \equiv 1$. Proceeding as before, we now obtain:

$$\begin{aligned}
P\{\tau = n+1\} &= \tilde{E}\left\{\prod_{i=1}^{n} e^{-(\lambda-\lambda_i)W_{i+1}} e^{R_i Y_{i+1}} \mathbf{1}_{\{\tau=n+1\}}\right\} \\
&= \tilde{E}\left\{\exp\left\{\sum_{i=0}^{n} \frac{\lambda_i-\lambda}{c_i}[c_i W_{i+1} - Y_{i+1}]\right\} \mathbf{1}_{\{\tau=n+1\}}\right\} \\
&\leq \tilde{E}\left\{\exp\left\{\frac{\lambda_0-\lambda}{c}\sum_{i=0}^{n}(c_i W_{i+1} - Y_{i+1})\right\} \mathbf{1}_{\{\tau=n+1\}}\right\}.
\end{aligned}$$

From (1) $c_i = \frac{d}{dt}U(T_i)$ is the initial slope of the surplus process at time T_i, so that $c_i W_{i+1} \leq X_{i+1}$ for each $i \leq \tau$ (see Figure 3). Thus $\sum_{i=0}^{n}(c_i W_{i+1} - Y_{i+1}) \leq U_{n+1} - u$, which yields the a.s. bound of 1 for the random variable inside the expectation above.

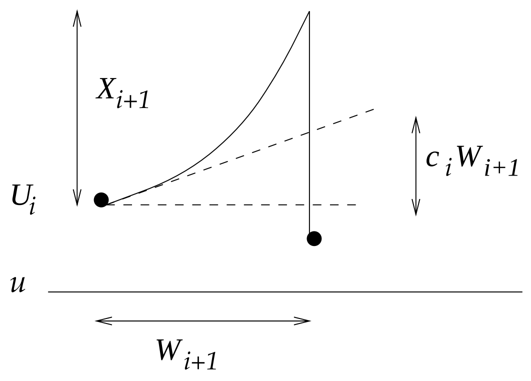

Figure 3: Linear approximation to the surplus process

Adding over n we obtain the estimator:

$$\psi(u,\lambda) = \tilde{E}\left[\exp\left\{\sum_{i=0}^{\tau-1}(\lambda_i - \lambda)\left[W_{i+1} - \frac{Y_{i+1}}{c_i}\right]\right\}\right]. \tag{9}$$

3 Derivatives of $\phi(X(t), t \geq 0)$

Several approaches have been proposed to estimate gradients using only one simulated trajectory of the system, as in Brémaud and Vázquez-Abad (1992), Fu and Hu (1992), Glasserman (1991), Glynn (1987), Pflug (1990), Reiman and Weiss (1989), among others. The main difficulty is that the functional $\psi(u,\lambda)$ depends on the path in a non-linear way and the process $N(t)$ is a.s. discontinuous in λ. The phantom RPA (Rare Perturbation Analysis) estimators are applicable to this problem. We will present our derivation of the method based on the results of Brémaud and Vázquez-Abad (1992), Baccelli and Brémaud (1993) and Vázquez-Abad (1999).

3.1 RPA for the Storage Process

Consider the general case $\delta \geq 0$. Let $\{X_\lambda(t), t > 0\}$ now denote the storage process (2) when claims arrive at rate λ, so that the ruin probability is given by (3). It follows (see Figure 1) that $D_n = \phi[X_\lambda(t); t \in [T_{n-1}, T_n)]$ for some additive functional ϕ, and $\sum_{k=1}^{N(t)} D_n = \phi[X(s); s \leq t]$.

Let $\{\eta_i\}$ be a sequence of i.i.d. Bernoulli variables with parameter $p = \Delta\lambda/\lambda$, also independent of $\{(T_i, Y_i)\}$. Think of $\Delta\lambda > 0$ as a "small" perturbation of the claim rate. Consider a process where the i-th claim is disregarded iff $\eta_i = 0$ (it "disappears" and it is called a "phantom claim"). Since the thinned arrival process thus obtained is a Poisson process with parameter $\lambda - \Delta\lambda$, then the corresponding *phantom* process has the same distribution as $\{X_{\lambda-\Delta\lambda}(t)\}$. We call *nominal* process the one for which all claims are honored: $\eta_i \equiv 1$.

Under some regularity assumptions on G, the stationary derivative is the (uniform) limit of the finite horizon one and the following interchange in the limits is justified by Theorem 4 of Vázquez-Abad and Kushner (1992), as explained in the arguments that follow.

$$\frac{\partial}{\partial\lambda}\psi(u,\lambda) = \lim_{t\to\infty} \lim_{\Delta\lambda\to 0} \left\{ \frac{E\phi[X_\lambda(s); s \leq t] - E\phi[X_{\lambda-\Delta\lambda}(s); s \leq t]}{t\,\Delta\lambda} \right\} \tag{10}$$

Consider the imbedded chain $\{X(T_n); n = 0, 1, 2, \ldots\}$ with state space $\mathbb{R}^+$. Under our assumption that $E[e^{sY}] < \infty$ for some $s > 0$, this Markov chain satisfies a Doeblin condition (see Revuz 1975) implying Harris recurrence. The phantom process for $\Delta\lambda > 0$ is stochastically dominated by the Harris recurrent process at λ, implying tightness of the set of invariant measures for $(\lambda - \Delta\lambda, \lambda]$. Using the Radon-Nikodym derivative of the transition probabilities of the phantom process with respect to the nominal, the derivatives of the first step transition probabilities are uniformly bounded on compact sets. Under these conditions, Theorem 4 in Vázquez-Abad and Kushner (1992) can be applied to establish (10). Furthermore, convergence is at a geometric rate along the subsequence $T_n \to \infty$, yielding exponential convergence in t.

Let $t < \infty$ be fixed and consider the finite difference in (10), called $\psi_{\Delta\lambda}(\lambda, t)$. Conditioning on the events that there are m phantoms $\eta_i = 0$ among the $N(t)$ claims,

$$\begin{aligned}\psi_{\Delta\lambda}(\lambda,t) &= E\Bigg\{\sum_{m=1}^{N(t)} \binom{N(t)}{m} \left(\frac{\Delta\lambda}{\lambda}\right)^m \left(1 - \frac{\Delta\lambda}{\lambda}\right)^{N(t)-m} \times \\ &\qquad E_m\left\{\frac{[\phi(X_\lambda(s); s \leq t] - \phi[X_{\lambda-\Delta\lambda}(s); s \leq t]}{t\Delta\lambda}\right\}\Bigg\}\end{aligned}$$

where E_m is the conditional expectation with respect to $\{\eta_i\}$ given $N(t)$ and m phantoms among the $N(t)$ claims. For any fixed t, as $\Delta\lambda \to 0$ the contribution of the sum over $m \geq 2$ in the previous expression vanishes a.s., since $N(t)$ has a bounded variance for finite λ, and $\phi[X_\lambda(s); s \leq t] \leq t$ a.s. (see the details in Vázquez-Abad (1999)). Therefore the only term that survives in the limit is $m = 1$, for which:

$$\frac{\partial}{\partial \lambda}\psi(u,\lambda) = \lim_{t\to\infty} \frac{1}{\lambda} E \sum_{j=1}^{N(t)} \left\{ \frac{\sum_{n=1}^{N(t)} [D_n(0) - D_n(j)]}{t} \right\} \tag{11}$$

where $D_n(0)$ is the amount of time bewteen claims $n-1$ and n that the process spends above level u, and $D_n(j)$ is the corresponding quantity in the system where claim j is the only one disappearing. To obtain (11), use the fact that given $N(t)$ and one phantom, any one of the $N(t)$ can be the chosen one with probability $N(t)^{-1}$, so that $E\{N(t)E_1\phi[X(s), s \leq t]\} = E\left(\sum_{j=1}^{N(t)} \phi[X_j(s), s \leq t]\right)$ and X_j denotes the phantom system where exactly and only claim j has disappeared.

If $E[\tau^4] < \infty$ then the rate of convergence of the derivative estimator is $\mathcal{O}(1/\sqrt{t})$. Indeed, $D_n(0) - D_n(j) = 0$ if n and j index claims belonging to different regenerative cycles (see Section 3.1 for details on the phantom processes). Furthermore, by definition $D_n(0) - D_n(j) \leq 2\tau_k(j)$ if $k(j)$ indexes the regenerative cycle where j belongs. Aggregating the double sum in (11) per cycles and calling $M(t) - 1$ the number of cycles completed by time t our estimator is bounded a.s. by $(1/t)\sum_{k=1}^{M(t)} \tau_k^2$ Using Wald's identity and the renewal theorem $(M(t)/t \to [E\tau]^{-1}$ a.s.) we can now bound the variance of our estmator:

$$\mathrm{Var}\left\{ \frac{1}{t}\sum_{k=1}^{M(t)} \tau_k^2 \right\} = \frac{4\mathrm{Var}(\tau^2)}{t} E\left\{ \left(\frac{M(t)}{t} \right) \right\} + \frac{2E[\tau^2]}{t} \mathrm{Var}\left\{ \left(\frac{M(t)}{t} \right) \right\}$$

which implies our assertion.

3.2 RPA with Importance Sampling

3.2.1 Phantom RPA

In this section we will denote by $\psi(u,\lambda)$ the ruin probability for the surplus process $\{U(t), t > 0\}$ of (1) for $\delta = 0$. We develop the general formulas for the Phantom RPA derivative estimation, although our conditions simplify when we choose $\lambda_0 = \lambda + cR$. Recall the definition of the cycle length (4) and identify now:

$$\phi[U(t); t \leq \tau] = {K_0}^\tau e^{\mu(cT_\tau - S_\tau)},$$

where $K_0 = (\lambda/K\lambda_0)$, so that $\psi(u,\lambda) = \tilde{E}_{\lambda_0}\{\phi[U(t); t \leq \tau]\}$, where $K_0 = (\lambda/K\lambda_0)$. Assume that $E[e^{\mu Y}] < \infty$. E_x will now denote the expectation of the process when accidents occur at a rate x, but all claims have the *fixed* distribution with density $\tilde{g}(y) = e^{\mu y}g(y)$, $\mu = (\lambda_0 - \lambda)/c$.

We assume that there exists $\epsilon > 0$ such that:

$$\sup_{\Delta\lambda < \epsilon} E_{\lambda_0 - \Delta\lambda}\left\{ \tau^3 \left(\frac{\lambda}{K\lambda_0} \right)^\tau \right\} < \infty. \tag{12}$$

To evaluate $\psi(u, \lambda - \Delta\lambda)$, apply now the change of measure in (5) using $\lambda_0 - \Delta\lambda$. Since μ remains the same as before, then:

$$\begin{aligned}\psi(u, \lambda - \Delta\lambda) &= E_{\lambda_0 - \Delta\lambda}\left[\left(\frac{\lambda - \Delta\lambda}{K(\lambda_0 - \Delta\lambda)}\right)^{\tau} e^{\mu(cT_\tau - S_\tau)}\right] \\ &= E_{\lambda_0 - \Delta\lambda}\left[{K_0}^{\tau} e^{\mu(cT_\tau - S_\tau)} + R(\Delta\lambda)\right]\end{aligned}$$

where the remainder term $R(\Delta\lambda)$ is:

$$R(\Delta\lambda) = e^{\mu(cT_\tau - S_\tau)}\left\{\left(\frac{\lambda - \Delta\lambda}{K(\lambda_0 - \Delta\lambda)}\right)^{\tau} - \left(\frac{\lambda}{K\lambda_0}\right)^{\tau}\right\}.$$

Consider the Poisson process $N(t)$ with rate λ_0 and define $\{\eta_i\}$ as a sequence of i.i.d. Bernoulli variables with parameter $1 - \lambda_0/\Delta\lambda$, as well as the corresponding phantom systems. Then the surplus process obtained when phantom claims are not considered has exactly the desired distribution for evaluating $\psi(u, \lambda - \Delta\lambda)$. Now $\tau(\eta)$ becomes a function of the sequence $\eta = \{\eta_i, i \le \tau(0)\}$, and we denote by $\tau(0)$ the nominal stopping time where no claims are disregarded. Notice that there is now stochastic domination of the nominal system by the phantoms, in the sense that $\tau(0) \le \tau(\eta)$ (ruin will happen no later in the nominal system than in a phantom where some claims have not been paid). Conditioning on the sets of m phantoms as before, the finite difference satisfies:

$$\begin{aligned}\psi_{\Delta\lambda}(\lambda_0) &= E_{\lambda_0}\Bigg\{\sum_{m=0}^{\tau(0)} \binom{\tau(0)}{m}\left(\frac{\Delta\lambda}{\lambda_0}\right)^m \left(1 - \frac{\Delta\lambda}{\lambda_0}\right)^{\tau(0)-m} \times \\ &E_m\left\{\frac{K_0^{\tau(0)} e^{\mu(cT_{\tau(0)} - S_{\tau(0)})} - K_0^{\tau(\eta)} e^{\mu(cT_{\tau(\eta)} - S_{\tau(\eta)})} - R(\Delta\lambda)}{\Delta\lambda}\right\}\Bigg\}.\end{aligned}$$

It can be shown (see Brémaud and Vázquez-Abad (1992), Vázquez-Abad (1999)) that

$$\sum_{m \ge 2} \binom{\tau(0)}{m}\left(\frac{\Delta\lambda}{\lambda_0}\right)^m \left(1 - \frac{\Delta\lambda}{\lambda_0}\right)^{\tau(0)-m} \le \frac{1}{2}\tau(0)^2\left(\frac{\Delta\lambda}{\lambda_0}\right)^2.$$

For $m \ge 1$ the contribution of the remainder term will vanish in passing to the limit, which is shown as follows. Notice that by definition of τ, $e^{\mu(cT_\tau - S_\tau)} < 1$ a.s. Next, let $f(\lambda) = \lambda/(C + \lambda)$ where $C = \lambda_0 - \lambda$ is kept constant. Using Taylor expansion with remainder, for any integer m:

$$\exists \xi \in (\lambda - \Delta\lambda, \lambda) \quad \text{s.t. } f(\lambda)^m = f(\lambda - \Delta\lambda)^m + \Delta\lambda m f(\lambda + \xi)^{m-1}\frac{\partial}{\partial\lambda} f(\lambda + \xi)$$

which in turn implies that $\|R(\Delta\lambda)/\Delta\lambda\| \le (1/\lambda - 1/\lambda_0)\tau(\eta)(K_0)^{\tau(\eta)}$ a.s. For $m = 1$, this term is multiplied by the probability of 1 phantom, itself bounded a.s. by $\tau(0)\Delta\lambda/\lambda_0$, while for $m \ge 2$ it is multiplied by a term bounded a.s. by $\tau(0)^2\Delta\lambda^2$. Our assertion now follows from assumption (12) and dominated convergence.

Using that $\tau(0) \le \tau(\eta)$ a.s, and $\phi \le 1$ a.s., from assumption (12) and dominated convergence, it follows that only the terms $m = 0, 1$ are left in passing to the limit $\Delta\lambda \to 0$.

For the term $m = 0$ (no phantoms occur) the nominal and phantom paths coincide. Therefore, only the contribution of the remainder term survives in the limit for $m = 0$. Recall that $\phi < 1$ a.s. and we have $(\lambda - \Delta\lambda)/K(\lambda_0 - \Delta\lambda) < \lambda/K\lambda_0$. As before, $C = (\lambda_0 - \lambda)$ is a fixed scalar. Then, using $(1 - \Delta\lambda/\lambda_0)^{\tau(0)} \to 1$ a.s., and dominated convergence, it follows from our assumption that

$$
\begin{aligned}
-\lim_{\Delta\lambda\to 0} E_{\lambda_0}\left\{\left(1 - \frac{\Delta\lambda}{\lambda_0}\right)^{\tau(0)} \frac{R(\Delta\lambda)}{\Delta\lambda}\right\} &= \\
&= E_{\lambda_0}\left\{ e^{\mu[cT(0)-S(0)]} \frac{\partial}{\partial\lambda}\left(\frac{\lambda}{K(C+\lambda)}\right)^{\tau(0)}\right\} \\
&= \frac{(\lambda_0 - \lambda)}{\lambda_0\lambda} E_{\lambda_0}\left\{\tau(0) {K_0}^{\tau(0)}\, e^{\mu[cT(0)-S(0)]}\right\}.
\end{aligned}
$$

Finally, the term $m = 1$ is treated in an analogous way as for the storage process, yielding:

$$
\begin{aligned}
\frac{\partial}{\partial\lambda}\psi(u) &= \frac{1}{\lambda_0} E_{\lambda_0}\left\{ \frac{(\lambda_0 - \lambda)}{\lambda}\tau(0) {K_0}^{\tau(0)}\, e^{\mu[cT(0)-S(0)]} \right. \\
&\quad \left. + \sum_{j=1}^{\tau(0)} \left({K_0}^{\tau(0)} e^{\mu[cT(0)-S(0)]} - {K_0}^{\tau(j)} e^{\mu[cT(j)-S(j)]}\right)\right\} \\
&= E_{\lambda_0}\left\{ \frac{\tau(0)}{\lambda} {K_0}^{\tau(0)} e^{\mu[cT(0)-S(0)]} - \frac{1}{\lambda_0}\sum_{j=1}^{\tau(0)} {K_0}^{\tau(j)} e^{\mu[cT(j)-S(j)]}\right\} \qquad (13)
\end{aligned}
$$

where $T(0) = T_{\tau(0)}$, $T(j) = T_{\tau(j)}$, $S(0) = S_{\tau(0)}$ and $S(j) = \sum_{i=1}^{\tau(j)} Y_i\eta_i$.

3.2.2 Virtual RPA

The Virtual RPA formula of Baccelli and Brémaud (1993) follows from a similar analysis as the phantom one, only this time we consider $\lambda + \Delta\lambda$, where $\Delta\lambda > 0$, so that:

$$
\psi(u, \lambda + \Delta\lambda) = E_{\lambda_0 + \Delta\lambda}\left[{K_0}^{\tilde\tau} e^{\mu(cT_{\tilde\tau} - S_{\tilde\tau})} + R(\Delta\lambda)\right]
$$

where the remainder term $R(\Delta\lambda)$ is now:

$$
R(\Delta\lambda) = e^{\mu(cT_{\tilde\tau} - S_{\tilde\tau})}\left\{\left(\frac{\lambda + \Delta\lambda}{K(\lambda_0 + \Delta\lambda)}\right)^{\tilde\tau} - \left(\frac{\lambda}{K\lambda_0}\right)^{\tilde\tau}\right\}
$$

where $\tilde\tau$ is the time of ruin for the process with Poisson arrivals at rate $\lambda_0 + \Delta\lambda$. This arrival process has the same distribution as a process where claims arrive from two independent sources: one which is Poisson at rate λ_0 and a "rare" source with rate $\Delta\lambda$. Claims coming from this source are called "virtual" claims. Naturally, for the process with only the Poisson arrivals, ruin cannot occur *before* it happens when we add the virtual claims. Given the *nominal process* $\{(W_i, Y_i), i = 1, \ldots, \tau\}$ with $W_i \sim \exp(\lambda_0)$, the probability of having

m virtual claims within $[0, T_\tau]$ is $e^{-\Delta\lambda T_\tau}(\Delta\lambda T_\tau)^m/m!$. Therefore, the finite difference will only have two nonvanishing terms as $\Delta\lambda \to 0$:

$$\begin{aligned}\Psi_{\Delta\lambda}(\lambda_0) &= E_{\lambda_0}\left\{\frac{1}{\Delta\lambda}\sum_{m=0}^{\infty} e^{-\Delta\lambda T_\tau}\frac{(\Delta\lambda T_\tau)^m}{m!}\times\right.\\ &\qquad\left. E_m\left[{K_0}^{\tilde\tau}e^{\mu(cT_{\tilde\tau}-S_{\tilde\tau})} - {K_0}^{\tau}e^{\mu(cT_\tau-S_\tau)} + R(\Delta\lambda)\right]\right\}\\ &= E_{\lambda_0}\left\{e^{-\Delta\lambda T_\tau}\left[\frac{R(\Delta\lambda)}{\Delta\lambda} + T_\tau E_1\left[{K_0}^{\tilde\tau}e^{\mu(cT_{\tilde\tau}-S_{\tilde\tau})} - {K_0}^{\tau}e^{\mu(cT_\tau-S_\tau)}\right]\right]\right\}\\ &\quad+\mathcal{O}(\Delta\lambda)\end{aligned}$$

where E_m now denotes the expectation conditioned on having m virtual claims. The arrival epochs of the virtual customers, given only m of them occur within $[0, T_\tau]$ follow the joint distribution of m independent uniform random variables in $[0, T_\tau]$ Taylor and Karlin (1994). The limit as $\Delta\lambda \to 0$ is:

$$\frac{\partial}{\partial\lambda}\psi(u) = E_{\lambda_0}\left\{{K_0}^{\tau}\left(\frac{\tau}{\lambda} - \frac{\tau}{\lambda_0} - T_\tau\right)e^{\mu(cT_\tau - S_\tau)} + T_\tau\, E_1\left[{K_0}^{\tilde\tau}e^{\mu(cT_{\tilde\tau}-S_{\tilde\tau})}\right]\right\} \tag{14}$$

where now E_1 is the expectation under a uniformly distributed arrival epoch of the virtual customer.

Remark: The virtual RPA method can of course be applied to the storage process as well, which we state without further detail. Nonetheless, for the storage process the phantom systems are dominated, that is, they are below the nominal process w.p.1 (see Section 3), while the virtual processes would dominate the nominal system, meaning that some of the virtual claims may cause two or more cycles to merge. Domination by the nominal system is a desired property, first because it allows us to bound the variance of the estimators in precise terms, but also for practical purposes when executing the programming codes. For this reason, we have chosen to present the virtual RPA for the surplus process, where with probability one, ruin cannot occur sooner in the phantom systems than in the nominal process.

3.2.3 Model with Interest

We will illustrate the Virtual RPA for the interest rate model (9), assuming that the claim distribution is exponential. Other distributions can be treated similarly, as long as they satisfy the assumptions. Define

$$h_n = \exp\left\{\sum_{i=0}^{n-1}(\lambda_i - \lambda)\left[W_{i+1} - \frac{Y_{i+1}}{c_i}\right]\right\}, \quad n = 1, 2, \ldots, \tau \tag{15}$$

so that $\{h_n\}$ is a adapted process, and $\psi(u, \lambda) = \tilde{E}[h_\tau]$, under the measure defined through (8). We can apply the Virtual RPA method by considering a process with arrival rate $\lambda + \Delta\lambda$, for $\Delta\lambda > 0$. The change of measure (8) of Section 1.1.2 prescribes using $\lambda_n = c_n\beta$. Instead, we use $\lambda_n + \Delta$ for the change of measure, to be able to write

$$\psi(u, \lambda+\Delta) = E_\Delta \left\{ \prod_{i=0}^{\tau-1} \left(\frac{\lambda+\Delta\lambda}{\lambda_i+\Delta\lambda} \right) \left(\frac{\lambda_i}{\lambda} \right) \exp\left[W_{i+1} - \frac{Y_{i+1}}{c_i} \right] \right\},$$

where, as before, $M_Y(R_i) = \lambda_i/\lambda$. We call this the "$\Delta\lambda$-process". Since $\Delta\lambda > 0$, then clearly this change of measure will also ensure $P_{\Delta\lambda}$-a.s. finite ruin epochs.

At each claim epoch $T_n^{\Delta\lambda}$, given $U_n^{\Delta\lambda}$, the next interarrival time for the $\Delta\lambda$-process has an exponential distribution with rate $\lambda_n + \Delta\lambda$, so it has the same distribution as the random variable $\min(W_{n+1}, V_{n+1})$, where $W_{n+1} \sim \exp(\lambda_n)$ and $V_{n+1} \sim \exp(\Delta\lambda)$ is independent of W_{n+1} and all previous random variables. Furthermore, it follows from the lack of memory of exponential distributions that the residual life of V_1 given $V_1 > T_n^{\Delta\lambda}$ is also exponential with parameter $\Delta\lambda$. Therefore the distribution of the $\Delta\lambda$-process is the same as the following process. Let $V_1 \sim \exp(\Delta\lambda)$ and call $\tau_1 = \min\{n : W_n > V_1 - T_{n-1}^{\Delta\lambda}\}$, where the distribution of $W_{n+1} \sim \exp(\lambda_n)$ depends on the state of the $\Delta\lambda$-process $U_n^{\Delta\lambda}$. Set $W_n^{\Delta\lambda} = W_n, n < \tau_1$, $W_{\tau_1}^{\Delta\lambda} = V_1$, $V_2 \sim \exp(\Delta)$ independent of all the other variables, and $\tau_2 = \min\{n > \tau_1 : W_n > V_2 - T_{n-1}^{\Delta\lambda}\}$, $W_n^{\Delta\lambda} = W_n, \tau_1 < n < \tau_2$, etc. Notice that the epochs τ_i are stopping times, adapted to the natural filtration of the $\Delta\lambda$-process.

Use now the fact that $V_1, V_2, \ldots$ are interarrival times of a Poisson process with rate $\Delta\lambda$. These are the virtual claims. Given the nominal process $\{(Y_i, W_i), i = 1, \ldots, \tau\}$, the probability of having m virtual claims is $e^{-\Delta\lambda T_\tau}(\Delta\lambda T_\tau)^m/m!$. For the term $m = 0$, with probability $e^{-\Delta\lambda T_\tau} \to 1$, the nominal and virtual paths coincide and the finite difference is only due to the remainder term:

$$\begin{aligned} \frac{R(\Delta\lambda)}{\Delta\lambda} &= \frac{1}{\Delta\lambda}\left[\prod_{i=0}^{\tau-1} \frac{\lambda+\Delta\lambda}{\lambda_i+\Delta\lambda} K_i - \prod_{i=0}^{\tau-1} \frac{\lambda}{\lambda_i} K_i \right] \to \frac{\partial}{\partial\lambda} \prod_{i=1}^{\tau-1} \frac{\lambda}{\lambda + c_i R_i} K_i \\ &= h_\tau \sum_{i=1}^{\tau-1} \frac{\lambda_i}{\lambda K_i} K_i \frac{\partial}{\partial\lambda}\left(\frac{\lambda}{\lambda+c_i R_i} \right) = h_\tau \sum_{i=1}^{\tau-1} \left(\frac{1}{\lambda} - \frac{1}{\lambda_i} \right) \end{aligned}$$

where $\lambda_i = \lambda + c_i R_i$, and $c_i, R_i, K_i = M_Y(R_i)\exp[R_i(c_i W_{i+1} - Y_{i+1}]$ are independent of $\Delta\lambda$, and $M_Y(R_i) = \lambda_i/\lambda$. For the term $m = 1$, the contribution of the remainder term vanishes, since the probability of one claim is of order $\mathcal{O}(\Delta\lambda)$. Similarly, the contribution vanishes as $\Delta\lambda \to 0$ for $m \geq 2$, since h_τ has finite expectation. The Virtual RPA formula becomes now:

$$\frac{\partial}{\partial\lambda}\psi(u) = \tilde{E}\left\{ \sum_{i=0}^{\tau-1} \left(\frac{1}{\lambda} - \frac{1}{\lambda_i} \right) h_\tau + T_\tau \, E_1 \left[\tilde{h}_{\tilde{\tau}} - h_\tau \right] \right\} \tag{16}$$

where as in (9) $\tilde{E}$ refers to the expectation under the measure determined by (8), and E_1 is the conditional expectation w.r.t. a uniformly distributed arrival time $T^* \sim U[0, T_\tau]$ of the virtual claim, $\tilde{h}_{\tilde{\tau}}$ is the corresponding statistics.

4 The Phantom and Virtual Estimators

We now describe how the derivatives can be estimated via the "phantom" estimators in (11), (13), (14) and (16). Our method is based on an efficient way to calculate these quantities in parallel, thus using common random numbers. The Phantom estimators for the model of the surplus process with interest is presented in detail first.

4.1 Phantom Storage Processes

We have chosen to illustrate the more difficult case when $\delta > 0$. The analysis for the linear case $\delta = 0$ follows in exactly the same manner and we leave it to the reader. Figure 1 shows one regenerative cycle of a typical path of the process. Let $\{X_n(0)\}$ denote the **nominal** process (2) sampled only at the event times when accidents occur. Then this process is Markovian, and from (2), it follows the recursions:

$$X_{n+1}(0) = \left(e^{-\delta W_{n+1}}[X_n(0) - \frac{c}{\delta}(e^{\delta W_{n+1}} - 1)]\right)^+ + Y_{n+1}$$

Furthermore, as in Michaud (1993), if r_{n+1} denotes the residual time needed for the ODE in (2) to attain level u if there were no further claims and the initial condition was set at $X_n(0)$, then:

$$r_{n+1} = \frac{1}{\delta} \ln \left[\frac{X_n(0) + c/\delta}{u + c/\delta}\right]$$

and

$$D_{n+1} = \begin{cases} 0 & \text{if } X_n(0) \le u \\ \min(W_{n+1}, r_{n+1}) & \text{otherwise} \end{cases}$$

where D_n is, as before, the amount of time within $[T_{n-1}, T_n)$ that the process stays above level u. Denote by $\{X_n(j)\}$ the corresponding values for the phantom process where claim j is absent, but where all other contingencies and events coincide. We are thus using common random numbers between two systems.

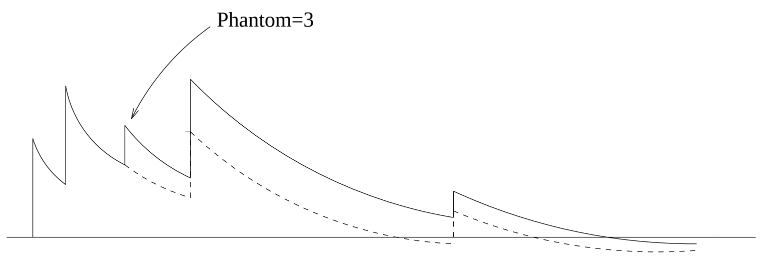

Figure 4: Pathwise construction of phantom system $j = 3$.

Define the *difference process* as:

$$d_n(j) = X_n(0) - X_n(j)$$

Clearly, $X_n(j) = X_n(0)$ for all $n < j$. At the epoch T_j, we set $X_j(j) = X_j(0) - Y_j$. The difference process then satisfies $d_n(j) = 0, j > n$, $d_j(j) = Y_j$. It is easy to see that $d_n(j), n = 1, 2, \ldots$ are decreasing a.s., and that $X_n(j) \leq X_n(0)$, as shown in Figure 4. Therefore if $X_n(0) = 0$, necessarily $X_n(j) = 0$ and by the next claim, the two processes will coincide. This means that $d_n(j) > 0$ only for n within the regenerative cycle where j belongs. The difference process can then be calculated recursively (see Figure 4) by taking into consideration only one cycle at a time:

$$d_{n+1}(j) = \begin{cases} e^{-\delta W_{n+1}} d_n(j), & \text{if } X_n(0) - \frac{c}{\delta}(e^{\delta W_{n+1}} - 1) > d_n(j) \\ e^{-\delta W_{n+1}} \left(X_n(0) - \frac{c}{\delta}(e^{\delta W_{n+1}} - 1) \right), & \text{otherwise} \end{cases}$$

since in the first situation the phantom storage process is positive in the whole interval, and otherwise $X_{n+1}(j) = Y_{n+1}$.

Remark: Notice that the evolution of the nominal process as well that of each of the phantom systems is independent of the value of u.

Our goal is to calculate $Z_n(j) = D_n - D_n(j)$. We use the difference process to do this. Notice first that if $X_n(0) \leq u$ then all nominal and phantom systems will remain below u for the whole interval $[T_n, T_{n+1})$ a.s, and $Z_{n+1}(j) = 0$. Otherwise, $X_n(0) > u$ and we distinguish two cases. If $d_n(j) > X_n(0) - u$ then, as shown in Figure 5, $D_{n+1}(j) = 0$ and therefore in this case $Z_{n+1}(j) = \min(W_{n+1}, r_{n+1})$.

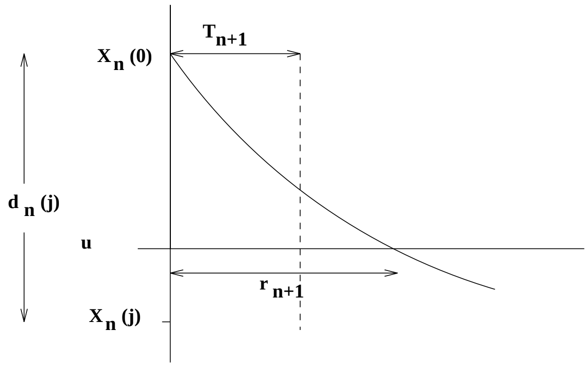

Figure 5: Phantom systems: case 1

The case where $d_n(j) < X_n(0) - u$ is illustrated in Figure 6. If W_{n+1} is smaller than the residual time until the j-th phantom system reaches level u, called $r_{n+1}(j)$, then $D_{n+1}(j) = D_{n+1} = W_{n+1}$, otherwise $D_{n+1}(j) = r_{n+1}(j)$ and $D_{n+1}(0) = \min(W_{n+1}, r_{n+1})$. In both this situations, we have $Z_{n+1}(j) = (\min(W_{n+1}, r_{n+1}) - r_{n+1}(j))^+$.

Finally, a straightforward calculation gives

$$r_{n+1}(j) = r_{n+1} + \frac{1}{\delta} \ln \left(1 - \frac{d_n(j)}{X_n(0) + c/\delta} \right)$$

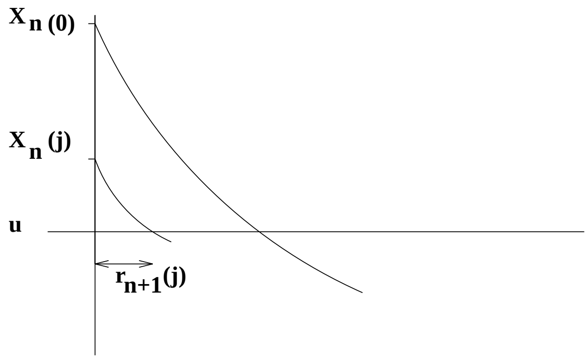

Figure 6: Phantom systems: case 2

4.2 Phantom Surplus Processes

The phantom systems resulting from disregarding any particular claim are quite easy to construct. We need to evaluate one cycle of the nominal surplus process under the new reference measure, containing a random number τ of claims. For each claim, we consider the corresponding phantom system, as illustrated in Figure 7.

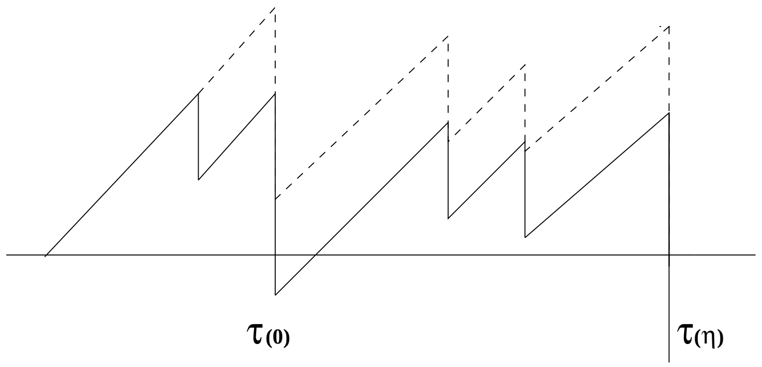

Figure 7: Pathwise construction of phantom system $j = 1$.

As soon as the claim amount $Y_j, j \leq \tau$ is generated, the difference $d_j(j) = Y_j$ between the nominal surplus and the phantom surplus is calculated. Since both processes evolve following the linear dynamics, then $d_n(j) = Y_j$ for all $n \leq \tau$. Having omitted the payment of one claim, clearly the phantom surplus is always larger than the nominal, which means that $\tau(j) \geq \tau(0)$. Using our previous notation, the amount by which the nominal process falls below zero at the moment of ruin is $S(0) - cT(0)$. If $S(0) - cT(0) > Y_j$, then $\tau(0)$ will also be the epoch of ruin for the j-th phantom system and we say that it "dies". At epoch $\tau(0)$ we calculate the contribution $(\tau(0)/\lambda){K_0}^{\tau(0)} e^{\mu[cT(0)-S(0)]}$. If there are any phantom systems still "alive", we keep generating W_n and Y_n and keep only one quantity $cT_n - S_n$ as would correspond to the nominal system. Again, at any moment, if $cT_n - S_n > Y_j$ for each living phantom indexed by $j \leq \tau(0)$, then this system dies and we can thus add its contribution $({K_0}^{\tau(j)}/\lambda_0)e^{\mu[cT(j)-S(j)]}$ to (13).

4.3 Virtual Surplus Processes

In order to implement (14), a direct approach prescribes generating one cycle of the surplus process under the measure with rate λ_0 and Esscher transform of the claims as in Section 2.2.1, keeping in memory all the values $\{(W_i, Y_i)\}$ of the trajectory. Once T_τ is known, generate $T^* \sim U[0, T_\tau]$ as the arrival epoch of the virtual customer, and its claim amount Y^* with density $Ke^{\mu y}g(y)$. To calculate the estimator (14) it is now necessary to backtrack and re-calculate the virtual trajectory once the virtual customer is added into place, to compute $\tilde{\tau}, T_{\tilde{\tau}}$ and $S_{\tilde{\tau}}$.

But backtracking a simulation is not numerically efficient. Instead, we propose to generate in advance the virtual claim size Y^*. There are two types of intervals: if $U(t) > Y^*$ then the virtual claim will not cause the ruin if $t < T^* \leq T_{\tau(n)}$, where $\tau(n)$ is the index of the first claim afterwards such that $U_{\tau(n)} < Y^*$, as illustrated in Figure 8 for intervals of type 1. Rather, $\tilde{\tau} = \tau(n)$ in this case. On the other hand, intervals of type 2 are those where the surplus process $U(t)$ is below Y^*. If T^* belongs to one of these intervals, then ruin is inmediate, $T_{\tilde{\tau}} = T^*, S_{\tilde{\tau}} = S_n$ and $\tilde{\tau} = n + 1$ where n is the index of the claim just prior to T^*.

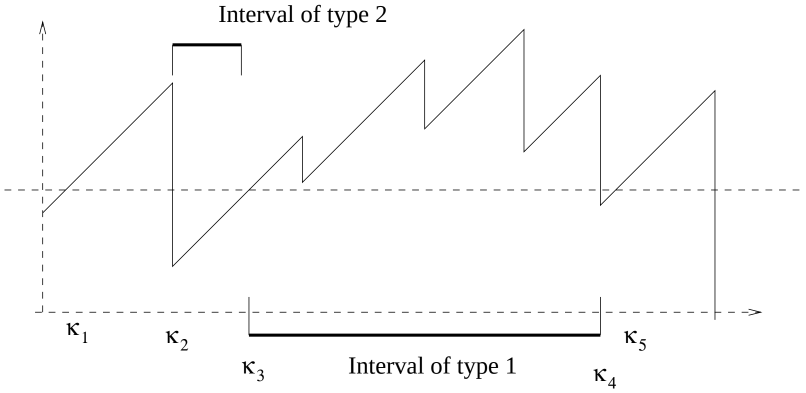

Figure 8: Virtual Estimation

Define the break points κ_i as follows: set $\kappa_0 = 0, \tau(1) = 0$. Suppose $u < Y^*$, set $\kappa_1 = \min(W_1, (Y^* - u)/c)$. While $U_n < Y^*, n \geq 1$ (which means that $\kappa_n = W_n$), and set $\tau(n) = \tau(n-1) + 1$, $Z(n) = (T_{\tau(n)-1}, S_{\tau(n)-1} - Y^*)$. Next, $\kappa_{n+1} = T_n + \min(W_{n+1}, (Y^* - U_n)/c)$. Once $\kappa_n < W_n$, we enter an interval of type 1. Set $\tau(n) = 1 + \min\{n \geq \tau(n-1) : U_n < Y^*\}$, so that $\kappa(n) = T_{\tau(n)-1}$. Until the $\tau(n) - 1$-st claim the process will remain above Y^* and virtual claims happening within this interval of time will all have the effect of causing the ruin at $\kappa(n)$. Set $Z(n) = (T_{\tau(n)}, S_{\tau(n)} - Y^*)$. Then repeat calculating intervals of type 2 with the associated Z_n. Call I_i the set of indeces n such that $(\kappa_n, \kappa_{n+1}]$ is an interval of type $i = 1, 2$.

When the simulation is over we know T_τ. A naïve approach would be to generate a uniform random variable on $[0, T_\tau]$ and look for the index n such that $\kappa(n) < T^* \leq \kappa(n+1)$. If this interval is of type 1, set $T_{\tilde{\tau}} = Z_1(n), S_{\tilde{\tau}} = Z_2(n) + Y^*, \tilde{\tau} = \tau(n)$, if it is of type 2, then set $T_{\tilde{\tau}} = T^*, S_{\tilde{\tau}} = Z_2(n) + Y^*, \tilde{\tau} = \tau(n)$.

A conditioning argument can further reduce the variance of the estimation

Ross (1997). Call $\rho_j = \kappa_j - \kappa_{j-1}$. Given Y^* and the nominal trajectory $\{(W_i, Y_i), i = 1, \ldots, \tau\}$, if T^* belongs to an interval of type 1, the desired statistics is independent of the actual value of T^*, while if the interval is of type 2, the conditional expectation can be evaluated analytically:

$$\begin{aligned} T_\tau E_1 \left[K_0{}^{\tilde{\tau}} e^{\mu(cT_{\tilde{\tau}} - S_{\tilde{\tau}})} \right] &= \sum_{j \in I_1} \rho_j K_0{}^{\tau(j)} e^{\mu(cZ_1(j) - Z_2(j))} \\ &\quad + \sum_{j \in I_2} \int_{\kappa_{j-1}}^{\kappa_j} \left[K_0{}^{\tau(j)} e^{\mu(ct - S_{Z_2(j)})} \, dt \right] \\ = \sum_{j \in I_1} \rho_j K_0{}^{\tau(j)} e^{\mu(cZ_1(j) - Z_2(j))} &+ \sum_{j \in I_2} \frac{K_0{}^{\tau(j)}}{\mu c} e^{-\mu Z_2(j)} (e^{\mu c \kappa_j} - e^{\mu c \kappa_{j-1}}) \end{aligned}$$

Unfortunately, for the interest model we cannot perform the integration to improve the efficiency of the method when using (16). Naturally, we can still condition the expectation according to the subinterval $[T_n, T_{n+1})$ where the virtual arrival may occur.

If $T^* > T_n^{\Delta\lambda}$, then the two processes coincide up to the n-th claim. On the other hand, since $T^* \sim U[0, T_\tau]$, $P[T_n < T^* \leq T_{n+1}] = W_{n+1}/T_\tau$, which yields:

$$T_\tau E_1[h_{\tilde{\tau}}] = \sum_{j=1}^{\tau} W_{j+1} h_j E^{(j)} \left[\prod_{i=j}^{\tau(j)-1} e^{(\lambda_i(j) - \lambda)\left[W_{i+1}(j) - \frac{Y_{i+1}(j)}{c_i(j)} \right]} \right]$$

where h_j is defined by (15). Now $E^{(j)}$ is the expectation conditioned on arrival of the virtual customer at $T^* \sim U[T_{j-1}, T_j]$, which naturally changes the evolution of the remainder of the process. In the simulations, we generate a uniform variable $W_j(j)$ in $(0, W_j]$ at each stage, and associate the claim $Y_j(j) = Y_j$ as in the nominal. From there on, we use common random numbers to generate W_{n+1} and $W_{n+1}(j)$ as well as Y_{n+1} and $Y_{n+1}(j)$, which will depend on the (distinct) state values U_n and $U_n(j)$. Under this construction, the virtual systems are dominated by the nominal trajectory.

5 Simulation Results

5.1 Functional Estimation

The storage process and its various phantoms are independent of the initial endowment u, as explained in Section 3.2. Therefore, a single run can be used to simultaneously estimate in parallel $\psi(\cdot, \lambda), \Psi(\cdot, \lambda)$ for a set of values of u. To illustrate the gain in computational effort, for $\lambda = 1.0$, estimating 10 values of u doing 10 separate simulations took 36.75 seconds of CPU time, while estimating the same 10 values in parallel took 12.49 seconds. These experiments were performed in a Sparc Station 10. In addition to saving computer time and memory, we were able to estimate the ruin probability and its derivative as functions of u using common random variables, which smoothes out the variability.

We illustrate functional estimation with the non-linear dynamics with interest rate $\delta = 0.05$ using the Storage Process with the RPA method. As shown in Gerber (1979), the ruin probability now satisfies Segerdahl's formula:

$$\psi(u,\lambda) = \frac{\Gamma\left(\frac{\lambda}{\delta}, \beta\left(\frac{c}{\delta} + u\right)\right)}{\Gamma\left(\frac{\lambda}{\delta}, \beta\frac{c}{\delta}\right) + \frac{\delta}{\lambda}\left(\frac{\beta c}{\delta}\right)^{\lambda/\delta} e^{-\beta c/\delta}}$$

where $\Gamma(a,b) = \int_b^\infty x^{a-1}e^{-x}\,dx$ is the incomplete gamma function. We do not have an explicit expression for the derivative $\Psi(u,\lambda)$.

Mathematica was used to plot the ruin probablility $\psi(u,\lambda)$ for a region in the (u,λ) plane. As for the derivative, direct differentiation caused numerical instabilities (aside from taking several minutes to compute). We had to use a sequence of finite differences, making the differences smaller yet avoiding numerical problems, until we were satisfied with our approximations, shown in Figure 9. Therefore, our "theoretical values" for the derivatives are actually numerical approximations. It should be noted that solving this way for the derivative using *Mathematica* took longer than running our simulations. The estimated surfaces are shown in Figure 10.

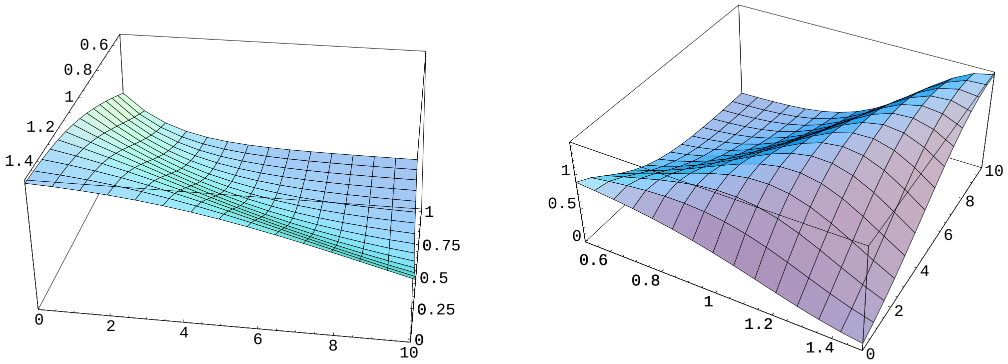

Figure 9: Theoretical Functions $\psi(u,\lambda)$ and $\Psi(u,\lambda)$.

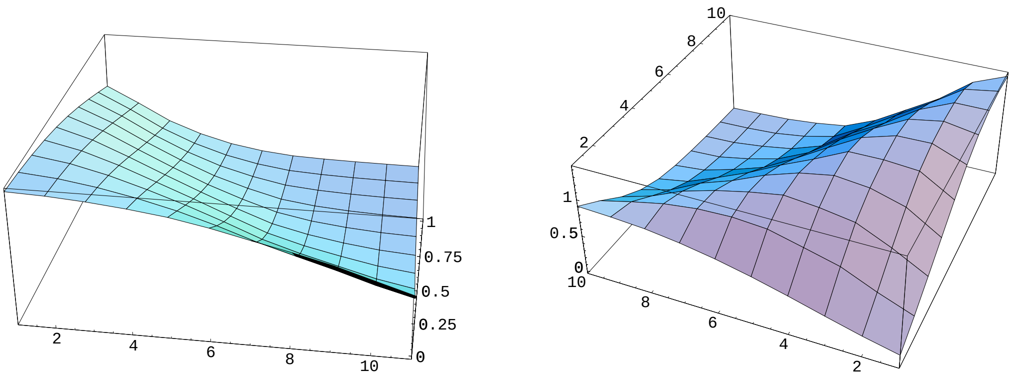

Figure 10: Estimated Functions $\hat{\psi}(u,\lambda)$ and $\hat{\Psi}(u,\lambda)$.

Finally, to illustrate a better measure of the over all efficiency of our estimators, we present in Figure 11 the estimated mean square error of both the ruin probability and of its derivative.

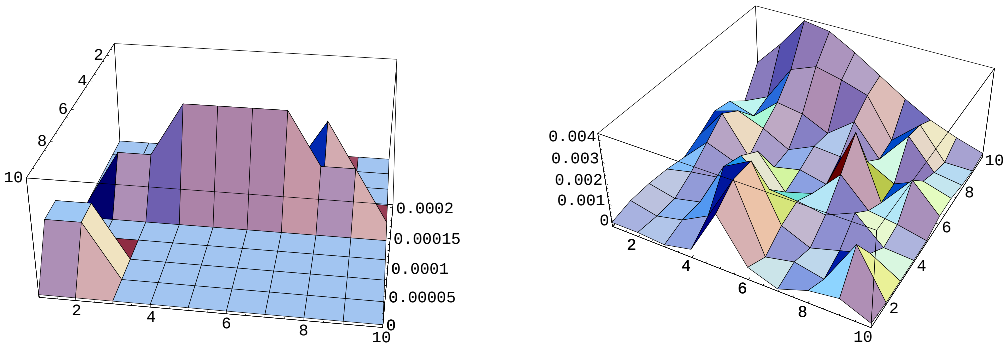

Figure 11: Estimated MSE's of $\hat{\psi}(u,\lambda)$ and $\hat{\Psi}(u,\lambda)$.

5.2 Confidence Intervals

For the surplus process with Importance Sampling a sequence of N i.i.d. samples was simulated. For each sample we simulated one trajectory of the surplus process under the reference measure until the time $\tau(0)$ of ruin, but we kept the simulation going until all the phantom systems also ruined. Since the estimators of both ψ and Ψ are then a sequence of i.i.d. variables Y_i, we used the sample variance to estimate the standard deviation as well as our level α confidence intervals:

$$\hat{S}_N(Y) = \frac{1}{N-1}\sum_{k=1}^{N}(Y_k - \bar{Y}_N)^2, \quad \bar{Y}_N \pm z_{1-\alpha/2}\sqrt{\frac{\hat{S}_N(Y)}{N}} \tag{17}$$

where $\bar{Y}_N$ is the sample mean and $z_{1-\alpha/2}$ is the upper $(1-\alpha/2)$ quantile of the normal distribution.

For the storage process, we use the regenerative method Bratley, Fox, and Schrage (1987) in order to estimate the (asymptotic) variance and the confidence intervals. Let C_i, Y_i denote, respectively, the cycle length and cumulative statistics $\sum_n D_n$ for estimating ψ (or $\sum_n Z_n$ for the derivative Ψ) over the i-th regenerative cycle. We used N regenerative cycles for the estimator. Define the statistics:

$$\begin{aligned}
\hat{\psi}_N &= \frac{\sum_{k=1}^{N} Y_k}{\sum_{k=1}^{N} C_k} = \frac{\bar{Y}_N}{\bar{C}_N}, \\
\hat{S}_N(C) &= \frac{1}{N-1}\sum_{k=1}^{N}(C_k - \bar{C}_N)^2, \\
\hat{S}_N(Y,C) &= \frac{1}{N-1}\sum_{k=1}^{N}(Y_k - \bar{Y}_N)(C_k - \bar{C}_N).
\end{aligned}$$

By the renewal theorem, $V_i = Y_i - \psi C_i$ are i.i.d. mean zero random variables. Since they have bounded variance, they satisfy the central limit theorem,

$$\frac{\sqrt{N}(\bar{Y}_N - \psi\bar{C}_N)}{\sqrt{\sigma_V^2}} \implies \mathcal{N}(0,1), \quad \text{as } N \to \infty$$

and $\hat{\sigma}_V^2 = \hat{S}_N(Y) - 2\hat{\psi}\hat{S}_N(Y,C) + (\hat{\psi})^2\hat{S}_N(C)$ is a consistent estimator of the variance of V_1, which in turn implies that for N large, the variance of $\hat{\psi} - \psi$ is approximately $(1/N)\hat{\sigma}_V^2/(\bar{C}_N)^2$. Our level α confidence intervals were therefore estimated using:

$$\hat{\psi} \pm z_{1-\alpha/2}\frac{\sqrt{\hat{\sigma}_V^2/N}}{\bar{C}_N} \tag{18}$$

In the next sections we present the numerical comparison of the proposed methods for estimating the Ruin Probability $\psi(u,\lambda)$ and its derivative $\Psi(u,\lambda)$. A common measure of comparison is the *efficiency* of the estimators. If we call Y_N an estimator obtained with N replications of the simulation, we define its efficiency as being te inverse of the product of its variance and mean CPU time:

$$E = \frac{1}{C_N(\mathrm{Var}(Y_N) + \beta(Y_N))} \tag{19}$$

where β_N is the bias of the estimator and C_N the mean CPU time associated. The Importance Sampling estimators are unbiased and the Storage Process estimators are strongly consistent. We therefore estimate the asymptotic efficiency assuming "large" sample sizes and estimating the efficiency by:

$$\hat{E} = \frac{1}{\hat{C}_N\,\hat{\mathrm{Var}}(Y_N)}$$

where the estimated variance is the same as the one used to build the Confidence Intervals.

In all our experiments we used Functional Estimation for the Storage Process, therefore we had to adjust the formula for the efficiency (19) in this case, multiplying by the number of different values of u reported (in this case, 4). This accounts for the fact that the CPU time of a single simulation yields the 4 estimations at once, while the Importance Sampling performs one simulation for each value of u and the CPU time is dependent on u.

5.3 The Linear Dynamics

We first report the simulation results for the linear dynamics, when $\delta = 0$. We used $Y_1 \sim \exp(1/\beta)$ for the claim amount distribution, $\beta = 1.0, \lambda = 1.0, c = 1.5$. For this model it is possible to calculate the analytical expression for the ruin probability Gerber (1979):

$$\begin{aligned}
\psi(u,\lambda) &= \frac{1}{\theta}\exp\left\{-\left(\frac{\theta}{1+\theta}\right)\frac{u}{\beta}\right\}\\
\Psi(u,\lambda) &\equiv \frac{\partial}{\partial\lambda}\psi(u,\lambda) = \frac{1}{\lambda}\psi(u,\lambda)\left[\frac{u}{\beta(1+\theta)} + \frac{1+\theta}{\theta}\right]
\end{aligned}$$

where $\theta = c/\lambda\beta - 1 > 0$ (otherwise $\psi(u,\lambda) = 1$).

For the surplus process under Importance Sampling we chose $\lambda_0 = \lambda + Rc$, so that $K_0 = 1$ and the likelihood ratio $\phi[U(t); t \le \tau] \le 1$ a.s. which bounds the variance of the estimator uniformly. We now proceed to justify this choice. For the exponential claim model, the new distribution is exponential with parameter $\alpha - \mu$, for $\alpha = 1/\beta$ provided that:

$$\frac{1}{\beta} > \frac{(\lambda_0 - \lambda)}{c} \qquad \Rightarrow \lambda_0 < \frac{c}{\beta} + \lambda$$

so that $\tilde{E}_{\lambda_0}[Y] = 1/(\alpha - \mu)$. We must choose λ_0 so that ruin is certain under the new measure, that is, we want $\lambda_0 > c/\tilde{E}_{\lambda_0}[Y]$, which yields:

$$\lambda_0 > \frac{c}{\beta} - (\lambda_0 - \lambda) \qquad \Rightarrow \lambda_0 > \frac{1}{2}\left(\frac{c}{\beta} + \lambda\right)$$

Finally, for this model $\lambda + Rc = c/\beta$, which always lies in the desired interval, since we assumed $\lambda < c/\beta$. The following table gives the simulation lengths.

Number of Cycles for estimating $\psi(u)$		
	Importance Sampling 100,000	Storage Process 300,000
Number of Cycles for estimating $\Psi(u)$		
Phantom IS 50,000	Virtual IS 70,000	Phantom SP 300,000

In Table 1 we give the results for $\hat{\psi}_X, \hat{\psi}_U$ and $\hat{\Psi}_X, \hat{\Psi}_U$ for the linear case. The estimators are identified by "X" or "U", whether they correspond to the storage process (SP) or to the surplus process under Importance Sampling (IS). Subindices "UP", "UV" and "XP" refer to the Phantom or Virtual RPA method. We report the confidence interval (C.I.) for $\alpha = 0.05$.

	$u = 0$	$u = 2$	$u = 4$	$u = 6$
Ruin Probability $\psi(u)$				
$\psi(u)$	0.66667	0.34228	0.17573	0.09022
$\hat{\psi}_U(u)$ Eff	0.6669 ± 0.0014 359,550	0.3422 ± 0.0007 1,369,808	0.1759 ± 0.0003 5,170,465	0.0904 ± 0.0002 19,640,353
$\hat{\psi}_X(u)$ Eff	0.6670 ± 0.0018 1,200,782	0.3420 ± 0.0027 532,937	0.1751 ± 0.0025 609,213	0.0898 ± 0.0020 977,867
Derivative of Ruin Probability $\Psi(u)$				
$\Psi(u)$	0.66667	0.79864	0.64435	0.45112
$\hat{\Psi}_{UP}(u)$ Eff	0.6713 ± 0.0153 16,359	0.7997 ± 0.0133 7,225	0.6394 ± 0.0089 12,045	0.4515 ± 0.0057 23,478
$\hat{\Psi}_{UV}(u)$ Eff	0.6698 ± 0.0120 26,286	0.8000 ± 0.0106 17,096	0.6465 ± 0.0070 25,512	0.4549 ± 0.0043 49,858
$\hat{\Psi}_{XP}(u)$ Eff	0.6669 ± 0.0019 817,048	0.7968 ± 0.0086 40,895	0.6389 ± 0.0121 20,895	0.4437 ± 0.0129 18,408

Table 1: Linear Dynamics: $\delta = 0$

The length of the cycles (and thus the computational effort) depends on u when using Importance Sampling, which naturally tends to work better for larger values of u, when estimating probabilities of a "rarer" event than for $u = 0$. For the Storage Process the situation is very different. On one hand, one single simulation yields all values of u, but naturally the variance of the estimator becomes larger (in general) for larger values of u. While it seems at first glance that Importance Sampling has smaller variance for all the estimators than the Indirect Estimation method, once the computing time is taken into account, it seems that the second estimator may be more efficient in some situations.

In addition, simulations using Importance Sampling strongly depend on the model and the particular claim distribution. It is not evident that for other distributions of Y_i, including empirical distributions, we can choose an optimal value of λ and a "good" change of measure for Y_i. In Vázquez-Abad and LeQuoc (1999) we experiment with Weibull and Pareto distributions as well as exponential and the set-up required by Importance Sampling is non negligible.

5.4 The Dynamics with Interest

We turn now to the case $\delta > 0$ and $\beta = E[Y] = 1.0$. As before, we use exponentially dsitributed claim amounts in order to compare with the theoretical values. We considered first $\lambda = 1.0$ and the two scenarios $c = 1.0$ and $c = 1.5$ with $\delta = 0.05$, using $N = 200,000$ cycles for SP and $N = 50,000$ cycles for IS. 95% confidence intervals are given in Table 2 for $c = 1.0$ and in Table 3 for $c = 1.5$.

	$u = 0$	$u = 2$	$u = 4$	$u = 6$
$c = 1.0$				
Ruin Probability $\psi(u)$				
$\psi(u)$	0.841108	0.547364	0.322416	0.173175
$\hat{\psi}_X(u)$	0.8413 ± 0.0015	0.5472 ± 0.0042	0.3219 ± 0.0062	0.1724 ± 0.0076
Eff	105,289	12,941	5,770	3,873
$\hat{\psi}_U(u)$	0.8428 ± 0.0024	0.5483 ± 0.0024	0.3216 ± 0.0016	0.1729 ± 0.0010
Eff	83,771	29,909	39,195	85,823
Derivative of Ruin Probability $\Psi(u)$				
$\Psi(u)$	0.575427	1.06984	1.04342	0.778886
$\hat{\Psi}_{XP}(u)$	0.5754 ± 0.0053	1.0731 ± 0.0099	1.0453 ± 0.0096	0.7782 ± 0.0072
Eff	8,015	2,304	2,428	4,381
$\hat{\Psi}_{UV}(u)$	0.5737 ± 0.0112	1.0742 ± 0.0111	1.0423 ± 0.0078	0.7803 ± 0.0048
Eff	3,852	1,424	1,856	3,803

Table 2: Exponential Dynamics: $\delta = 0.05$

These results seem to indicate that, while estimation of ruin probabilities may be more efficient using Importance Sampling, especially for larger values

	$u = 0$	$u = 2$	$u = 4$	$u = 6$
$c = 1.5$				
Ruin Probability $\psi(u)$				
$\psi(u)$	0.619915	0.264757	0.106251	0.040303
$\hat{\psi}_X(u)$	0.6199 $\pm$ 0.0025	0.2642 $\pm$ 0.0049	0.1060 $\pm$ 0.0061	0.0404 $\pm$ 0.0065
Eff	290,963	77,637	52,587	45,639
$\hat{\psi}_U(u)$	0.6211 $\pm$ 0.0022	0.2657 $\pm$ 0.0010	0.1067 $\pm$ 0.0005	0.0403 $\pm$ 0.0002
Eff	830,833	1,150,919	3,761,661	17,284,735
Derivative of Ruin Probability $\Psi(u)$				
$\Psi(u)$	0.568656	0.542398	0.332780	0.168018
$\hat{\Psi}_{XP}(u)$	0.5689 $\pm$ 0.0038	0.5405 $\pm$ 0.0037	0.3304 $\pm$ 0.0022	0.1675 $\pm$ 0.0011
Eff	129,857	143,880	385,081	1,498,020
$\hat{\Psi}_{UV}(u)$	0.5695 $\pm$ 0.0073	0.5423 $\pm$ 0.0047	0.3313 $\pm$ 0.0023	0.1680 $\pm$ 0.0011
Eff	72,466	56,363	137,896	507,052

Table 3: Exponential Dynamics: $\delta = 0.05$

of u, this is not the case when estimating the derivatives. Recall that the virtual method was constructed for the particular example of exponential claims. When other distributions are used, it is not clear how the method should be adapted. In particular, evaluation of the implicit equation for R_i is required at each step of the simulation, a task that may be numerically very heavy. On the other hand, the Phantom RPA method with the Storage Process is very robust and does not require any special model dependent code.

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