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PREVENTIVE REPLACEMENT FOR MULTICOMPONENT SYSTEMS: AN OPPORTUNISTIC DISCRETE TIME DYNAMIC PROGRAMMING MODEL

by

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ABSTRACT: A preventive replacement problem is formulated in discrete time for a multicomponent system having non identical elements. The components are assumed to be s-independent but there is a possibility of economies of scale in the replacement activity. The Dynamic Programming equation is obtained and some useful properties of the Bellman value function are stated. An algorithm for the numerical computation of optimal and suboptimal replacement strategies is proposed. A numerical illustration is developped.

RESUME: On définit un modèle de remplacement préventif en temps discret pour un système à plusieurs composantes non identiques. Les composantes sont stochastiquement indépendantes mais des économies d'échelle sont possibles lorsque plusieurs remplacements se font simultanément. On obtient l'équation de la Programmation Dynamique et quelques propriétés de la fonction de Bellman sont énoncées. On propose un algorithme permettant d'obtenir numériquement des stratégies optimales ou sous-optimales. Un exemple numérique est traité.

1. INTRODUCTION

In this paper an optimal preventive replacement problem (OPR) is formulated in a discrete time setting for a multicomponent system having non identical elements.

If the failure rate of each element is only a function of its age, it is theoretically possible to obtain an OPR strategy through the use of dynamic programming [8]. However the curse of dimensionality occurs very early as the cardinality of the state set rises quickly with the number of elements comprising the system.

The case of a system with identical elements has already been considered in [3, 4]. Various properties of the Bellman functional (or optimal cost-to-go function) permitted a substantial reduction in the size of the state set and of the action set. Furthermore it has been possible to notice that suboptimal preventive replacement (SOPR) strategies, much simpler to implement than OPR strategies, could be in some cases almost as efficient as the former ones.

In the present paper the case of non identical elements is considered. An example motivating the present research is given by the optimal maintenance of multimodule aircraft engines. In this case each module j is characterized by a fixed age limit T_j and a constant failure rate. In order to replace an element, either upon failure or due to its age limit, several other elements have to be removed. A cost reduction is thus possible if one can group some of the component replacements.

In section 2 the dynamic programming equation to be solved in order to obtain the OPR strategy is given and an algorithm is proposed for its numerical solution.

In section 3, a numerical example is worked out and in section 4 a comparison is made with some simple suboptimal strategies.

2. THE DYNAMIC PROGRAMMING APPROACH

We consider a multicomponent system observed at discrete sampled times: $t_o=0$, $t_1=\Delta t,\ldots,$ $t_\sigma=\sigma\Delta t,\ldots,$ $t_n=n\Delta t$. At time $t_o=0$ the whole system is new. We assume that the components have independent life-time distributions characterized by discrete non-decreasing failure rate functions:

$$p_i(r_i)\equiv Pr\ [\text{component } i \text{ fails during the next } \Delta t \mid \text{its age is } r_i\Delta t],$$

$$i=1,\ldots,m$$

At a sampled time t_σ every failed component must be replaced by a new one (Emergency Replacement, E.R.) and some components still operative may be preventively replaced (P.R.). The advantage of performing P.R.s stems from the following cost structure: If the set $R\subseteq M\equiv\{1,2,\ldots,m\}$ of components have to be replaced, the cost of the intervention is given by:

$$C(R) = \sum_{i\in R} C_i + \sum_{i\in W(R)} c_i$$

where C_i is the replacement cost of component i, W(R) is the set of components to remove in order to replace the set R, and c_i is the removal cost of component i.

It is assumed that replacements, if any, are instantaneous. Hence we will make a distinction between the state of the system at time t_σ before any intervention, and the state of the system at time $t_\sigma^+ = t_\sigma + 0$, just after a possible intervention. At t_σ the state is a vector $x \equiv (x_i)_{i \in M}$ whose component x_i is equal to "d" if component i is failed, and gives the age expressed in number of Δt periods otherwise. Thus $x \in X \equiv \prod_{i=1}^{m} (\mathbb{N} \cup \{d\})$. Notice that no component x_i can be equal to zero. At t_σ^+ the state is a vector $r \in X_o \equiv \prod_{i=1}^{m} (\{0\} \cup \mathbb{N})$, whose component r_i is equal to zero if the component i is replaced and is equal to x_i otherwise.

The probability that the system will be in state x at $t_{\sigma+1}$ given that it is in state r at t_σ^+ is given by:

$$Pr(x|r) = \prod_{i=1}^{m} p_i(x_i|r_i)$$

where

$$P_i(x_i|r_i) = \begin{cases} p_i(r_i) & \text{if } x_i = d \\ 1 - p_i(r_i) & \text{if } x_i = r_i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Assuming a discount factor $\beta \in (0,1)$ per time period Δt, the dynamic programming approach can be used to characterize the optimal expected "cost-to-go" functionals $J_k(x)$ giving the minimal discounted expected cost on the k remaining time periods when the system is observed to be in state x at time $t_\sigma = (n-k)\Delta t$. We obtain:

$$J_k(x) = T(J_{k-1})(x) \equiv \min_{\{R \mid H \subseteq R \subseteq M\}} \{C(R) + Z_k\left(\sum_{i \in M-R} x_i e_i\right)\}$$

where $H \equiv \{i \mid x_i = d\}$ is the set of failed elements and

$$Z_k(r) = \begin{cases} 0 & \text{if } k=0 \\ \beta \sum_{x \in X} P(x|r)\, J_{k-1}(x) & \text{if } k \geq 1 \end{cases}$$

It can be proved [5] that $J_k(x)$ is non-decreasing w.r.t. each component x_i. Furthermore it can be shown that optimality precludes preventive replacements in the absence of failures (i.e. when $H=\emptyset$).

When the time horizon $n\Delta t$ tends to infinity the minimal discounted expected cost $J(x)$ when the system starts in state x is obtained as the limit of $J_k(x)$ when $k\to\infty$. Also there exists an optimal stationary strategy $\{\theta(x)\}_{x\in X}$ which gives the set $\theta(x)$ of components to replace when the system is observed to be in state x. It is also shown in [5] that the following "control limit" property holds:

$$\theta(x) = M \Rightarrow \theta(\tilde{x}) = M \qquad \forall\ \tilde{x} \geq x.$$

For the case of only two components (m=2), this property corresponds to the control-limit rule stated in [2], [7], [8] and [9]: for each component i, there is a threshold ℓ_i such that when the other component is failed, it is optimal to also replace i if and only if $x_i \geq \ell_i$.

Notice that in general, however, the preceding property cannot be extended to the case where $\theta(x) \neq M$. In section 3, an example is given illustrating that it is possible to have $\theta(\tilde{x}) < \theta(x)$ even if $\tilde{x} > x$.

A complete study of the form of the optimal strategy for systems having identical components can also be found in [3].

Using the monotonicity property of $J(x)$, the following algorithm yields an approximation of the optimal strategy.

Algorithm:

① Choose $\delta \geq 0$, $N \in \mathbb{N}$, set $k=0$ and $J_o(x) \equiv 0$.

② Repeat N times: Set k: = k+1 and define $J_k(x)$ as a monotone non-decreasing function such that:

$$\sup_{x \in X} \left| J_k(x) - T(J_{k-1})(x) \right| \leq \delta$$

③ Set J_k: = $J_k + \frac{\gamma_1+\gamma_2}{2}$ where

$$\gamma_1 \leq \frac{\beta}{1-\beta} \left[\inf_{x \in X} \{J_k(x) - J_{k-1}(x)\} - \delta\right]$$

$$\gamma_2 \geq \frac{\beta}{1-\beta} \left[\sup_{x \in X} \{J_k(x) - J_{k-1}(x)\} + \delta\right]$$

④ Define V: $X \to \mathbb{R}$ as a monotone non-decreasing function such that:

$$\sup_{x \in X} \left| V(x) - T(J_k)(x) \right| \leq \delta$$

and set $\delta_o = \sup_{x \in X} \left| V(x) - J_k(x) \right|$

if $\frac{\delta_o \beta + \delta}{1-\beta} + \text{Min} \left\{ \frac{\beta(\gamma_2-\gamma_1)}{2} + \delta, \frac{\gamma_2-\gamma_1}{2} + \delta_o \right\} > \varepsilon$

then return to step ②.

⑤ Stop. The argument of the minimum in the definition of $T(J_k)$ yields a stationary ε-optimal strategy. ■

Theorem: For any $\varepsilon > 0$, there exists $\delta > 0$ such that the preceding algorithm converge in a finite number of iterations.

The proof of this theorem follows closely similar proofs given in [1].

3. NUMERICAL ILLUSTRATION

We consider a system consisting of 4 components (see fig. 1). Component 4 must be removed to give access to component 3, which must be removed to give access to components 1 or 2. We assume that each component has a constant failure rate and an age limit of 8 periods.

Other pertinent values are:

i	1	2	3	4
c_i	.5	1.5	1.0	4.0
C_i	3.0	2.0	2.0	3.0
$p_i(r_i)$ $r_i \leq 7$	.10	.10	.08	.08
$p_i(8)$	1	1	1	1
W({i})	4,3,1	4,3,2	4,3	4

Assuming a discount factor $\beta=0.9$, $\varepsilon=0.1$, $\delta=0$, the DP Algorithm yields $V(0)=32.2$ as the expected discounted cost to go when the system is new. In what follows, the form of the optimal strategy is illustrated by giving the optimal action for a few states of the system. Here, $(x_1\ x_2\ x_3\ x_4)$ denotes a state and x_i is encircled when component i has to be replaced.

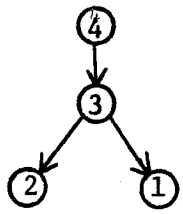

Figure 1. Accessibility structure of the system

Example A

Component	1	2	3	4
	ⓓ	1	⑥	⑥
State	ⓓ	6	6	6
	ⓓ	⑦	⑥	⑥

Example B

Component	1	2	3	4
	1	1	⑥	ⓓ
State	6	6	6	ⓓ
	⑦	⑦	⑥	ⓓ
	⑦	⑥	1	ⓓ

In example A, component 1 is failed so that components 3 and 4 (both of age 6) must be removed. If component 2 is of age 1, one replaces components 3 and 4 to obtain a quite new system. But if component 2 is of age 6, only the failed one is replaced. Finally, if component 2 is of age 7, then it is optimal to replace all the system.

This example shows that $\tilde{x} > x$ does not imply $\theta(\tilde{x}) \geq \theta(x)$. This counterintuitive result can be explained as follows: when the system is in state (d 1 6 6), replacing components 3 and 4 lead to a state where all components are quite new. When the system is in state (d 6 6 6), the same action would leave component 2 at age 6 and thus with a very short remaining life. To move away significantly the expected time of the next intervention, all the system must be replaced. But this is more expensive and it happens that the best action is to replace only the failed component and wait for the next sampled time.

In example B, component 4 is failed and it is not compulsory to remove any other component. However, when component 4 is removed, the other components are more easily accessible so that it is sometimes advantageous to do preventive replacements at the same occasion.

In general, the optimal strategy is very complicated, thus there is an incentive for considering simpler suboptimal strategies.

4. CLASSES OF SIMPLE SUBOPTIMAL STRATEGIES

One simple strategy is to replace only the failed elements. Such a strategy yields, on the preceding example, an expected discounted cost of V(0)=33.7, i.e. a 4.5% increase w.r.t. the optimal strategy.

A better suboptimal strategy is the following one: for each component i, select a threshold $\ell_i>0$. Then, at any sample time, in addition to the set H of failed components, replace every component i in W(H) whose age is greater or equal to ℓ_i. For our example, $\ell_3=\ell_4=6$ gives the best strategy in that class which yields V(0)=33.2, still a 3% increase w.r.t. the optimal strategy.

This suboptimal strategy could be refined as follows, so that it also takes advantage of the increased accessibility to some components when others are removed: for each i, define an age threshold for each "level" of accessibility to i. As an illustration, in the preceding example, component 2 should be replaced if either

$x_2 \geq \ell_{22}$ and component 4 is removed, $x_2 \geq \ell_{23}$ and component 3 is removed or if $x_2 = d$. Here, ℓ_{21}, ℓ_{22} and ℓ_{23} are the fixed age thresholds for components 2. A similar set of thresholds can be defined for each component and a strategy is thus defined.

When the curse of dimensionality prevents us to find the optimal strategy or even approximate it, one may look for a good suboptimal strategy of the latter form by using repeated simulation. The idea is to move in the space of all possible sets of thresholds in an intelligent manner and find "the best" or a "very good" point in that space.

However, this approach does not yield necessarily a satisfactory solution. For our example, the best strategy of this form is still different from the optimal one for many states.

CONCLUSION

Optimal preventive replacement strategies have been considered for a multicomponent system with non identical elements and a discrete time dynamic programming algorithm has been proposed for the approximate computation of optimal strategies.

The numerical illustration shows that the control limit rules, valid for the single or two-component system, cannot be readily generalized to larger systems. Experiments with suboptimal strategies having the control-limit structure show that it is not always possible to obtain a good approximation of the optimal cost-to-go with such strategies.

Further research is needed for providing efficient algorithms which would permit one to solve the optimal stochastic control problem in continuous time. Refs [4] and [6] give some preliminary results based on the theory of optimal control for jump processes.

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