

MEASURE THEORY (in the smallest nutshell you can find)

Objective: Show that if the entries of a matrix $A \in \mathbb{R}^{m \times n}$ are drawn independently from $\mathcal{U}(0,1)$. Then, with probability one (almost surely):

- A will be full rank.
- All the eigenvalues of A will be distinct.
- A is diagonalizable.

If $A \in \mathbb{R}^{m \times R}$ and $B \in \mathbb{R}^{R \times n}$ with $R \leq \min(m, n)$ with entries drawn IID from $\mathcal{U}(0,1)$, then w/ probability 1 $\text{rank}(AB) = R$

Rank: + For tensors things are more complex, e.g., the typical rank depends on the dimension (see Kolda & Bader, TABLE 3.3)

Lebesgue Measure over \mathbb{R} : extension of the notion of length of an interval.

The Lebesgue measure μ satisfies:

- For every interval I , $\mu(I) = \text{length}(I)$

$$\text{ex: } \mu([0, 2]) = 2 - 0 = 2$$

$$\mu(]0, 2]) = \mu(]0, 2[) = 2$$

- If $A \subset B \subset \mathbb{R}$, then $0 \leq \mu(A) \leq \mu(B) \leq \infty$
- If $\{S_n\}$ is a countable collection of subsets of \mathbb{R} , we have

$$\mu\left(\bigcup_{n \in \mathbb{N}} S_n\right) \leq \sum_{n \in \mathbb{N}} \mu(S_n)$$

$$\text{ex: } \text{If } a \in \mathbb{R}, \mu(\{a\}) = 0$$

Prop: If $A \subset \mathbb{R}$ is a countable set, then $\mu(A) = 0$

Proof: A is countable $\Rightarrow A = \bigcup_{n \in \mathbb{N}} \{a_n\}$ where each $a_n \in \mathbb{R}$

$$\text{Then } \mu(A) = \mu\left(\bigcup_{n \in \mathbb{N}} \{a_n\}\right) \leq \sum_{n \in \mathbb{N}} \mu(\{a_n\}) = 0$$

What does this imply? $\rightarrow \mu(\mathbb{N}) = 0 \quad \mu(\mathbb{Q}) = 0$

Def: A distribution P over \mathbb{R} is (absolutely) continuous distribution (wrt the Lebesgue measure) if $\mu(S) = 0 \Rightarrow P(x \in S) = 0$

ex: $\mathcal{U}(0,1)$ and $\mathcal{U}([0,1])$ are continuous

$P: x \mapsto \begin{cases} 1 & \text{if } x = \pi \\ 0 & \text{otherwise} \end{cases}$ is not continuous

\hookrightarrow If $x \sim \mathcal{U}(0,1)$, $P(x \in \mathbb{Q}) = 0$

Def: We say a property holds almost everywhere if it holds for all elements in a set except a subset of measure 0.

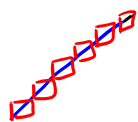
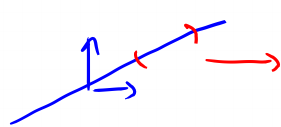
Ex: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: x \mapsto \begin{cases} 0 & \text{if } x \in \mathbb{N} \\ f(x) & \text{otherwise} \end{cases}$. Then, f and g are equal almost everywhere. (Since $\mu(\{x \mid f(x) \neq g(x)\}) = \mu(\mathbb{N}) = 0$)

• If $f: [a, b] \rightarrow \mathbb{R}$ is monotonic then it is differentiable almost everywhere.

→ The Lebesgue measure can be naturally extended to \mathbb{R}^n (use hypercubes instead of intervals)

Prop: Let $L \subset \mathbb{R}^2$ be a line, then $\mu(L) = 0$.

Proof:



→ Any segment of the line can be covered by n squares of area $\frac{1}{n^2}$

Prop: Any k -dimensional subspace of \mathbb{R}^n , with $k < n$, has Lebesgue measure 0.

Thm (See [CARON & TRAYNOR, 2005])

A polynomial function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is either identically 0 (i.e. $f(x) = 0$) or non-zero almost everywhere.

↳ This implies that if $f \neq 0$ is a polynomial, then $\mathbb{P}(f(x) = 0) = 0$ for any continuous distribution \mathbb{P} .

Corollary: Let $A \in \mathbb{R}^{m \times m}$ be a random matrix with entries drawn iid from $\mathcal{U}(0,1)$, then $\mathbb{P}(\text{rank}(A) = m) = 1$.

Proof: • $\text{rank}(A) = m \Leftrightarrow A$ is invertible $\Leftrightarrow \det(A) \neq 0$
• determinant is a polynomial of the entries of A and it is not the zero function.

$$\Rightarrow \mathbb{P}(\text{rank}(A) = m) = \mathbb{P}(\det(A) \neq 0) = 1$$

↑
polynomial not 0 \Rightarrow det is non zero almost everywhere.

Corollary: Let $T \in \mathbb{R}^{d \times d \times d}$ be a random tensor whose entries are drawn iid from $\mathcal{P}(0,1)$, then with probability one, the multilinear rank of T is (d, d, d) .

Proof: (sketch)

$$T_{ijk} \underset{\substack{\uparrow \\ \text{iid}}}{\sim} \mathcal{P}(0,1)$$



\cong

$$(T_{(1)})_{i,j} \underset{\substack{\uparrow \\ \text{iid}}}{\sim} \mathcal{P}(0,1)$$

\times

