THE CASE OF THE SWAPPING AND OF THE DISAPPEARING PHANTOMS

Felisa J. Vázquez-Abad

Department of Computer Science and Operations Research
University of Montreal
P.O. Box 6128, Downtown Satation,
Montreal, Quebec H3C 3J7
E-mail: vazquez@iro.umontreal.ca

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Abstract

Suppose that we want to optimize a long term or finite horizon average for a stochastic DES, where the control parameter is a probability vector. A regular gradient search procedure cannot be implemented without additional mechanisms to ensure feasibility of the new control values. The Kuhn-Tucker conditions establish an equilibrium equation whose zeroes correspond to local optima. This formulation gives rise to a stochastic approximation algorithm that no longer needs adjustments. Instead of the regular gradients, we must now estimate a particular generalized gradient.

This paper introduces two versions of the Phantom Rare Perturbation Analysis (RPA) method. The version of the Swapping Phantoms implements RPA directly to estimate this generalized gradient. The version of the Disappearing Phantoms indirectly recovers the desired gradient estimation via a surrogate approach. Both methods are designed to work on-line, and we compare and discuss several schemes using CRN or independent streams.

1 Formulation of the problem

Consider the problem of minimizing a cost function $F(\theta)$, which is a function of a long term average defined for a given stochastic discrete event system. The rate at which events occur is λ, and the Point process describing the occurrences of events is denoted by $N(t)$. At each event a particular action from amongst V possibilities must be taken, and the control strategy is randomized: θ_i is the probability that action i will be chosen, independently of everything else in the system. We call Θ the feasible state space, or:

$$\Theta = \{\theta \in I\!R^V : \theta_i \geq 0, \quad \sum_{i=1}^{V} \theta_i = 1\}.$$

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Define $F'_k(\theta) = \partial F(\theta)/\partial\theta_k$. Call $\mathcal{G}_k$ the *generalized gradient operator* defined by:

$$\mathcal{G}_k[F(\cdot)] = \frac{\partial F(\cdot)}{\partial \theta_k} - \sum_{j=1}^{V} \theta_j \frac{\partial F(\cdot)}{\partial \theta_j} \tag{1}$$

Then, a value of θ^* such that $\sum_{k=1}^{V} \theta^*_k = 1$ and

$$\mathcal{G}_k[F(\theta^*)] = \begin{cases} = 0 & \text{if } \theta^*_k > 0 \\ \geq 0 & \text{if } \theta^*_k = 0 \end{cases} \text{ for } k = 1, .., V \tag{2}$$

satisfies the Kuhn-Tucker conditions. Under convexity of $F(\theta)$, if $\bar{\theta}$ satisfies (2), then $\bar{\theta} = \theta^*$.

Lemma 1 *Consider the ODE:*

$$\frac{d\vartheta_k(t)}{dt} = -\vartheta_k(t)\mathcal{G}_k[F(\vartheta(t))] \tag{3}$$

If the starting point is a probability vector: $\vartheta_k(0) \in \Theta$, then $\vartheta(t)$ remains always as a probability vector. Furthermore, if $\vartheta(0) \in \Theta$ is such that $\vartheta_k(0) > 0$ for each $k = 1, \ldots, V$ and it is not a local maximum of $F(\cdot)$, then the stable points of (3) are local minima. If the function $F(\cdot)$ has a unique minimum, then (3) has an asymptote at $\theta^* = \min_{\theta\in\Theta} F(\theta)$.

The proof of this lemma is in [1]. The relevance of this result is the following. If we can measure $\mathcal{G}_k[F(\theta)]$ then we can approximate the ODE to search for the optimum. In the context of a stochastic system, if we can estimate the generalized gradient, then under some regularity conditions we can use stochastic approximation [1] and the ensuing control values $\vartheta(\cdot)$ will also be asymptotically optimal.

In this paper we propose two phantom RPA estimators of the generalized gradient, generalizing [2] and [3] for the case of multi-valued decisions. In order to do so, we shall focus on a particular model example.

2 The model example

Consider a system in which a machine processes items at different speeds $v_k; k = 1, \ldots, V$. Items arrive at the machine following a renewal process $N(t)$. Let $\eta_i(\theta)$ be the speed chosen by the machine for the i-th item processed. The service times of consecutive customers $\{S_i\}$ are independent random variables with $E[S_i|\eta_i(\theta) = v_k] = \mu_k^{-1}$. The associated operating cost per unit time is c_k. We assume $\mu_k < \mu_j, c_k < c_j$ if $k < j$, thus faster modes of operation are more costly. Although we may know which speeds are faster in the average, in practice we may not know the distribution of the consecutive service times. The problem is to select the speeds of operation of the machines that minimize the waiting time at the lowest cost. The simplest strategy is the randomized strategy, in which $P\{\eta_i(\theta) = v_k\} = \theta_k$.

Assume for simplicity that $\lambda < \mu_1$, ensuring stability of the process for all $\theta \in \Theta$. Then, the limiting probabilities exist, the process is ergodic, and the invariant measure is unique for each θ. The performance function to be optimized is $F(\theta) = W(\theta) + C(\theta)$, where $W(\theta)$ is the stationary average waiting time in the system and $C(\theta)$ is the mean stationary cost resulting from the use of the machine and can be calculated analytically, $C(\theta) = \lambda \sum_{k=1}^{V} c_k\theta_k/\mu_k$. From the structure of the model, $F(\cdot)$ is continuously differentiable. This problem may be stated as a minimization problem of the function $F(\theta)$ under the feasibility constraints:

$$\begin{array}{lrcl} \text{Minimize} & F(\theta) & = & W(\theta) + C(\theta) \\ \text{subject to} & \displaystyle\sum_{i=1}^{V} \theta_i & = & 1 \end{array}$$

If we do not know the exact values of μ_k, the gradient of $C(\theta)$ can be strongly consistently estimated via sample averages. Since $F(\theta) = W(\theta) + C(\theta)$, it suffices to build estimators of the generalized gradient in (3) for the waiting time.

In the remaining sections we present such estimators, using the so-called Phantom RPA (Rare Perturbation Analysis) method.

3 Parallel phantom systems

Call A_i the interarrival time between customers i and $i + 1$, and let $\{u_i\}$ be a sequence of i.i.d uniform variates. The service of customer i is

$$S_i(u_i, \eta_i) = \sum_{k=1}^{V} G_k^{-1}(u_i)\mathbf{1}_{\{\eta_i = v_k\}}$$

where $G_k(\cdot)$ is the service distribution of speed v_k. The discrete event process is described via Lindley's equations. If R_i is the total time a customer remains in the system and W_i its waiting time, then Lindley's equations yield:

$$\begin{aligned} W_{i+1} &= \max(0, R_i - A_i) \\ R_{i+1} &= S_{i+1}(u_i, \eta_i) + W_{i+1} \end{aligned}$$

The above equations hold regardless of the way we choose the speeds of the machine, since we are using the information on the sequence of decisions $\eta = \{\eta_i\}$ itself. Use the notation $\eta_i(0)$ for the decisions of the *nominal system*, taken according to $P\{\eta_i(0) = v_k\} = \theta_k$, independently of $\{(A_j, u_j); j \le i\}$. Fix the index k to estimate a sensitivity, let $\Delta > 0$ be a small number, and $\tilde{\Delta} \in I\!R^V$ be a vector with the value Δ as the k-th component. The other components satisfy $\tilde{\Delta}_l = -p_l\Delta, p_l > 0, l \neq k$ and shall be determined later, according to the two gradient estimators that we shall describe.

Definition 1 *Given a trajectory of the nominal system $(A_i, u_i, \eta_i(0))$, and a sequence $\tilde{\eta}$ of "phantom decisions", define the* parallel $\tilde{\eta}$-phantom system *as the system whose trajectory is determined using the same sequence (A_i, u_i) as the nominal system, but where the decision of the i-th customer is $\tilde{\eta}_i$.*

Consider now the $\tilde{\eta}$-phantom system, where $\tilde{\eta}_i$ is assigned as follows:

$$\begin{aligned} P\{\tilde{\eta}_i = v_l|\eta_i(0) = v_l\} &= 1, \quad l \neq k \\ P\{\tilde{\eta}_i = v_k|\eta_i(0) = v_k\} &= 1 - \frac{\Delta}{\theta_k} \\ P\{\tilde{\eta}_i = v_l|\eta_i(0) = v_k\} &= \frac{p_l\Delta}{\theta_k}, \quad l \neq k \end{aligned}$$

Then the decisions of the $\tilde{\eta}$-phantom system satisfy $P\{\tilde{\eta}_i = v_l\} = \theta_l - \tilde{\Delta}_l$, for all $l = 1, \ldots, V$ and $\tilde{\eta}$ has a distribution according to $\tilde{\theta} = \theta - \tilde{\Delta}$.

For any sequence $\eta = \{\eta_i\}$ of decisions, we define the cumulative waiting time over the first M customers as $\varphi_M(\eta) = \sum_{i=1}^{M} W_i$. The finite difference is defined by:

$$D_\Delta(M) = \frac{\varphi_M(\eta(0)) - \varphi_M(\tilde{\eta})}{\Delta} \tag{4}$$

Given M customers, the number of phantoms in the sequence $\{\tilde{\eta}_i, i = 1, \ldots, M\}$ has a Binomial distribution, since each customer is chosen as a phantom customer with probability $P\{\tilde{\eta}_i \neq \eta_i(0)\} = P\{\tilde{\eta}_i \neq v_k|\eta_i(0) = v_k\}P\{\eta_i(0) = v_k\} = \Delta$, independently of previous ones. Therefore,

$$\begin{aligned} E[D_\Delta(M)] &= E\left[M\Delta(1-\Delta)^{M-1}E^{(1)}[D_\Delta(M)]\right] \\ &+ E\left[\sum_{m=2}^{M} \binom{M}{m} \Delta^m (1-\Delta)^{M-m} E^{(m)}[D_\Delta(M)]\right] \end{aligned}$$

where $E^{(m)}$ is the expectation w.r.t. $\tilde{\eta}$, conditioning on having exactly m phantoms. As shown in [3], $\sum_{m=2}^{M} \binom{M}{m} \Delta^m (1-\Delta)^{M-m} \le M^2\Delta^2$ bounds the second term. Since $\varphi_M(\eta) \le \sum_{i=1}^{M}\sum_{j=1}^{M} S_i$ for any sequence of decisions, and since $E[S_i] \le 1/\mu_V < \infty$, the dominated convergence theorem establishes the phantom RPA formula for any finite M:

$$\lim_{\Delta\to 0} E[D_\Delta(M)] = \frac{1}{\theta_k} E\left\{ \sum_{j=1}^{M} I_k(j)\, [\frac{\varphi_M(\eta(0) - \varphi_M(\eta(j))}{M}] \right\} \quad (5)$$

where $I_k(j) = \mathbf{1}_{\{\eta_j(0)=v_k\}}$. The sequence $\eta(j)$ is defined by

$$\begin{aligned} \eta_i(j) &= \eta_i(0), \quad i \neq j, \\ P\{\eta_j(j) = v_l | \eta_j(0) = v_l\} &= 1, \quad l \neq k \\ P\{\eta_j(j) = v_l | \eta_j(0) = v_k\} &= p_l. \end{aligned}$$

Equation (5) follows from equiprobability of all the sequences of phantom decisions that differ only in one component with $\eta(0)$.

3.1 The Swapping Phantoms

Consider $\tilde{\Delta}_l = -\theta_l \Delta/(1-\theta_k)$ and let $\tilde{\theta} = \theta - \tilde{\Delta}$ as before. Simple algebraic manipulations show that

$$\lim_{\Delta\to 0} \frac{F(\theta) - F(\tilde{\theta})}{\Delta} = \frac{1}{1-\theta_k} \mathcal{G}_k[F(\theta)] \quad (6)$$

In order to see this, rewrite:

$$\begin{aligned} F(\theta) - F(\tilde{\theta}) = \\ & -F(\tilde{\theta}_1, \tilde{\theta}_{-1}) + F(\theta_1, \tilde{\theta}_{-1}) \\ & -F(\theta_1, \tilde{\theta}_2, \tilde{\theta}^{(-2)}) + F(\theta_1, \theta_2, \tilde{\theta}^{(-2)}) \\ & - \dots \\ & -F(\theta^{(V-1)}, \tilde{\theta}_V) + F(\theta^{(V)}) \\ & +F(\theta) - F(\theta_1, \dots, \tilde{\theta}_k, \dots, \theta_V) \end{aligned}$$

where $\tilde{\theta}^{(-i)}$ is the vector $(\tilde{\theta}_{i+1}, \dots, \tilde{\theta}_V)$ and $\theta^{(i)}$ is the vector with components $\theta_i, i \neq k$ and $\theta_k^{(i)} = \tilde{\theta}_k$. Now use $\delta\theta_i = \Delta\theta_i/(1-\theta_k)$,

$$\begin{aligned} \lim_{\Delta\to 0} & \frac{F(\theta^{(i-1)}, \tilde{\theta}_i, \tilde{\theta}^{(-i)}) - F(\theta^{(i)}, \tilde{\theta}^{(-i)})}{\Delta} \\ &= \frac{\theta_1}{(1-\theta_k)} \lim_{\delta\theta_i \to 0} \frac{F(\theta_i + \delta\theta_i) - F(\theta_i)}{\delta\theta_i} \\ &= \frac{\theta_1}{(1-\theta_k)} \frac{\partial}{\partial\theta_i} F(\theta) \end{aligned}$$

Finally,

$$\begin{aligned} \lim_{\Delta\to 0} & \frac{F(\theta) - F(\theta_1, \dots, \tilde{\theta}_k, \dots, \theta_V)}{\Delta} = \\ &= \frac{1}{(1-\theta_k)} \left[\frac{\partial}{\partial\theta_k} F(\theta) - \theta_k \frac{\partial}{\partial\theta_k} F(\theta) \right] \end{aligned}$$

which establish (6).

Using this parameter $\tilde{\Delta}$ to determine $\tilde{\eta}$, the ensuing phantom customers in the $\tilde{\eta}$-phantom system swap speeds compared to the corresponding nominal decisions, choosing one of the remaining ones according to their original proportion. We then use (5) where the phantom decision sequences are chosen with $p_l = \theta_l/(1-\theta_k)$. This approach helps us define the corresponding estimators to drive the process towards the limiting ODE (3).

Proposition 1 *Assume that the service times are dominated by $\{S_i(\bar{\eta})\}$, for some random sequence $\bar{\eta}$, and that the dominating queueing process satisfies $E[\mathbf{N}^3(\bar{\eta})] < \infty$. Then:*

$$\frac{\theta_k}{(1-\theta_k)} \mathcal{G}_k[W(\theta)] = \lim_{M\to\infty} E\left\{ \sum_{j=1}^{M} I_k(j)\, \frac{[\varphi_M(\eta(0)) - \varphi_M(\eta(j))]}{M} \right\} \quad (7)$$

Proof: Using (6), (5) is an unbiased estimator of $(1-\theta_k)^{-1}\mathcal{G}_k[F_M(\theta)]$, where

$$F_M(\theta) = \frac{1}{M} \sum_{i=1}^{M} E_\theta[W_i]$$

is the finite horizon average waiting time of the first M customers, and the expectation E_θ is with respect to the underlying probability when θ is used as the decision probabilities.

From the construction of the phantom systems, $W_i(0) = W_i(j)$ for all $i \leq j$. When the system $\bar{\eta}$ finishes the busy period where j belongs, both the nominal and the j-th phantom system with decisions $\{\eta(j)\}$ finish. After this, their evolution is identical, since j was the only decision that is different. Therefore $E|\varphi_M(\eta(0)) - \varphi_M(\eta(j))| \leq 2E[\sum_{i=1}^{\mathbf{N}_j(\bar{\eta})} W_i(\bar{\eta})] < E[\mathbf{N}^2(\bar{\eta})\mathbf{B}_j(\bar{\eta})] < CE[\mathbf{N}^3(\bar{\eta})] < \infty$, where $\mathbf{N}_j(\bar{\eta})$ is the number of customers in the busy period of the dominating system where customer j belongs and $\mathbf{B}_j(\bar{\eta})$ is the corresponding length. The convergence as $M \to \infty$ of the r.h.s. of (7) follows now from this uniform bound.

To apply (6), we need only to verify that the interchange between the expectation and the limit $M \to \infty$ is valid. Since the invariant measure is unique and ergodic, $\mathcal{G}_k[W(\theta)] < \infty$, and the generalized gradient is a linear operator, this is established if

$$\frac{\partial}{\partial\theta_k} W(\theta) = \lim_{M\to\infty} \frac{\partial}{\partial\theta_k} F_M(\theta) \quad \text{a.s.}$$

for every initial distribution of the process, which in turn follows from [4] for our model, where all the n-step transition probabilities of the process are polynomials in $(\theta_i, i = 1, \dots, k)$ and therefore continuously differentiable. ∎

The practical problem is the required condition for domination. In our numerical examples, we used Poisson arrivals and uniformly distributed service distributions for each speed. In general, for an $M/G/1$ queue, Takács method (see [5]) gives a functional relationship between the moment generating function of the service time and the one of the number in a busy period. If the m-th moment of the service distribution is bounded, so is $E[\mathbf{N}^m]$, which follows from the above argument, although we omit the details of the proof. In our case, the moment generating function of the service times exists and thus all moments of the service times are bounded.

3.2 The Disappearing Phantoms

There are several practical problems with the Swapping Phantoms. First, the requirement that the system be stable if we always choose the slowest speed may be too restrictive, especially when we do not know explicitly the service distributions. Second, the method requires knowledge of the service distributions, in order to "swap" speeds. Recall that it requires assigning the services of the phantom customers as $S_i(u_i, v_l)$ with probability $p_l = \theta_l/(1-\theta_k)$. Naturally, if we knew the service distribution, we could evaluate the analytical solution and find the optimal setting without need for adaptive control. Even worse, the service distributions and input rates may vary in an unpredictable manner, in which case stochastic approximation can be used to track the optimal value. In such cases the Swapping Phantom method has to either approximate or estimate the distributions in some manner. We propose now a more robust estimator based only on the direct observation of the system.

Using a uniformization argument, consider the alternative model for the original process, where customers of class k arrive according to a renewal process with rate $\theta_k\lambda$. They require an amount of service that has distribution $G_k(\cdot)$. In this model, we can use $\tilde{\theta} = \theta - \tilde{\Delta}$ with $p_l = 0, l \neq k$, meaning that some of the customers, the phantom ones, are not allowed entrance to the machine and thus "disappear" from the system. The corresponding system will have a total incoming rate of $\tilde{\lambda} = \lambda - \Delta\theta_k\lambda$. We will presently show that:

$$\begin{aligned}\mathcal{D}_k(\lambda,\theta) &= \lim_{\Delta\to 0}\frac{F(\lambda,\theta) - F(\tilde{\lambda},\tilde{\theta})}{\Delta}\\ &= \frac{\partial F(\lambda,\theta)}{\partial\theta_k} + \lambda\frac{\partial F(\lambda,\theta)}{\partial\lambda} - \sum_{j=1}^{V}\frac{\partial F(\lambda,\theta)}{\partial\theta_j} \qquad (8)\end{aligned}$$

and therefore the generalized gradient can be expressed as:

$$\mathcal{G}_k[F(\lambda,\theta)] = \mathcal{D}_k[F(\lambda,\theta)] - \sum_{i=1}^{V}\theta_i\mathcal{D}_i[F(\lambda,\theta)] \qquad (9)$$

In order to see why this result holds, notice first that since the rate $\lambda_k = \theta_k\lambda$, then $\tilde{\lambda} = \lambda - \Delta\lambda$ and consequently, the corresponding system satisfies:

$$\begin{aligned}\tilde{\theta}_i &= \frac{\lambda\theta_i}{\lambda - \Delta\lambda} = \theta_i + \delta_i\\ \tilde{\theta}_k &= \frac{\lambda_k - \Delta\theta_k\lambda}{\lambda - \Delta\lambda} = \theta_k - \delta_k\end{aligned}$$

where:

$$\delta_i = \theta_i\Delta + \mathcal{O}(\Delta^2), \quad \delta_k = (1-\theta_k)\Delta + \mathcal{O}(\Delta^2)$$

Using $\delta\lambda = \Delta\theta_k\lambda$, we then have:

$$\begin{aligned}\lim_{\Delta\to 0}\frac{F(\lambda,\theta) - F(\tilde{\lambda},\tilde{\theta})}{\Delta} &= \lambda\frac{\partial}{\partial\lambda}F(\lambda,\theta) + \\ &+\lim_{\Delta\to 0}\frac{F(\lambda,\theta) - F(\lambda,\tilde{\theta})}{\Delta}\end{aligned}$$

Call $\tilde{\theta}^{(-i)} = (\tilde{\theta}_{i+1},\ldots,\tilde{\theta}_V)$ and proceed as before, rewriting:

$$\begin{aligned}\frac{F(\theta) - F(\tilde{\theta})}{\Delta} &= \theta_1\frac{F(\theta_1,\tilde{\theta}^{(-1)}) - F(\tilde{\theta}_1,\tilde{\theta}^{(-1)})}{\delta_1}\\ &\quad+\frac{F(\theta) - F(\theta_1,\tilde{\theta}^{(-1)})}{\Delta} + \mathcal{O}(\Delta) =\\ &= -\theta_1\frac{\partial}{\partial\theta_1}F(\theta) +\\ &\quad+\theta_2\frac{F(\theta_1,\theta_2,\tilde{\theta}^{(-1)}) - F(\theta_1,\tilde{\theta}_2,\tilde{\theta}^{(-1)})}{\delta_2}\\ &\quad+\frac{F(\theta) - F(\theta_1,\theta_2,\tilde{\theta}^{(-1)})}{\Delta} + \mathcal{O}(\Delta)\\ &= -\sum_{i\neq k}\theta_i\frac{\partial}{\partial\theta_i}F(\theta) + (1-\theta_k)\frac{\partial}{\partial\theta_k}F(\theta)\\ &= \frac{\partial}{\partial\theta_k}F(\theta) - \sum_{i=1}^{V}\theta_i\frac{\partial}{\partial\theta_i}F(\theta)\end{aligned}$$

In the case of the disappearing phantoms, Lindley equations can be calculated in parallel to the nominal system using $S_j(j) \equiv S_j(\eta_j(j), u_j) \equiv 0$.

Proposition 2 *Assume that $E[\mathbf{N}(0)^3] < \infty$. Then:*

$$\theta_k\mathcal{D}_k[W(\theta)] = \lim_{M\to\infty} E\left\{\sum_{j=1}^{M} I_k(j)\,\frac{[\varphi_M(\eta(0)) - \varphi_M(\eta(j))]}{M}\right\} \qquad (10)$$

Proof: Since $S_j(j) = 0$, the nominal system dominates all of the possible phantoms systems and $\mathbf{N}(\tilde{\eta}) \leq \mathbf{N}(\eta(0))$ a.s., which requires then the condition $E[\mathbf{N}(\eta(0))^3] < \infty$ in order to establish convergence. Notice that in this case it is no longer required that $\lambda/\mu_1 < 1$, but only that $\lambda < \sum_k \theta_k/\mu_k$. As before, the first statement follows from the a.s. convergence of the derivatives of the finite horizon averages to those of the stationary averages.

It suffices now to see that the limit is the correct one. In this case the RPA formula is obtained via a thinning of a Poisson process, where the phantom system now has less customers. Following the same arguments as before, we obtain the unbiased estimator for the directional derivative (8) of $F_M(\theta)$:

$$\hat{D}_k = \sum_{j=1}^{M} \left[\frac{\varphi_M(0)}{M} - \frac{\tilde{\varphi}_M(j)}{M-1} \right]$$

where $\tilde{\varphi}_{M-1}(j) = \sum_{i \neq j} W_i(j)$ does not include the waiting time of the phantom customer (since it disappeared from the queue). After some algebra, we obtain:

$$\begin{aligned} \hat{D}_k &= \sum_{j=1}^{M} \frac{[\varphi_M(0) - \tilde{\varphi}_M(j)]}{M-1} \\ &\quad - \frac{1}{M(M-1)} \sum_{j=1}^{M} \varphi_M(0) \\ &= \sum_{j=1}^{M} \frac{[\varphi_M(0) - \varphi_M(j)]}{M-1} \end{aligned}$$

where we have used $\sum_i W_i(0) = \sum_j W_j(j)$, so that the last equation uses $\varphi(j)$ considering also the waiting time of phantom customers. ■

4 Parallel Computation

Clearly, the evolution of the $\eta(j)$-phantom system can be evaluated in parallel to the nominal system using Lindley's equation. Our estimators require computation of:

$$\sum_{j=1}^{M} I_k(j) \sum_{i=1}^{M} \frac{[W_i(0) - W_i(j)]}{M}$$

where $W_i(j)$ is the waiting time of the i-th customer in the $\eta(j)$-phantom system, and $j = 0$ corresponds to the nominal system. For given j, call

$$d_i(j) = W_i(0) - W_i(j), i = 1, \ldots, M$$

the difference process (in waiting time). Since a $\eta(j)$-phantom system has only cutomer j as a phantom, then $W_i(j) = W_i(0)$ for all $i \leq j$. The first difference occurs because the service times are different: in the case of the swapping phantoms, we generate a different service speed for customer j only, and set $d_{j+1}(j)$ to the corresponding value, which may be positive or negative. In the case of the disappearing phantoms we set $d_{j+1}(j) = S_j > 0$, the service time in the nominal queue.

After initializing the difference process the subsequent values can be obtained recursively, since they form a Markov Chain, adapted to the queueing process, as follows:

- If $d_{j+1}(j) > 0$ then

$$d_{i+1}(j) = \begin{cases} d_i(j) & \text{if } W_{i+1}(0) > d_i(j) \\ W_{i+1}(0) & \text{otherwise} \end{cases}$$

- If $d_{j+1}(j) < 0$ then

$$d_{i+1}(j) = \begin{cases} d_i(j) & \text{if } W_{i+1}(0) > 0 \\ -W_{i+1}(j) & \text{otherwise} \end{cases}$$

where $W_{i+1}(j) = \max(0, (R_i(0) - A_i) - d_i(j))$ is the waiting time in the phantom queue. The above equations follow from Lindley's equations: all arrival times are equal in both the nominal and the $\eta(j)$-phantom system. If the initial difference is positive, customers wait less in the phantom system. The gain in sojourn time of customer $i+1$ is identical to that of i if this gain is not larger than $W_{i+1}(0)$, otherwise $i+1$ will start a busy period in the phantom queue, which sets the difference to $W_{i+1}(0)$. The perturbation ends when the nominal queue finishes the busy period, since then both nominal and phantom systems regenerate.

If the initial difference is negative, meaning a larger service in the phantom queue, then a nominal busy period may merge into several phantom ones. While the nominal busy period is not finished the same additional waiting time is carried from customer to customer. Suppose now that a nominal busy period finishes with customer i, but that the added wait causes the phantom service completion to take place after the arrival to the nominal empty queue, in other words, $R_i(j) > A_i$. This means that the phantom queue has not finished a busy period, but now the added wait is reduced to the quantity $R_i(j) - A_i$. From Lindley's equations, this is $R_i(0) - d_i(j) - A_i$, since for $i > j$ the service times S_i are the same in both systems. Now it is at the end of the phantom busy period that the perturbation ends.

From the recursion above it is straightforward to calculate the generalized gradient in one run. We only have to calculate the statistics for parallel phantom systems, one being created at each arrival. While for the disappearing phantoms domination helps in reducing the computational effort, in the case of the swapping phantoms we must rearrange the data structures at the start of each nominal busy period to keep only those phantom systems that are still "alive", that is, those whose perturbation has not finished. In this manner, we may have many parallel phantom systems interacting at any one moment in our program.

5 Comparing the Estimators

We compare the performance of the estimators studying several implementations. The case of the

$\lambda = 0.05$, $\rho = 0.40$				
Method	$k = 1$	$k = 2$	$k = 3$	CPU Time
Theoretical	20.22	-5.04	-5.08	–
Disap	20.18 $\pm$ 0.31	-5.03 $\pm$ 0.08	-5.06 $\pm$ 0.08	7 secs.
Swap (CRN)	20.36 $\pm$ 0.31	-5.05 $\pm$ 0.05	-5.05 $\pm$ 0.07	13 secs.
Swap (IND)	20.13 $\pm$ 0.32	-5.05 $\pm$ 0.05	-5.05 $\pm$ 0.08	11 secs.
$\lambda = 0.10$, $\rho = 0.80$				
Method	$k = 1$	$k = 2$	$k = 3$	CPU Time
Theoretical	246.55	-61.55	-61.79	–
Disap	246.1 $\pm$ 9.786	-61.47 $\pm$ 2.479	-61.60 $\pm$ 2.465	12 secs.
Swap (CRN)	246.7 $\pm$ 10.043	-61.33 $\pm$ 2.037	-61.22 $\pm$ 2.210	18 secs.
Swap (IND)	254.2 $\pm$ 12.923	-62.37 $\pm$ 2.390	-62.77 $\pm$ 2.296	20 secs.

Table 1: Comparison

swapping phantoms can of course be implemented using CRN or independent random variates for the swapping speed. If we were not simulating but rather observing the system and had no knowledge of the service distribution, an empirical rule can be applied to determine the swapping speed. We then compare the efficiency with the disppearing phantoms, which pose no such problems: whether observing or simulating a system, the corresponding phantom system can be uniquely reproduced even if we have no knowledge of the underlying distributions.

The numerical results were obtained via simulations of the system for $V = 3$ speeds. The arrival rate is $\lambda = 0.05$ and 0.10 and all service distributions are uniform between (a_k, b_k) as given in Table 2.

Speed k	Uniform(a_k, b_k)		θ_k
1	1	38	0.2
2	3	7	0.5
3	4	6	0.3

Table 2: Data for Simulations

The results are in Table 1, where the Disappearing Phantoms Method is compared to the Swapping Phantoms using common random numbers for the phantom services (CRN) and using independent service times (IND).

In addition to being model independent, the disappearing phantom method has the advantage of requiring less CPU time. This reduction is due to the computational burden added by the swaping phantom customers that have higher service times, creating longer busy periods than in the nominal system. The disappearing phantoms are always dominated and calculations can be made much faster. As for the variance, CRN have less variance than the independent swapping phantoms, as expected, while the disappearing phantoms may have variance reduction for some components of the generalized gradient only. The gain in variance reduction may be dependent on the component as well as the parameter values, as usual with indirect estimation.

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