

The Case of the Swapping and of the Disappearing Phantoms

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1.1 The Multi-Valued Decision Process

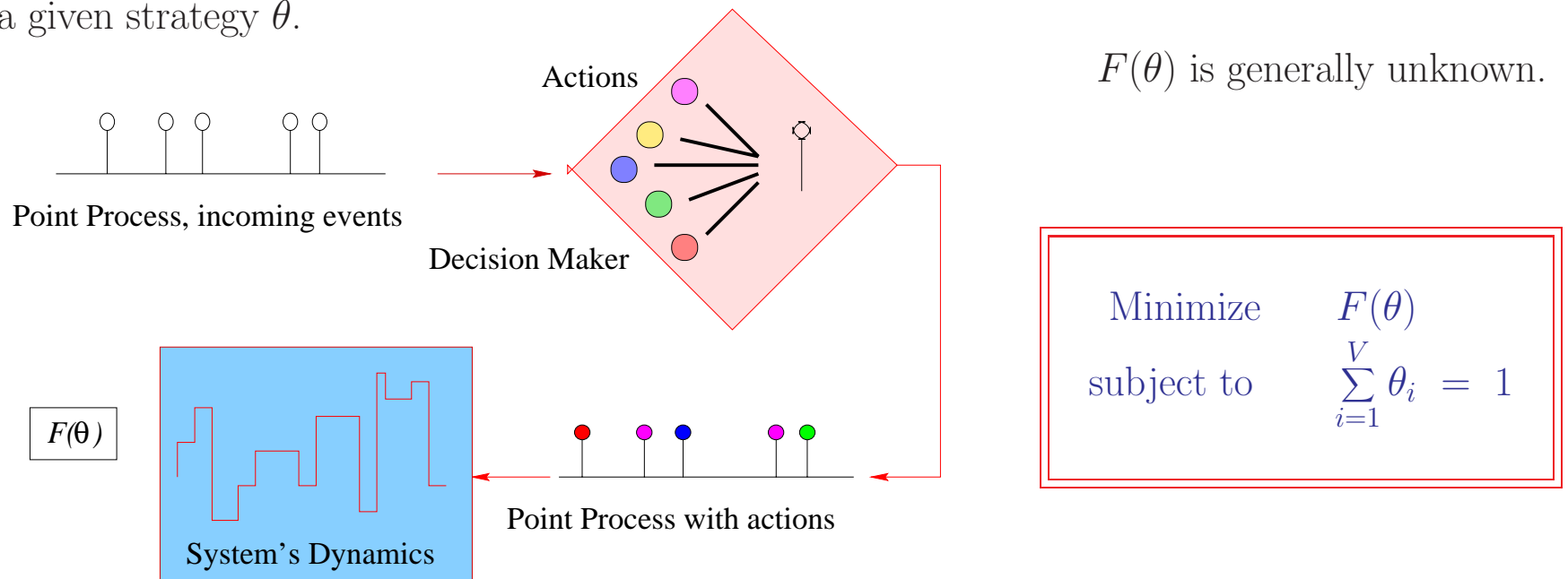
Model: *Decision events* occur according to a Point process $N(t)$, with rate λ . One amongst V actions must be taken upon arrival of such an event.

Control Strategy: The decision strategy prescribes *randomized* independent decisions:

θ_i : Probability that action i will be taken upon arrival of the n -th event.

Decision Space : $\Theta = \left\{ \theta \in \mathbb{R}^V \text{ such that } \theta_i \geq 0, i = 1, \dots, V, \text{ and } \sum_{i=1}^V \theta_i = 1 \right\}.$

Long Term Cost per Unit Time: $F(\theta)$ is the long term average cost rate associated with a given strategy θ .



We call $\theta^* = \arg \min_{\theta \in \Theta} F(\theta)$ the optimal value (which may not be unique).

1.2 Optimality of the Probability Vector θ

Definition: The *Generalized Gradient Operator* is defined by:

$$\mathcal{G}_k[F(\cdot)] = \frac{\partial F(\cdot)}{\partial \theta_k} - \sum_{j=1}^V \theta_j \frac{\partial F(\cdot)}{\partial \theta_j}$$

A value of $\bar{\theta}$ such that $\sum_{k=1}^V \bar{\theta}_k = 1$ and

$$\mathcal{G}_k[F(\bar{\theta})] = \begin{cases} = 0 & \text{if } \bar{\theta}_k > 0 \\ \geq 0 & \text{if } \bar{\theta}_k = 0 \end{cases} \text{ for } k = 1, \dots, V \quad (1)$$

satisfies the Kuhn-Tucker conditions. Under convexity of $F(\theta)$, if $\bar{\theta}$ satisfies (1), then $\bar{\theta} = \theta^*$.

Lemma 1 Consider the ODE:

$$\frac{d\vartheta_k(t)}{dt} = -\vartheta_k(t) \mathcal{G}_k[F(\vartheta(t))] \quad (2)$$

If the starting point is a probability vector: $\vartheta_k(0) \in \Theta$, then $\vartheta(t)$ remains always as a probability vector. Furthermore, if $\vartheta(0) \in \Theta$ is such that $\vartheta_k(0) > 0$ for each $k = 1, \dots, V$ and it is not a local maximum of $F(\cdot)$, then the stable points of (2) are local minima.

If the function F has a unique minimum, then (2) has an asymptote at $\theta^* = \arg \min_{\theta \in \Theta} F(\theta)$.

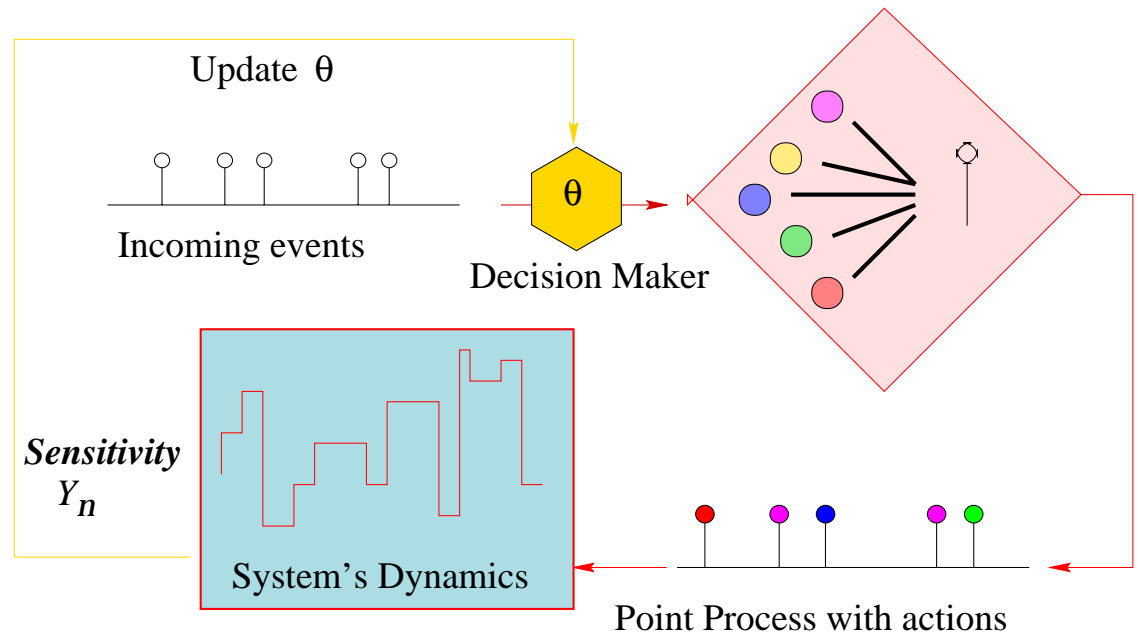
1.3 Estimation Problem: Formulation

Approximating the solution to the ODE: *stochastic approximation*

$$\theta_k(n + 1) = \theta_k(n) - \epsilon_n \theta_k(n) Y_n, \quad Y_n \approx \mathcal{G}_k[F(\vartheta(t))]$$

The *sensitivity estimator* Y_n is a noisy measurement of the desired *sensitivity*.

For the stochastic approximation to be asymptotically optimal, we need to *estimate the generalized gradient*.



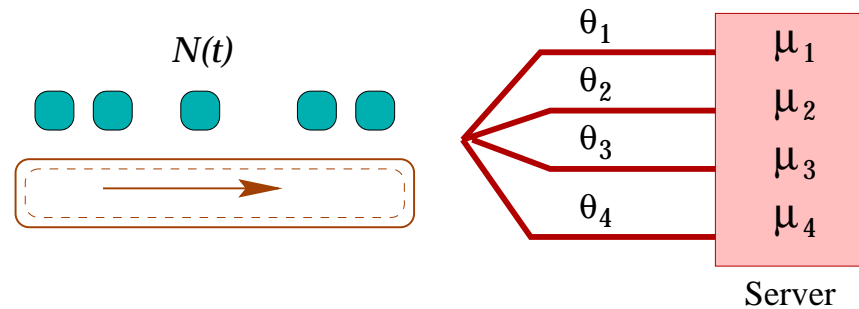
How to estimate the sensitivities?

IPA: not applicable, SF: generally large variances.

Extension of the Phantom RPA method to multivalued decisions.

2.1 A Queueing Example

- Items arrive at a queue in front of a machine, where they wait to be processed according to a FCFS rule. The (stationary) *arrival process* $N(t)$ has rate λ .
- A machine can process items at V different speed settings. We call $\eta_i \in \{v_1, \dots, v_V\}$ the particular *decision* or speed chosen for the i -th processed item.
- The distribution of the random processing times is conditional to the chosen speed, with $\mu_k^{-1} = E[S_i | \eta_i = v_k]$. We order the settings: $\mu_k \leq \mu_j$ if $k < j$ (v_1 is the fastest in average).
- The *operating cost* assumes that it costs c_k dollars per unit of processing time when the machine works at speed v_k , and $c_k < c_j$ if $k < j$, (faster modes more costly, no switching cost).



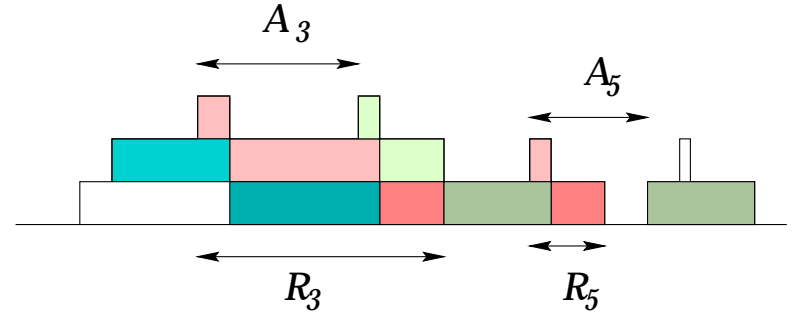
$$\text{Minimize } F(\theta) = W(\theta) + \lambda \sum_{k=1}^V \frac{c_k \theta_k}{\mu_k}, \text{ subject to } \sum_{i=1}^V \theta_i = 1.$$

\implies Estimation of $\mathcal{G}_k[W(\theta)]$.

2.2 Parallel Δ -Phantom Systems

Lindley's recursions, for $i \geq 0$, with $W_1 = 0$:

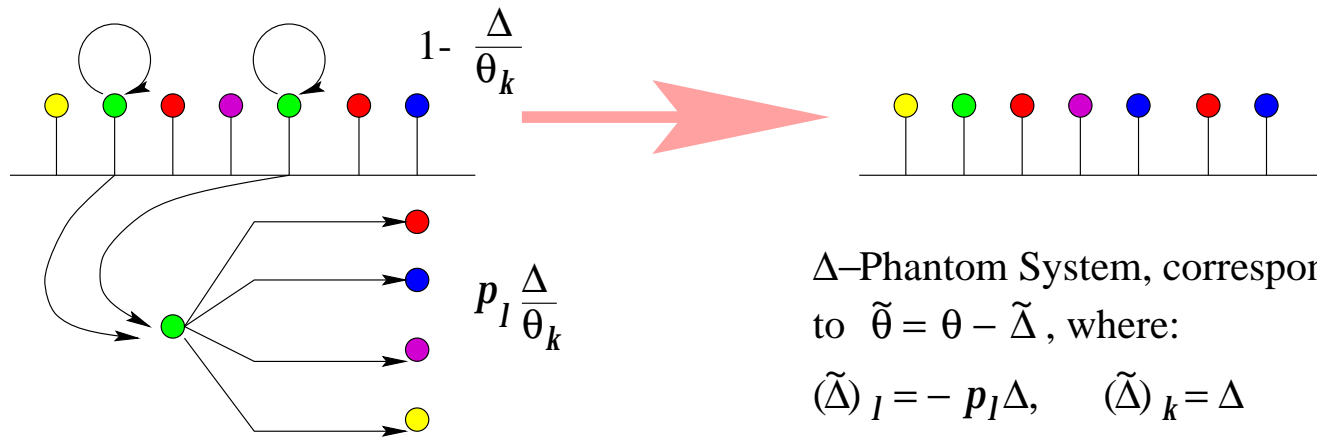
$$\begin{aligned} R_{i+1} &= S_{i+1}(u_{i+1}, \eta_{i+1}) + W_{i+1} = \\ &= \sum_{k=1}^V G_k^{-1}(u_{i+1}) \mathbf{1}_{\{\eta_{i+1}=v_k\}} + W_{i+1} \\ W_{i+1} &= \max(0, R_i - A_i) \end{aligned}$$



Definition: Given a *trajectory* (A_i, u_i, η_i) , let $\tilde{\eta}$ be a sequence of *phantom decisions* s.t.

$$\begin{aligned} P\{\tilde{\eta}_i = v_l | \eta_i(0) = v_l\} &= 1, \quad l \neq k \\ P\{\tilde{\eta}_i = v_k | \eta_i(0) = v_k\} &= 1 - \frac{\Delta}{\theta_k}, \quad P\{\tilde{\eta}_i = v_l | \eta_i(0) = v_k\} = \frac{p_l \Delta}{\theta_k}, \quad l \neq k \end{aligned}$$

where $p_l \geq 0$ are to be determined later. The Δ -*phantom system* is defined by the queueing system that follows the trajectory driven by the sequence $(A_i, u_i, \tilde{\eta}_i)$.



Δ -Phantom System, corresponds to $\tilde{\theta} = \theta - \tilde{\Delta}$, where:

$$(\tilde{\Delta})_l = -p_l \Delta, \quad (\tilde{\Delta})_k = \Delta$$

2.3 The General RPA Formula

Finite Horizon: Let $\varphi_M(\eta) = \sum_{i=1}^M W_i$ be the total wait of the first M customers.

Finite Difference in θ : Consider now the expected value of the *finite difference*:

$$D_\Delta(M) = \frac{\varphi_M(\eta(0)) - \varphi_M(\tilde{\eta})}{\Delta}, \text{ where } \tilde{\eta} \text{ yields the } \Delta\text{-Phantom system.}$$

Given M customers, the number of phantom decisions in $\{\tilde{\eta}_i, i = 1, \dots, M\}$ has a **Binomial distribution**, since each decision is chosen as a phantom one with probability $P\{\tilde{\eta}_i \neq \eta_i(0)\} = \Delta$, independently of previous ones. Therefore,

$$E[D_\Delta(M)] = E\left[M\Delta(1-\Delta)^{M-1}E^{(1)}[D_\Delta(M)]\right] + E\left[\sum_{m=2}^M \binom{M}{m} \Delta^m(1-\Delta)^{M-m}E^{(m)}[D_\Delta(M)]\right]$$

where $E^{(m)}$ is the expectation w.r.t. $\tilde{\eta}$, conditioning on having exactly m phantoms.

The RPA Limit: The finite horizon phantom RPA formula is:

$$\lim_{\Delta \rightarrow 0} E[D_\Delta(M)] = \frac{1}{\theta_k} E\left\{ \sum_{j=1}^M I_k(j) \left[\frac{\varphi_M(\eta(0)) - \varphi_M(\eta(j))}{M} \right] \right\}$$

where $I_k(j) = \mathbf{1}_{\{\eta_j(0)=v_k\}}$. The sequence $\eta(j)$ is defined by $\eta_i(j) = \eta_i(0), i \neq j$, and:

$$P\{\eta_j(j) = v_l | \eta_j(0) = v_l\} = 1, l \neq k, P\{\eta_j(j) = v_l | \eta_j(0) = v_k\} = p_l.$$

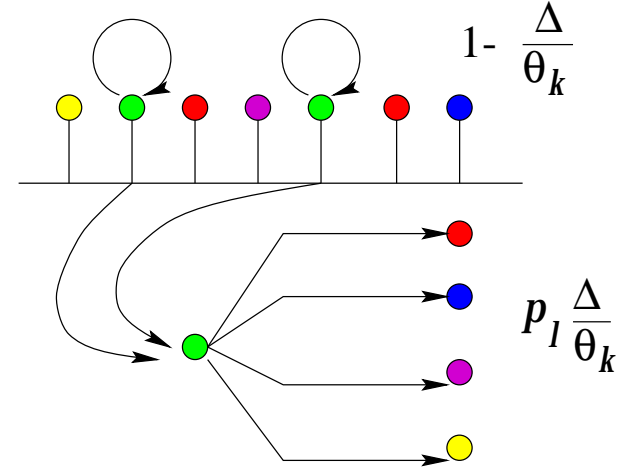
Use equiprobability of all the sequences of phantom decisions with only one distinct from $\eta(0)$.

3.1 The Swapping Phantoms: Main Result

Chosen phantom decisions are swapped:

$$p_l = -\frac{\theta_l}{(1 - \theta_k)}, l \neq k$$

The ensuing phantom customers in the $\tilde{\eta}$ -phantom system *swap speeds* compared to the corresponding nominal decisions, choosing one of the remaining ones according to their *original proportion*.



Proposition 1 Assume that the service times are dominated by $\{S_i(\bar{\eta})\}$, for some random sequence $\bar{\eta}$, and that the dominating queueing process satisfies $E[\mathbf{N}^3(\bar{\eta})] < \infty$. Then:

$$\frac{\theta_k}{(1 - \theta_k)} \mathcal{G}_k[W(\theta)] = \lim_{M \rightarrow \infty} E \left\{ \sum_{j=1}^M I_k(j) \frac{[\varphi_M(\eta(0)) - \varphi_M(\eta(j))]}{M} \right\}$$

where $I_k(j) = \mathbf{1}_{\{\eta_j(0)=v_k\}}$. The sequence $\eta(j)$ is defined by $\eta_i(j) = \eta_i(0), i \neq j$, and:

$$P\{\eta_j(j) = v_l | \eta_j(0) = v_l\} = 1, l \neq k, P\{\eta_j(j) = v_l | \eta_j(0) = v_k\} = p_l.$$

3.2 The Swapping Phantoms: Sketch of Proof

First Step: Using this definition for the Δ -Phantom system, simple algebraic manipulations show that

$$\lim_{\Delta \rightarrow 0} \frac{F(\theta) - F(\tilde{\theta})}{\Delta} = \frac{1}{1 - \theta_k} \mathcal{G}_k[F(\theta)]$$

In order to see this, set $V = 3, k = 2$:

$$F(\theta) - F(\tilde{\theta}) = F(\theta_1, \theta_2, \theta_3) - F(\theta_1, \theta_2, \tilde{\theta}_3) + F(\theta_1, \theta_2, \tilde{\theta}_3) - F(\theta_1, \tilde{\theta}_2, \tilde{\theta}_3) + F(\theta_1, \tilde{\theta}_2, \tilde{\theta}_3) - F(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$$

Now use $\delta\theta_i = \tilde{\theta}_i - \theta_i = \Delta\theta_i/(1 - \theta_k), i \neq k$ to rewrite:

$$\lim_{\Delta \rightarrow 0} \frac{F(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3) - F(\theta_1, \tilde{\theta}_2, \tilde{\theta}_3)}{\Delta} = \frac{\theta_1}{(1 - \theta_k)} \lim_{\delta\theta_1 \rightarrow 0} \frac{F(\theta_1 + \delta\theta_1, \tilde{\theta}_2, \tilde{\theta}_3) - F(\theta_1, \tilde{\theta}_2, \tilde{\theta}_3)}{\delta\theta_1} = \frac{\theta_1}{(1 - \theta_k)} \frac{\partial F(\theta)}{\partial \theta_1}$$

and similarly for $i = 3$. For $k = 2$ we have:

$$\lim_{\Delta \rightarrow 0} \frac{F(\theta_1, \theta_2, \tilde{\theta}_3) - F(\theta_1, \tilde{\theta}_2, \tilde{\theta}_3)}{\Delta} = \frac{\partial F(\theta)}{\partial \theta_k} = \frac{1}{(1 - \theta_k)} \left[\frac{\partial F(\theta)}{\partial \theta_k} - \theta_k \frac{\partial F(\theta)}{\partial \theta_k} \right]$$

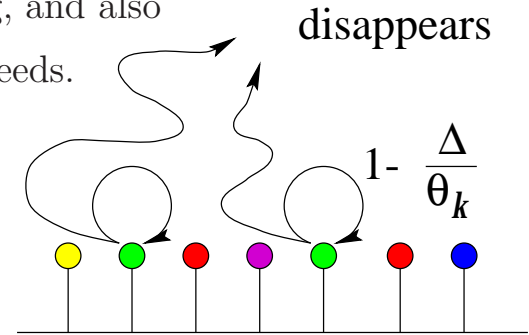
which establishes our claim.

Second Step: Use Dominated Convergence to interchange the limits of the finite differences (RPA formula) and the finite horizon approximation to the stationary average.

3.3 The Disappearing Phantoms: Main Result

- Swapping requires *domination*: system stability under slowest setting, and also
- requires *knowledge of the service distributions*, in order to “swap” speeds.

Uniformization argument: customers of class k arrive at rate $\theta_k \lambda$. Service distribution is class dependent. Use now $p_l = 0, l \neq k$, meaning that some of the customers, the phantom ones, are not allowed entrance to the machine and thus *disappear* from the system.



$$\text{Call: } \mathcal{D}_k(\lambda, \theta) = \frac{\partial F(\lambda, \theta)}{\partial \theta_k} + \lambda \frac{\partial F(\lambda, \theta)}{\partial \lambda} - \sum_{j=1}^V \theta_j \frac{\partial F(\lambda, \theta)}{\partial \theta_j}$$

so that:

$$\mathcal{G}_k[F(\lambda, \theta)] = \mathcal{D}_k[F(\lambda, \theta)] - \sum_{i=1}^V \theta_i \mathcal{D}_i[F(\lambda, \theta)]$$

Proposition 2 Assume that $E[\mathbf{N}(0)^3] < \infty$. Then:

$$\theta_k \mathcal{D}_k[W(\theta)] = \lim_{M \rightarrow \infty} E \left\{ \sum_{j=1}^M I_k(j) \frac{[\varphi_M(\eta(0)) - \varphi_M(\eta(j))]}{M} \right\}$$

where $I_k(j) = \mathbf{1}_{\{\eta_j(0)=v_k\}}$. The sequence $\eta(j)$ is defined by $\eta_i(j) = \eta_i(0), i \neq j$, but now $\eta_j(j) = \emptyset$, meaning that its service time will be set to zero in the phantom queue.

3.4 The Disappearing Phantoms: Sketch of Proof

First Step: Since the rate $\lambda_k = \theta_k \lambda$, then $\tilde{\lambda} = \lambda - \Delta \lambda$ and consequently, the corresponding system satisfies:

$$\tilde{\theta}_i = \frac{\lambda \theta_i}{\lambda - \Delta \lambda} = \theta_i + \delta_i, \tilde{\theta}_k = \frac{\lambda_k - \Delta \theta_k \lambda}{\lambda - \Delta \lambda} = \theta_k - \delta_k, \text{ where: } \delta_i = \begin{cases} \theta_i \Delta + \mathcal{O}(\Delta^2) & \text{if } i \neq k \\ (1 - \theta_k) \Delta + \mathcal{O}(\Delta^2) & i = k \end{cases}$$

Using $\delta \lambda = \Delta \theta_k \lambda$, we then have:

$$\lim_{\Delta \rightarrow 0} \frac{F(\lambda, \theta) - F(\tilde{\lambda}, \tilde{\theta})}{\Delta} = \lambda \frac{\partial}{\partial \lambda} F(\lambda, \theta) + \lim_{\Delta \rightarrow 0} \frac{F(\lambda, \theta) - F(\lambda, \tilde{\theta})}{\Delta}$$

Proceed as before to write down:

$$\lim_{\Delta \rightarrow 0} \frac{F(\lambda, \theta) - F(\lambda, \tilde{\theta})}{\Delta} = - \sum_{i \neq k} \theta_i \frac{\partial F(\lambda, \theta)}{\partial \theta_i} + (1 - \theta_k) \frac{\partial F(\lambda, \theta)}{\partial \theta_k} = \frac{\partial F(\lambda, \theta)}{\partial \theta_k} - \sum_{i=1}^V \theta_i \frac{\partial F(\lambda, \theta)}{\partial \theta_i}$$

therefore:

$$\lim_{\Delta \rightarrow 0} \frac{F(\lambda, \theta) - F(\tilde{\lambda}, \tilde{\theta})}{\Delta} = \frac{\partial F(\lambda, \theta)}{\partial \theta_k} + \lambda \frac{\partial F(\lambda, \theta)}{\partial \lambda} - \sum_{j=1}^V \theta_j \frac{\partial F(\lambda, \theta)}{\partial \theta_j} = \mathcal{D}_k(\lambda, \theta)$$

Second Step: Use Dominated Convergence to interchange the limits of the finite differences (RPA formula) and the finite horizon approximation to the stationary average.

4.1 Implementation: Parallel Computation

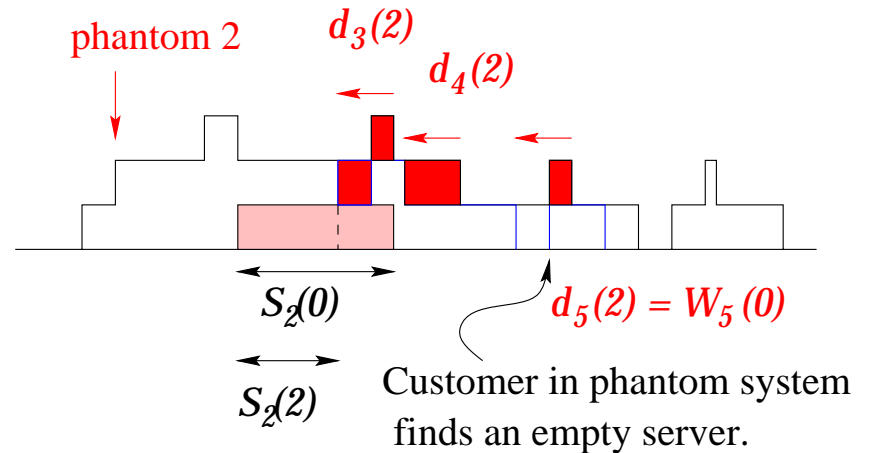
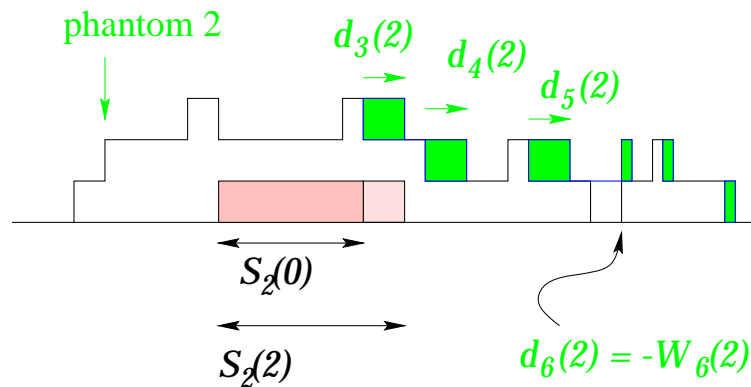
Our estimators require computation of:

$$\sum_{j=1}^M I_k(j) \frac{1}{M} \sum_{i=1}^M d_i(j), \text{ where } d_i(j) \equiv W_i(0) - W_i(j)$$

$W_i(j)$ is the waiting time of the i -th customer in the j -phantom system. We simulate only *one path* of the nominal system. To each customer we associate a phantom system.

If $d_{j+1}(j) > 0$ then

$$d_{i+1}(j) = \begin{cases} d_i(j) & \text{if } W_{i+1}(0) > d_i(j) \\ W_{i+1}(0) & \text{otherwise} \end{cases}$$



If $d_{j+1}(j) < 0$ then

$$d_{i+1}(j) = \begin{cases} d_i(j) & \text{if } W_{i+1}(0) > 0 \\ -W_{i+1}(j) & \text{otherwise} \end{cases}$$

$$W_{i+1}(j) = [(R_i(0) - A_i) - d_i(j)]^+.$$

4.2 Implementation: Simulation Results

The numerical results were obtained via simulations of the system for $V = 3$ speeds. The arrival rate is $\lambda = 0.05$ and 0.10 and all service distributions are uniform between (a_k, b_k) as given in:

Speed k	Uniform(a_k, b_k)		θ_k
1	1	38	0.2
2	3	7	0.5
3	4	6	0.3

The results for two different utilization factors are given below.

$\lambda = 0.05, \rho = 0.40$				
Method	$k = 1$	$k = 2$	$k = 3$	CPU Time
Theoretical	20.22	-5.04	-5.08	-
Disap	20.18 ± 0.31	-5.03 ± 0.08	-5.06 ± 0.08	7 secs.
Swap (CRN)	20.36 ± 0.31	-5.05 ± 0.05	-5.05 ± 0.07	13 secs.
Swap (IND)	20.13 ± 0.32	-5.05 ± 0.05	-5.05 ± 0.08	11 secs.

$\lambda = 0.10, \rho = 0.80$				
Method	$k = 1$	$k = 2$	$k = 3$	CPU Time
Theoretical	246.55	-61.55	-61.79	-
Disap	246.1 ± 9.786	-61.47 ± 2.479	-61.60 ± 2.465	12 secs.
Swap (CRN)	246.7 ± 10.043	-61.33 ± 2.037	-61.22 ± 2.210	18 secs.
Swap (IND)	254.2 ± 12.923	-62.37 ± 2.390	-62.77 ± 2.296	20 secs.