Learning and Planning in POMDP with Weighted Automata

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Theoretically Inclined Machine Learning Seminar

Outline

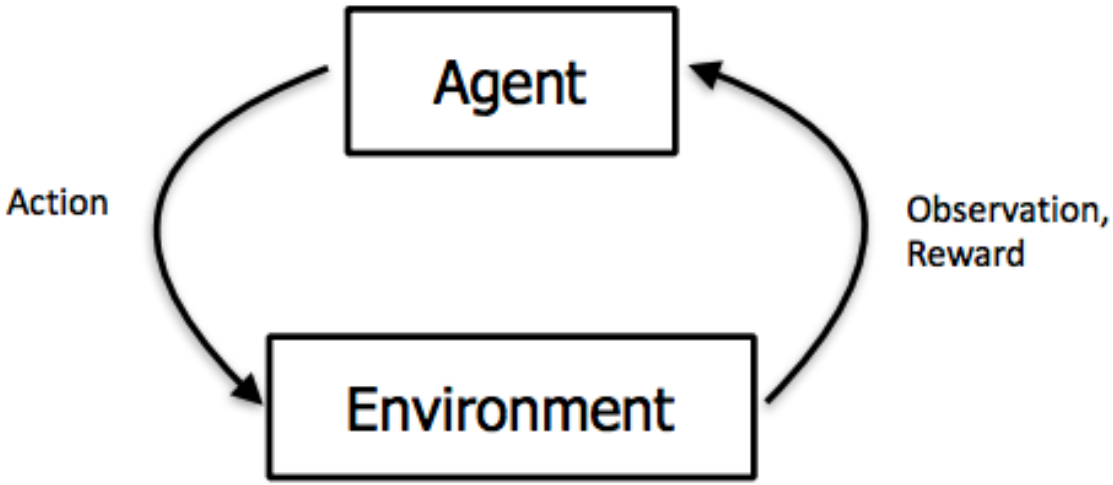
What I will talk about today is based on joint work with Tianyu Li, Bogdan Mazoure and Doina Precup:



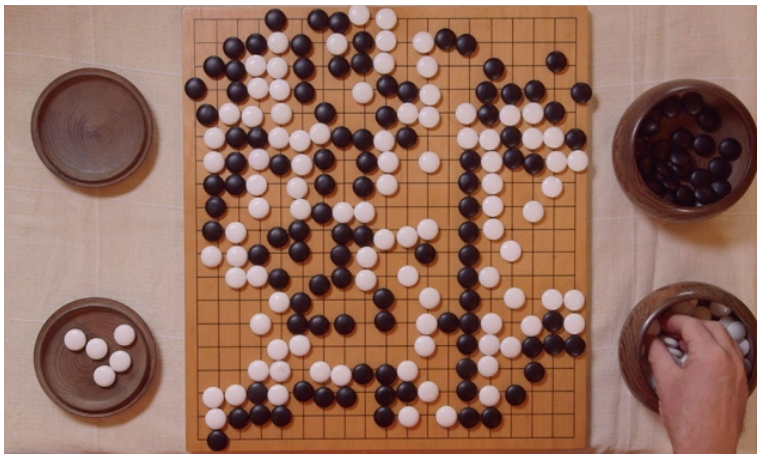
- Li, T., Mazoure, B., Precup, D., Rabusseau, G. (2020, June). *Efficient Planning under Partial Observability with Unnormalized Q Functions and Spectral Learning*. In AISTATS 2020.

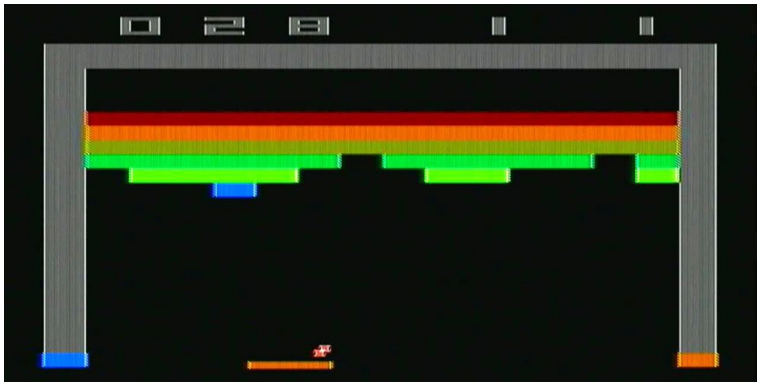
Introduction

Reinforcement Learning

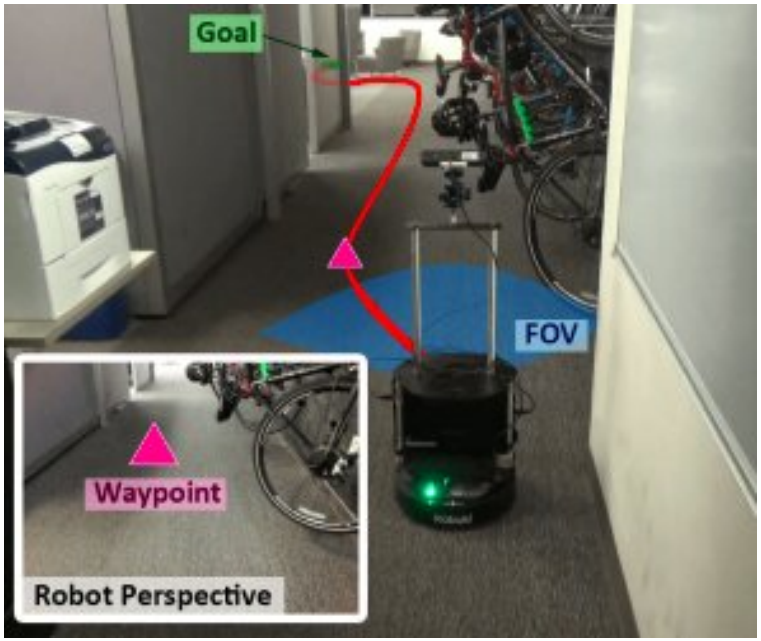


Fully Observable Environment

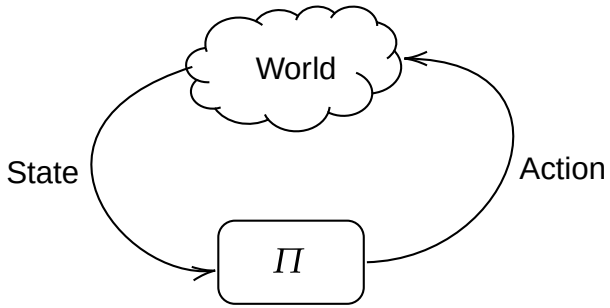




Partially Observable Environment



General Approaches



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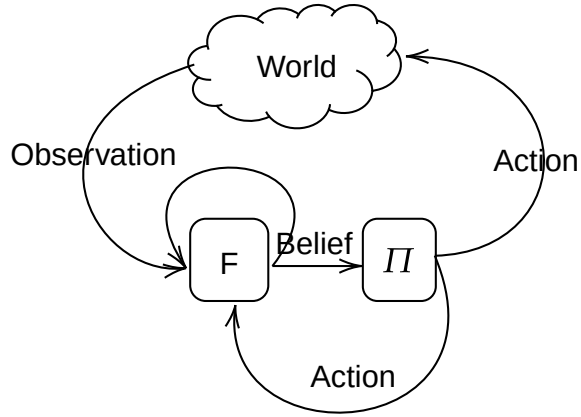

Partially observable environment

General Approach for Partially Observable Environment

1. Learning:
   - Learn the function F to estimate the belief states given a history sequence h.
   - Learn a reward model (mapping belief states to rewards / Q-value)
2. Planning: Policy acting optimally w.r.t. belief states estimator & reward model

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Examples:

- The family of predictive state representations (PSRs) [(Littman and Sutton, 2002; Singh et al., 2003, 2004; Boots et al., 2011)]
  - Separate the reward information from the belief estimation
  - Inefficient state representation for planning
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- Deep Q-Learning [(Mnih et al., 2013; Hausknecht and Stone, 2015)]
  - No theoretical guarantees
  - Partial observability is often ignored (e.g. state = last 10 observations)
  - Sample inefficient

Our Contribution

- Propose an algorithm that incorporate reward information into the belief estimation
- Sample efficient
- Consistent algorithm (spectral learning)

Background

Background

Partially Observable Markov Decision Processes

Partially Observable Markov Decision Processes (POMDPs)

A POMDP is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{O}, \mathbf{P}, \mathbf{R}, \mathbf{O}, \mathbf{b}_0 \rangle$ where

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Goal

Find a policy $\pi : \mathcal{H} \to \mathcal{A}$ maximizing the expected discounted return

$$V(\pi) = \mathbb{E}\left[\sum_{t \geq 0} \gamma^t R(s_t, a_t) \middle| \pi, \mathbf{b}_0 \right]$$

where $\mathcal{H} = (\mathcal{A} \times \mathcal{O})^*$ is the set of all histories and γ is a discount factor.

Background

Weighted Finite Automata

String Weighted Automata (WA)

- Notations:
  - Σ: a finite alphabet (e.g. $\{a, b\}$)
  - Σ^*: strings on Σ (e.g. $abba$, ab, λ, ...)
  - λ: the empty string.

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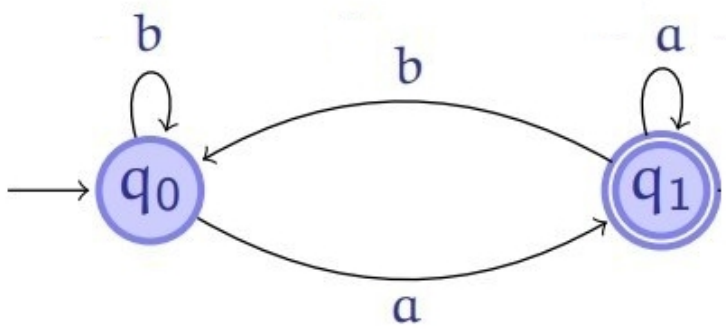
- Recall: a Deterministic Finite Automaton (DFA) recognizes a *language* (subset of Σ^*).



$\hookrightarrow$ a DFA computes a function $f : \Sigma^* \to \{\top, \bot\}$.

Weighted Automata: States and Weighted Transitions

Example with 2 states and alphabet $\Sigma = \{a, b\}$

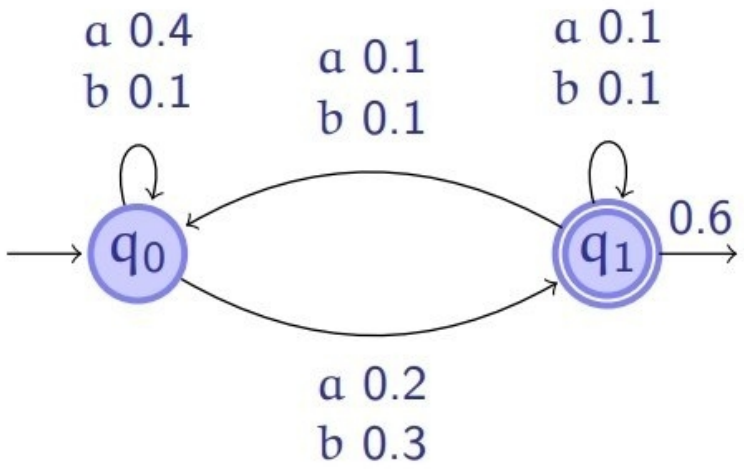


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image credits: B. Balle, X. Carreras, A. Quattoni - ENMLP'14 tutorial

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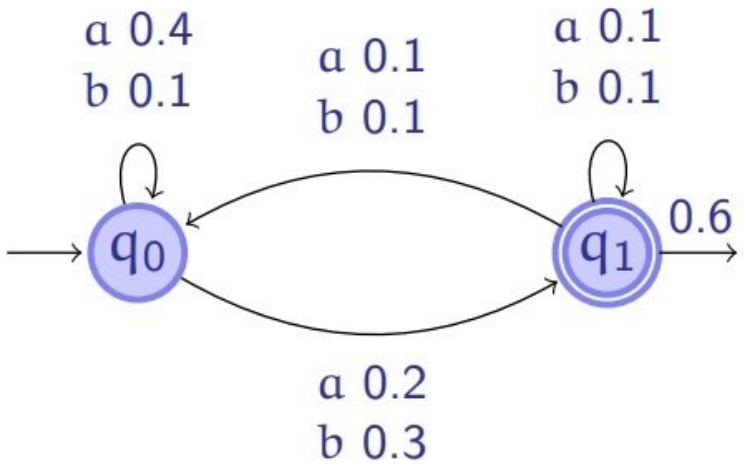

Operator Representation

$$\boldsymbol{\alpha} = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix} \quad \mathbf{A}^{a} = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}$$

$$\boldsymbol{\omega} = \begin{bmatrix} 0.0 \\ 0.6 \end{bmatrix} \quad \mathbf{A}^{b} = \begin{bmatrix} 0.1 & 0.3 \\ 0.1 & 0.1 \end{bmatrix}$$

$$f(ab) = 0.4 \times 0.3 \times 0.6 + 0.2 \times 0.1 \times 0.6 = 0.084$$

$$= \boldsymbol{\alpha}^{\top} \mathbf{A}^{a} \mathbf{A}^{b} \boldsymbol{\omega}$$

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$\boldsymbol{\alpha} \in \mathbb{R}^n$ initial weights vector
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- A computes a function $f_A : \Sigma^* \to \mathbb{R}$ defined by

$$f_A(\sigma_1 \sigma_2 \cdots \sigma_k) = \boldsymbol{\alpha}^\top \mathbf{A}^{\sigma_1} \mathbf{A}^{\sigma_2} \cdots \mathbf{A}^{\sigma_k} \boldsymbol{\omega} = \boldsymbol{\alpha}^\top \mathbf{A}^{\sigma_1 \sigma_2 \cdots \sigma_k} \boldsymbol{\omega}$$

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Theorem ((Singh et al., 2004))

For any POMDP, there exists a WFA A computing the trajectory probabilities. I.e., such that $f_A(h) = \mathbb{P}(h)$ for all $h \in \mathcal{H}$, where $\mathcal{H} = (\mathcal{A} \times \mathcal{O})^*$ is the set of all histories.

Background

Spectral Learning of WFAs

Hankel matrix

- Σ a finite alphabet of size d (e.g. $\{a, b\}$)
- Σ^* strings on Σ (e.g. *abba*)
- A WFA computes a function $f : \Sigma^* \to \mathbb{R}$:

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- $\mathbf{H}_f \in \mathbb{R}^{\Sigma^* \times \Sigma^*}$: Hankel matrix of $f : \Sigma^* \to \mathbb{R}$
  - *Definition*: prefix p, suffix $s \Rightarrow (\mathbf{H}_f)_{p,s} = f(ps)$

$$\begin{array}{c} \\ \lambda \\ a \\ b \\ aa \\ ab \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccc} \lambda & a & b & aa & ab & bb & \dots \end{array} \\ \left[\begin{array}{ccccccc} f(\lambda) & f(a) & f(b) & f(aa) & f(ab) & \dots & \dots \\ f(a) & f(aa) & f(ab) & f(aaa) & f(aab) & \dots & \dots \\ f(b) & f(ba) & f(bb) & f(baa) & f(bab) & \dots & \dots \\ f(aa) & f(aaa) & f(aab) & f(aaaa) & f(aaab) & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right] \end{array}$$

Hankel matrix

- We consider the case where inputs are sequences of discrete symbols:
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Spectral Learning of WFA

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- Fundamental theorem [Carlyle and Paz, 1971; Fliess 1974]:

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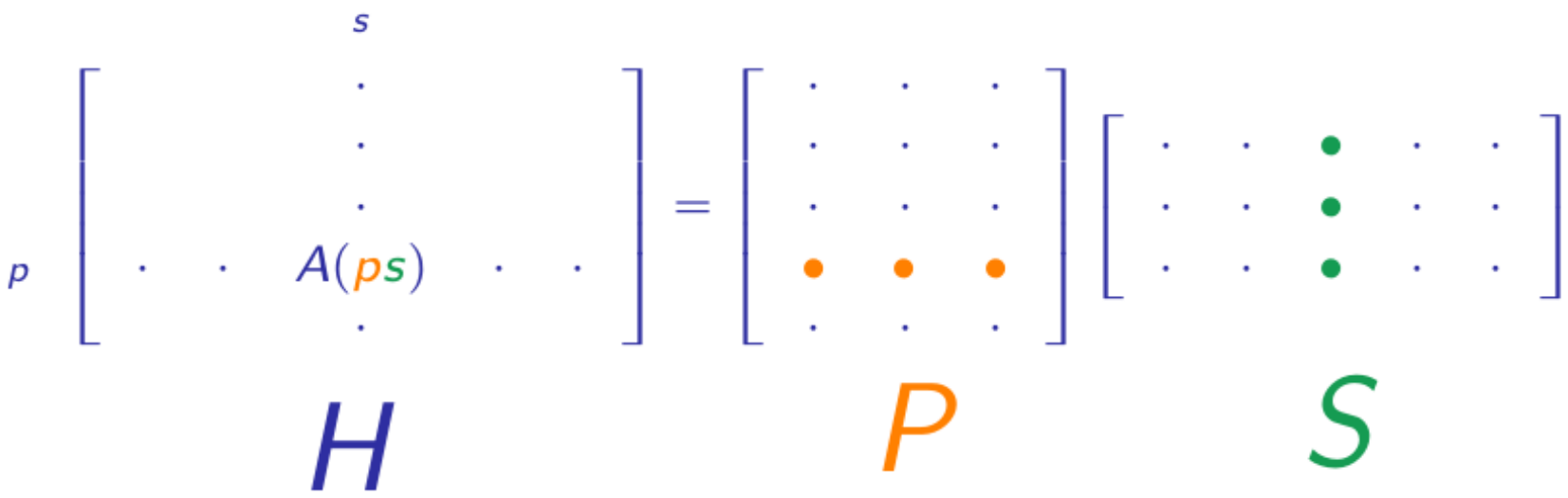
$\hookrightarrow$ Proof is constructive! From a low rank factorization of $\mathbf{H}_f$ we can recover a WFA computing f...

Hankel matrix and WA

Theorem (Fliess '74)
The rank of a *real* Hankel matrix H equals the minimal number of states of a WFA recognizing the weighted language of H

$$A(p_1 \cdots p_t s_1 \cdots s_{t'}) = \alpha^\top A_{p_1} \cdots A_{p_t} A_{s_1} \cdots A_{s_{t'}} \beta$$



slide credits: B. Balle

Hankel matrices

In addition to the Hankel matrix

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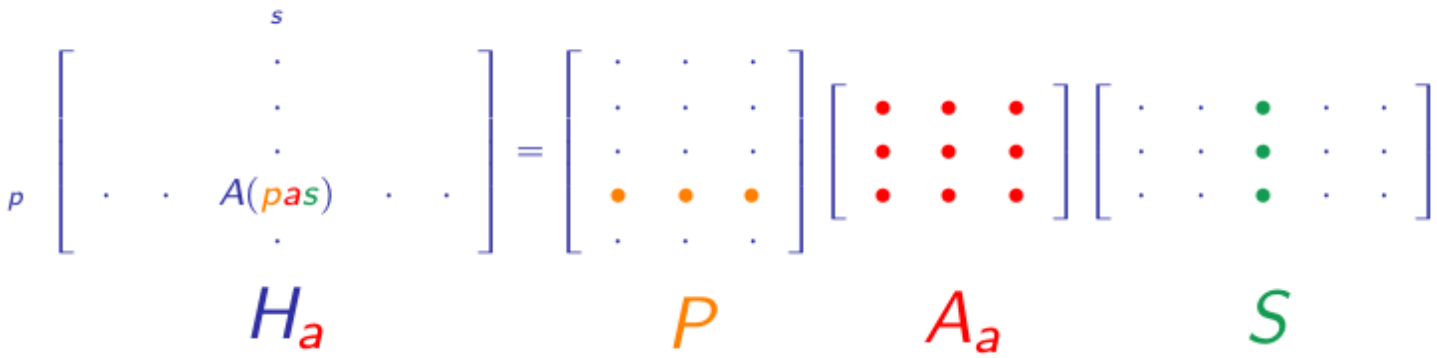
For each $\sigma \in \Sigma$ we also define

$$\mathbf{H}_\sigma = \begin{array}{c} \\ a \\ b \\ aa \\ ab \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & aa & ab & \dots \end{array} \\ \begin{bmatrix} f(a\sigma a) & f(a\sigma b) & \dots & \dots & \dots \\ f(b\sigma a) & f(b\sigma b) & \dots & \dots & \dots \\ f(aa\sigma a) & f(aa\sigma b) & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

Hankel matrix: spectral learning

$$H_a(p, s) = A(pas)$$

$$A(p_1 \cdots p_t a s_1 \cdots s_{t'}) = \alpha^\top A_{p_1} \cdots A_{p_t} A_a A_{s_1} \cdots A_{s_{t'}} \beta$$



$$H = P\ S \qquad H_a = P\ A_a\ S \qquad A_a = P^+\ H_a\ S^+$$

slide credits: B. Balle

Spectral Learning of Weighted Automata (in a nutshell)

1. Choose a set of prefixes and suffixes, $\mathcal{P}, \mathcal{S} \subset \Sigma^*$.
2. Estimate the Hankel sub-blocks $\mathbf{H}$ and $\mathbf{H}_\sigma \in \mathbb{R}^{\mathcal{P} \times \mathcal{S}}$ for each $\sigma \in \Sigma$

$$\mathbf{H} = \begin{array}{c} \\ a \\ b \\ aa \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} f(aa) & f(ab) \\ f(ba) & f(bb) \\ f(aaa) & f(aab) \end{bmatrix} \end{array} \qquad \mathbf{H}_\sigma = \begin{array}{c} \\ a \\ b \\ aa \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} f(a\sigma a) & f(a\sigma b) \\ f(b\sigma a) & f(b\sigma b) \\ f(aa\sigma a) & f(aa\sigma b) \end{bmatrix} \end{array}$$

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3. Perform rank n decomposition $\mathbf{H} = \mathbf{P}\mathbf{S}$
4. WA with initial/final weights $\boldsymbol{\alpha} = \mathbf{P}_{\lambda,:}$, $\boldsymbol{\omega} = \mathbf{S}_{:,\lambda}$ and transition matrices $\mathbf{A}^\sigma = \mathbf{P}^\dagger \mathbf{H}_\sigma \mathbf{S}^\dagger$ is a minimal WFA for f.

Spectral Learning of Weighted Automata (in a nutshell)

1. Choose a set of prefixes and suffixes, $\mathcal{P}, \mathcal{S} \subset \Sigma^*$.
2. Estimate the Hankel sub-blocks $\mathbf{H}$ and $\mathbf{H}_\sigma \in \mathbb{R}^{\mathcal{P} \times \mathcal{S}}$ for each $\sigma \in \Sigma$

$$\mathbf{H} = \begin{array}{c} \\ a \\ b \\ aa \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} f(aa) & f(ab) \\ f(ba) & f(bb) \\ f(aaa) & f(aab) \end{bmatrix} \end{array} \quad \mathbf{H}_\sigma = \begin{array}{c} \\ a \\ b \\ aa \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} f(a\sigma a) & f(a\sigma b) \\ f(b\sigma a) & f(b\sigma b) \\ f(aa\sigma a) & f(aa\sigma b) \end{bmatrix} \end{array}$$

3. Perform rank n decomposition $\mathbf{H} = \mathbf{P}\mathbf{S}$
4. WA with initial/final weights $\boldsymbol{\alpha} = \mathbf{P}_{\lambda,:}$, $\boldsymbol{\omega} = \mathbf{S}_{:,\lambda}$ and transition matrices $\mathbf{A}^\sigma = \mathbf{P}^\dagger \mathbf{H}_\sigma \mathbf{S}^\dagger$ is a minimal WFA for f.

$\rightarrow$ Efficient and consistent learning algorithms for weighted automata [Hsu et al., 2009; Bailly et al. 2009; Balle et al., 2014, ...].

Spectral Learning: when does it work?

Theorem (Exact case)

If the set of prefixes and suffixes $\mathcal{P}, \mathcal{S} \subset \Sigma^*$ are such that

$$\text{rank}(\mathbf{H}_{\mathcal{P},\mathcal{S}}) = \text{rank}(\mathbf{H}_f) < \infty$$

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- By a continuity argument, one can show that the result approximately holds when we are given noisy estimates of the Hankel matrices.

$\hookrightarrow$ When f is a probability distribution, we get an unbiased and consistent estimator! [c.f. work of B. Balle]

Learning and Planning in POMDP with WFA

Learning and Planning with WFA: Previous Approaches

Previous approaches based on spectral learning relies on 2 stages:

1. Learn a model of the POMDP dynamics
   - Learn a WFA computing $\mathbb{P}(h)$ from sampled trajectories
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Our approach: *directly* learn a reward model from trajectories

Learning and planning with the Q function

Q function: Given policy π, the value of taking action a_t after history h_t is

$$Q^{\pi}(h_t, a_t) = \mathbb{E}[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots | h_t, a_\tau = \pi(h_\tau), \tau > t]$$

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Policy improvement: Acting greedily w.r.t. Q^{π} results in a better policy $\tilde{\pi}$, i.e., $V(\tilde{\pi}) \geq V(\pi)$.

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$$\pi_0 \xrightarrow{evaluate} Q^{\pi_0} \xrightarrow{improve} \pi_1 \xrightarrow{evaluate} Q^{\pi_1} \xrightarrow{improve} \cdots \rightarrow \pi^*$$

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Main Contribution

We design a consistent learning algorithm to perform policy evaluation and improvement from sampled trajectories.

Goal

Given a policy π, we want to act greedily w.r.t. the Q-function. I.e., for any history h we want find the action a s.t.

$$a = \arg\max_{a \in \mathcal{A}} Q^{\pi}(h, a)$$

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$$\begin{aligned} Q^{\pi}(h,a) &= \mathbb{E}(r_t + \gamma r_{t+1} + \cdots + \gamma^i r_{t+i} + \cdots | ha) \\ &= \sum_{z \in \mathcal{H}} \sum_{o \in \mathcal{O}} \gamma^{|z|} \tilde{R}(haoz) \mathbb{P}(haoz|ha) \\ &= \frac{\sum_{o \in \mathcal{O}} \sum_{z \in \mathcal{H}} \gamma^{|z|} \tilde{R}(haoz) \mathbb{P}(haoz)}{\mathbb{P}(ha)} \\ &= \frac{\sum_{o \in \mathcal{O}} \sum_{z \in \mathcal{H}} \gamma^{|z|} \tilde{R}(haoz) \mathbb{P}(haoz)/\pi(a|h)}{\mathbb{P}(h)} := \frac{\tilde{Q}^{\pi}(h,a)}{\mathbb{P}(h)} \end{aligned}$$

Lemma

Given a policy π, let $\tilde{Q}^\pi$ be the un-normalized Q-function defined by

$$\tilde{Q}^\pi(h,a) = \sum_{o\in\mathcal{O}} \sum_{z\in\mathcal{H}} \gamma^{|z|} \tilde{R}(haoz)\mathbb{P}(haoz)/\pi(a|h)$$

where $\tilde{R}(h) = \sum_{s\in\mathcal{S}} \mathbb{P}(s|h)R(s)$ is the expected immediate reward.
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Given a policy π, we want to compute the un-normalized Q-function

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> Let $A = (\boldsymbol{\alpha}, \{\mathbf{A}^{\sigma}\}_{\sigma \in \Sigma}, \boldsymbol{\omega})$ be a WFA and $0 < \gamma < 1$ such that $\rho(\gamma \sum_{\sigma \in \Sigma}) < 1$. Then,
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$$\sum_{x \in \Sigma^*} \gamma^{|x|} f_A(x) = \sum_{x \in \Sigma^*} \gamma^{|x|} \boldsymbol{\alpha}^{\top} \mathbf{A}^{x} \boldsymbol{\omega} = \boldsymbol{\alpha}^{\top} (\mathbf{I} - \gamma \sum_{\sigma \in \Sigma} \mathbf{A}^{\sigma})^{-1} \boldsymbol{\omega}$$

Main Results

Theorem

For any POMDP, there exists a WFA A computing the function $h \mapsto \tilde{R}(h)\mathbb{P}(h)$.

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Let the WFA $B = \langle \beta, \{\mathbf{B}^{ao} \mid (a,o) \in \mathcal{A} \times \mathcal{O}\}, \boldsymbol{\tau} \rangle$ be a WFA computing the function $h \mapsto \tilde{R}(h)\mathbb{P}(h)$.
Then, for any policy π, the UQF for a history $h \in \mathcal{H}$ and an action $a \in \mathcal{A}$ can be computed by:

$$\tilde{Q}^{\pi}(h,a) = \frac{1}{\pi(a|h)} \beta^{\top} \boldsymbol{B}^{h} \left(\sum_{o \in \mathcal{O}} \boldsymbol{B}^{ao} \right) \left(\mathbf{I} - \gamma \sum_{(a',o') \in \mathcal{A} \times \mathcal{O}} \mathbf{B}^{a'o'} \right)^{-1} \boldsymbol{\tau}$$

Spectral Learning Algorithm for the UQF

1. Collect trajectories from the policy π
2. Estimate WFA $B = \langle \beta, \{\mathbf{B}^{ao} \mid (a,o) \in \mathcal{A} \times \mathcal{O}\}, \boldsymbol{\tau} \rangle$ computing the function $\tilde{R}(h)\mathbb{P}(h)$ from data (spectral learning algorithm).
3. Construct the WFA A with initial weight vector $\boldsymbol{\alpha} = \beta$, transition matrices $\mathbf{A}^{ao} = \mathbf{B}^{ao}$ and final weight vector

$$\boldsymbol{\omega} = \left(\mathbf{I} - \gamma \sum_{\sigma \in \Sigma} \mathbf{B}_\sigma\right)^{-1} \boldsymbol{\tau}$$

4. For any $h \in \Sigma^*$, $a \in \mathcal{A}$, the UQF is computed by:

$$\tilde{Q}^{\Pi}(h,a) = \frac{1}{\pi(a|h)} \sum_{o \in \mathcal{O}} \boldsymbol{\alpha}^\top \mathbf{A}^h \mathbf{A}^{ao} \boldsymbol{\omega}$$

5. Return the policy:

$$\pi : h \mapsto \arg\max_{a \in \mathcal{A}} \tilde{Q}^{\Pi}(h,a)$$

Experimental Results

Grid World Navigation

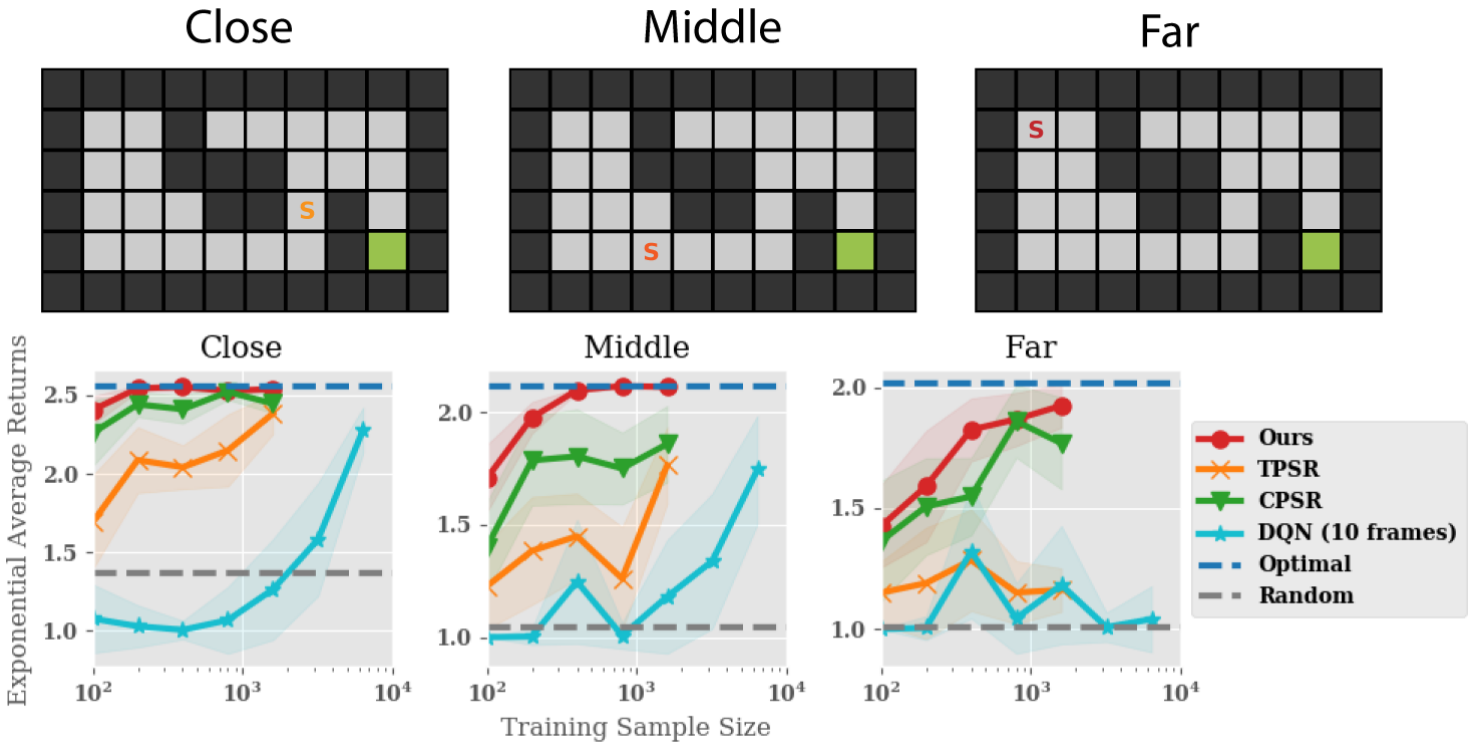

Figure: Experiments on three grid world tasks. The agent only receives knowledge about the number of the surrounding walls in four direction .

S-PocMan Domain (Hamilton et al., 2014)

- Pacman is unaware of the maze structure.
- Receives a 4-bit observation describing the wall configuration at its current location.
- Receives a 4-bit observation indicating whether a ghost is visible via direct line of sight

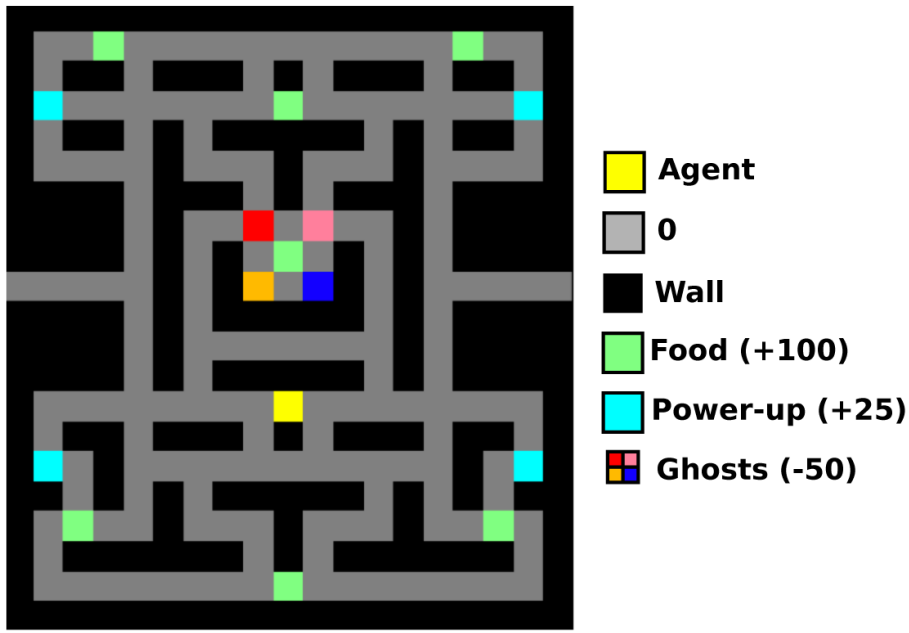

Figure: S-PocMan domain

S-PocMan Domain

Method	Fitted-Q Iterations	Time (s)	Returns
UQF	-	**2**	**-92**
CPSR	400	489	-101
	100	116	-109
	50	60	-150
	10	15	-200

Table: Training time for one policy iteration and averaged accumulated discounted rewards on S-PocMan trained on 500 trajectories.

Conclusion and Future Work

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- We proposed a sample efficient and consistent learning and planning algorithm for POMDP
- Our approach integrates planning and learning in one step
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Future Work:

- Extending the approach to continuous actions/observations (Rabusseau et al., 2018)
- Scale up the approach to larger problems leveraging connections with tensor networks
  - Li, T., Precup, D., Rabusseau, G. *Connecting Weighted Automata, Tensor Networks and Recurrent Neural Networks through Spectral Learning.* arXiv preprint arXiv:2010.10029 (2020).

Thanks for listening!
Questions?

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