Tensor Factorization via Matrix Factorization

(Kuleshov et al., 2015)

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Outline

1. Introduction
2. Tensor Factorization via Matrix Factorization(TFMF)
3. Simultaneous diagonalization
4. Experiments
5. Conclusion

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Introduction

Given a tensor $\mathcal{T} \in \mathbb{R}^{d \times d \times d}$ with the following CP-decomposition:

$$\hat{\mathcal{T}} = \sum_{i=1}^{k} \pi_i \mathbf{a_i} \otimes \mathbf{b_i} \otimes \mathbf{c_i} + \text{noise},$$

our goal is to estimate the factors $\mathbf{a_i}, \mathbf{b_i}, \mathbf{c_i}$ and the factor weights $\pi \in \mathbb{R}^k$.

- To solve this, we saw ALS, gradient-based approaches in class
- **This presentation** $\Rightarrow$ Tensor Factorization via Matrix Factorization (TFMF)

The core idea

Projection $\mathcal{T}$ along a vector w to do eigendecomp. repeated L times

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Tensor Factorization via Matrix Factorization(TFMF)

TFMF algorithm overview

1. **Input** : L random vectors w, a tensor $\mathcal{T}$
2. Project $\mathcal{T}$ onto a set of random vectors w_L producing $\mathcal{M}$ matrices
3. Simultaneously diagonalize $\mathcal{M}$ producing CP decomp. factors estimates $\tilde{u}_{\mathcal{I}}$
4. Refine by repeating with the factor estimates instead of the random vectors
5. **Output** : CP factor matrices $\tilde{u}_{\mathcal{I}}$

- **Application**: to orthogonal, non-orthogonal and asymmetric tensors of arbitrary order.
- **Novelty**: Simultaneous matrix diagonalization.

Factors u ?

- When $a_i = b_i = c_i = u_i \; \forall i \Rightarrow$ symmetric factorization ! We have :

$$\sum_i \pi_i u_i^{\otimes 3} \tag{1}$$

- Project along a vector w !

$$\sum_i \pi_i (w^T u_i) u_i^{\otimes 2} \tag{2}$$

- Estimate u_i by eigendecomposition of Eq. 2 $\Rightarrow \tilde{u}_i$
- The error $||u_i - \tilde{u}_i||_2$ is *noise sensitive*
- Sensitivity $\approx$ smallest diff. between eigenvalues : the *eigengap*

$$\max_{j \neq i} \frac{1}{|\lambda_i - \lambda_j|}$$

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Solving the *eigengap* with multiple projections (orth. case)

- Using *L random* projections we have the matrices M_ℓ

$$\sum_i \pi_i (w_\ell^T u_i) u_i^{\otimes 2} \tag{3}$$

- The set of matrices M_ℓ has common eigenvectors $\Rightarrow$ simultaneous diagonalization !
- The error bound then follows

$$||\tilde{u}_i - u_i||_2 \leq \left(\frac{2\sqrt{2||\pi||_1 \pi_{max}}}{\pi_i^2} + \frac{C(\delta)}{\pi_i}\right)\epsilon + o(\epsilon) \tag{4}$$

with $C(\delta) = \mathcal{O}(\log(kd/\delta\sqrt{\frac{d}{L}})$

$\Rightarrow$ The bigger L the lower the error bound !

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- After a first pass w/ random W the paper proposes to use $\tilde{u}$ as the projection
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Simultaneous diagonalization

- Symmetric matrices $\mathbf{M_1}, \cdots, \mathbf{M_L} \in \mathbb{R}^{d \times d}$ as:

$$\mathbf{M_l} = \mathbf{U}\Lambda_l\mathbf{U}^T + \epsilon\mathbf{R_l}.$$

- $U \in \mathbb{R}^{d \times k}$ is common, $\Lambda_l \in \mathbb{R}^{k \times k}$ and ϵR_l are individual.

- **Goal**: find inverse factors $\mathbf{V^{-1}} \in \mathbb{R}^{d \times d}$ such that $\mathbf{V^{-1}M_lV^{-T}}$ is nearly diagonal.
- Optimizing objective function to find $\mathbf{V}$:

$$F(\mathbf{X}) \triangleq \sum_{l=1}^{L} \text{off}(\mathbf{X^{-1}M_lX^{-T}}), \qquad \text{off}(\mathbf{A}) = \sum_{i \neq j} \mathbf{A}_{ij}^2.$$

  $\Rightarrow$ this penalizes the off-diagonal terms!

- Use Jacobi & QRJ1D

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For asymmetric and higher-order tensors

Asymmetric tensors:

- The l-th projection($\mathbf{M_l}$) of an asymmetric tensor has the following form:

$$\mathbf{M_l} = \sum_i \lambda_i \mathbf{u_{il}} \mathbf{v_{il}}^T = \mathbf{U} \Lambda_l \mathbf{V}^T,$$

  where Λ_l is diagonal but not necessarily positive matrix, and $\mathbf{U}, \mathbf{V}$ are common but not necessarily orthogonal.

- for each $\mathbf{M_l}$ we define another matrix $\mathbf{N_l}$ as:

$$\mathbf{N_l} = \begin{bmatrix} 0 & \mathbf{M_l}^T \\ \mathbf{M_l} & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{V} & \mathbf{V} \\ \mathbf{U} & -\mathbf{U} \end{bmatrix} \begin{bmatrix} \Lambda_l & 0 \\ 0 & -\Lambda_l \end{bmatrix} \begin{bmatrix} \mathbf{V} & \mathbf{V} \\ \mathbf{U} & -\mathbf{U} \end{bmatrix}^T.$$

- The $\mathbf{N_l}$ are symmetric matrices with common (in general, non-orthogonal) factors.

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Higher order tensors:

- For higher order tensor (say fourth order):

$$\mathcal{T} = \sum_i \pi_i \mathbf{a_i} \otimes \mathbf{b_i} \otimes \mathbf{c_i} \otimes \mathbf{d_i},$$

- We first determine $\mathbf{a_i}, \mathbf{b_i}$ by projecting into matrices:

$$\mathbf{T} = \sum_i \pi (\omega^T \mathbf{c_i})(u^T \mathbf{d_i}) \mathbf{a_i} \otimes \mathbf{b_i},$$

- Then determine $\mathbf{c_i}, \mathbf{d_i}$ by projecting along the first two components.

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Convergence properties

- Convergence depends on the choice of joint diagonalization subroutine.
- Theoretically:
  - Convergence to Local minimum at a quadratic rate guaranteed.
  - Convergence to Global minimum is an open question!
- Empirically, convergence to global minima achieved.

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Experiments

- Examining convergence to global minima in orthogonal setting (Jacobi Algo.): $\Rightarrow$ Using 1000 random starting points, getting the same solution!

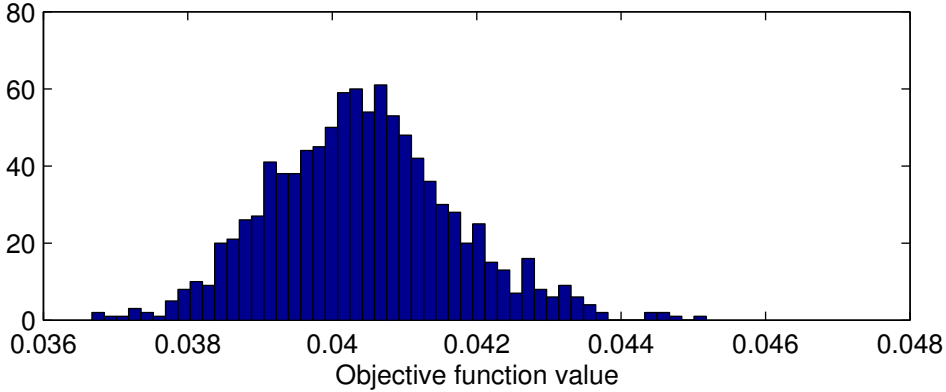

Figure 1: Histogram of objective function values, in orthogonal setting

Experiments

- Plotting histogram for different ϵ values ($\epsilon = 0, \epsilon = 1e-4, \epsilon = 1e-3$)

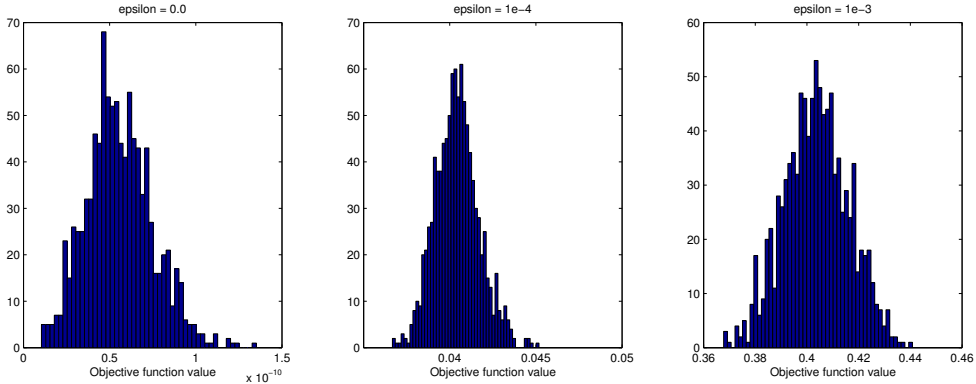

Figure 2: Comparing Histograms for different ϵ sizes, in orthogonal setting

- For small enough ϵ convergence is guaranteed.

Experiments

- Examning convergence to global minimum in Non-orthogonal setting:

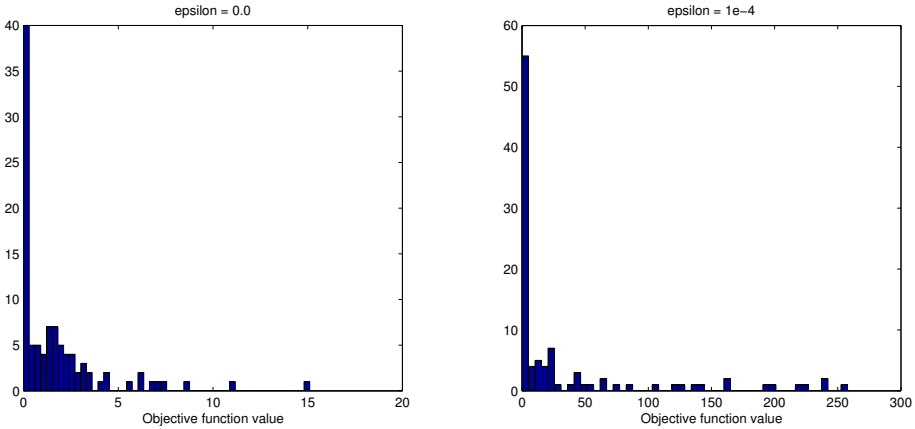

Figure 3: Histograms when μ is big

Experiments

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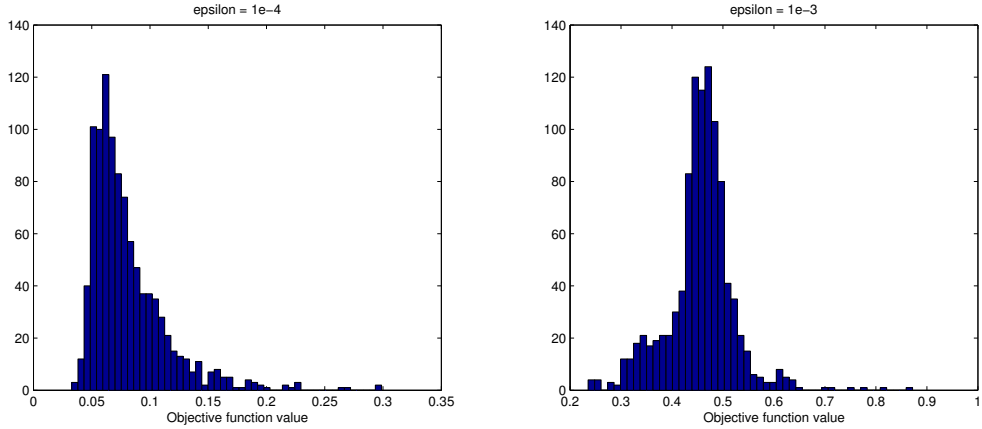
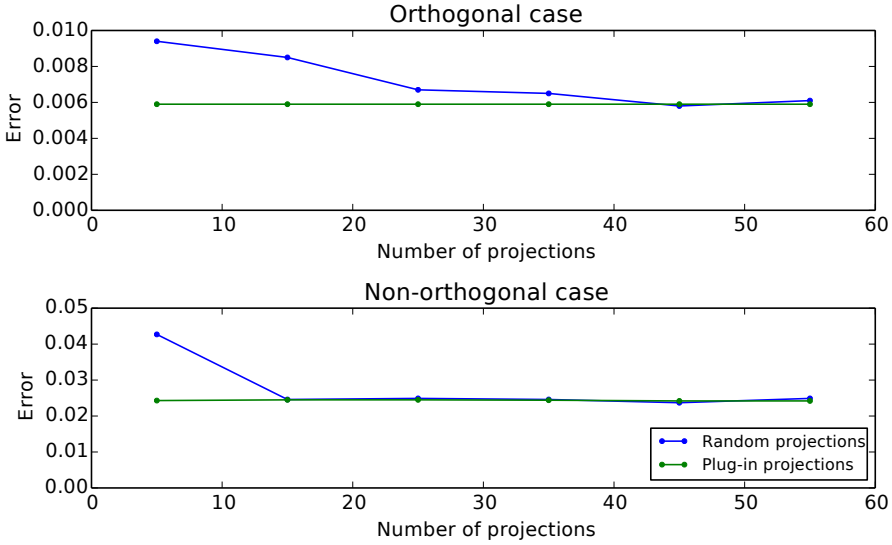


Figure 4: Histogram when μ is small

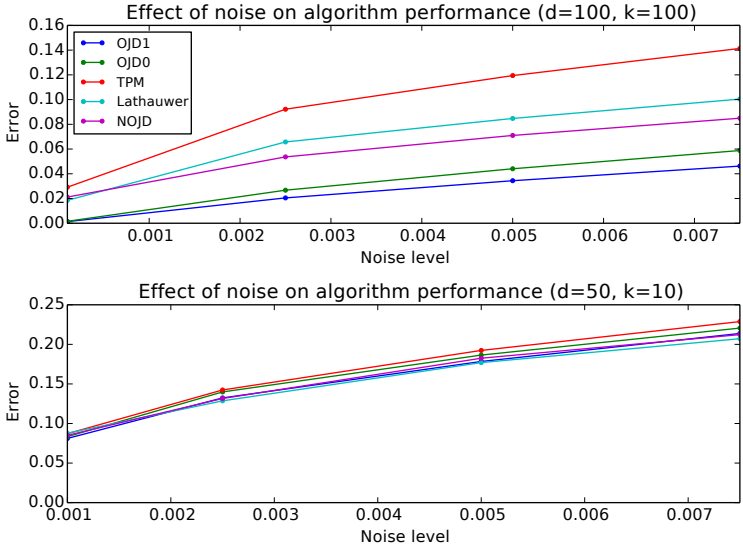
Experiments

- Comparing random vs. plugin projection



Experiments

- Performance comparison:



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Conclusion

- TFMF, another take on CP decomposition
- TFMF = random projections + simultaneous diagonalization + plugin estimates
- Works for orthogonal, non-orthogonal, symmetric, asymetric, high order tensors
- Is more accurate that state-of-the-art.