

1 **CONSTRUCTION-FREE MEDIAN QUASI-MONTE CARLO RULES**
2 **FOR FUNCTION SPACES WITH UNSPECIFIED SMOOTHNESS**
3 **AND GENERAL WEIGHTS** *

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5 **Abstract.** We study quasi-Monte Carlo (QMC) integration of smooth functions defined over
6 the multi-dimensional unit cube. Inspired by a recent work of Pan and Owen, we study a new
7 *construction-free median QMC rule* which can exploit the smoothness and the weights of function
8 spaces adaptively. For weighted Korobov spaces, we draw a sample of r independent generating
9 vectors of rank-1 lattice rules, compute the integral estimate for each, and approximate the true
10 integral by the median of these r estimates. For weighted Sobolev spaces, we use the same approach
11 but with the rank-1 lattice rules replaced by high-order polynomial lattice rules. A major advantage
12 over the existing approaches is that we do not need to construct good generating vectors by a
13 computer search algorithm, while our median QMC rule achieves almost the optimal worst-case
14 error rate for the respective function space with *any smoothness and weights*, with a probability that
15 converges to 1 exponentially fast as r increases. Numerical experiments illustrate and support our
16 theoretical findings.

17 **Key words.** Numerical integration; quasi-Monte Carlo; rank-1 lattice rule; high-order polyno-
18 mial lattice rule; weighted function space; median; construction-free

19 **AMS subject classifications.** 65D30, 65D32, 41A55, 46E35

20 **1. Introduction.** We consider numerical integration of functions defined over
21 the s -dimensional unit cube $[0, 1]^s$. For an integrable function $f : [0, 1]^s \rightarrow \mathbb{R}$, we
22 denote the integral of f by

$$23 \quad I_s(f) := \int_{[0,1]^s} f(\mathbf{x}) \, d\mathbf{x}.$$

24 The *quasi-Monte Carlo* (QMC) method approximates $I_s(f)$ by the equally-weighted
25 average of function evaluations over a deterministic N -element point set $P_{N,s} \subset [0, 1]^s$:

$$26 \quad Q_{P_{N,s}}(f) = \frac{1}{N} \sum_{\mathbf{x} \in P_{N,s}} f(\mathbf{x}).$$

27 The *worst-case error* for a given normed function space \mathcal{F} and point set $P_{N,s}$ is

$$28 \quad e^{\text{wor}}(Q_{P_{N,s}}; \mathcal{F}) = \sup_{f \in \mathcal{F}; \|f\| \leq 1} |Q_{P_{N,s}}(f) - I_s(f)|$$

29 where $\|f\|$ is the norm of f in this space. The key to success of the QMC method
30 lies in a proper choice of the point set depending on a target class of functions. One
31 wants to construct point sets for which this worst-case error is small and converges
32 at the fastest possible rate as a function of N , for the given space \mathcal{F} . In *randomized*
33 *QMC*, the point set $P_{N,s}$ is randomized in a way that $Q_{P_{N,s}}(f)$ becomes an unbiased

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estimator of $I_s(f)$ and one wishes to minimize its variance [24, 25, 29]. In this paper, we focus on deterministic QMC point sets only.

There are two main families of QMC point sets: digital nets and sequences [10, 35] and lattice point sets [35, 42]. We refer the reader to [7, 29, 31] for further introductory details. In this paper, we consider rank-1 lattice point sets for Korobov spaces of periodic functions, and high-order polynomial lattice point sets [9, 34, 40] (which are a special type of digital nets) for Sobolev spaces of non-periodic functions. Each point set from these types is defined by an s -dimensional generating vector, with integer coordinates in the ordinary lattice case and with polynomial coordinates in the polynomial lattice case.

In both cases, the weighted spaces of functions are defined by selecting a positive smoothness parameter α and a positive weight γ_u for each subset of coordinates $u \subseteq \{1, \dots, s\}$, with $\gamma_\emptyset = 1$. The parameter α determines how smooth the admissible functions are required to be. For the Korobov spaces, it tells the minimal rate at which the Fourier coefficients of f are required to decay, and when it is an integer it corresponds to the minimal number of square-integrable mixed partial derivatives of f with respect to each coordinate; see [36, Appendix A]. For the Sobolev spaces, α is a positive integer which also imposes integrability conditions on the partial derivatives of f . The weights γ_u act as constant importance factors given to the subsets of coordinates [11, 44]. A larger γ_u means that the projection of f over the subset of coordinates in u can have a larger variation in some sense, so that more importance should be given to the uniformity of the points over this projection.

It is known that the best possible QMC point sets cannot provide a better convergence rate than $\mathcal{O}(N^{-\alpha})$ for the worst-case error for these two function spaces. On the other hand, there are effective search algorithms which, for a given α and a given selection of weights γ_u , can construct good rank-1 lattice or polynomial lattice point sets for the function spaces determined by these parameters, and for which the worst-case error converges as $\mathcal{O}(N^{-\alpha+\epsilon})$ for any $\epsilon > 0$ [10, 21]. Software that implement such algorithms is also freely available [26, 28, 37]. These algorithms typically use a greedy component-by-component (CBC) construction approach proposed originally by [19], then re-introduced and popularized by [43]. With the CBC approach, the generating vector is determined one coordinate at a time by optimizing a figure of merit that depends only on this new coordinate and the previous ones, and where the previous coordinates can no longer be changed.

In general, the number of weights γ_u to specify is $2^s - 1$. When s is large, specifying all these weights individually becomes impractical, so it is common practice to “parameterize” the weights by a smaller number of parameters, usually linear instead of exponential in s . The most popular forms of parameterizations are the product weights, the order-dependent weights, and their combination. For the product weights, one specifies a weight $\gamma_j > 0$ for each coordinate $j = 1, \dots, s$, and the γ_u 's are defined as $\gamma_u = \prod_{j \in u} \gamma_j$. For the order-dependent weights, γ_u depends only on the cardinality of u : $\gamma_u = \Gamma_{|u|}$, where $\Gamma_1, \Gamma_2, \dots, \Gamma_s$ are selected positive constants. Their combination gives the product and order-dependent (POD) weights, for which $\gamma_u = \Gamma_{|u|} \prod_{j \in u} \gamma_j$ [33]. The main reason why the most popular choices of weights have this form is that the existing search algorithms are truly efficient for large s only when the weights have this specific POD form [11, 23, 26, 28, 33, 38]. Then, by using a fast-CBC approach that speeds up the search by exploiting a fast Fourier transform [28, 38], one can find a generating vector that gives a worst-case error of $\mathcal{O}(N^{-\alpha+\epsilon})$ in $\mathcal{O}(sN \log N)$ operations for ordinary lattices and $\mathcal{O}(\alpha s N \log N)$ for polynomial lattices (with interlacing).

84 Although this form is convenient, the restriction to POD weights is **limiting**: for
 85 a given application, the appropriate weights may be quite far from the POD form. In
 86 this case, imposing POD implies that the point sets are constructed with the wrong
 87 weights. Moreover, even without constraints imposed on the form of weights, finding
 88 or approximating appropriate weights and the appropriate α for a given application is
 89 generally very difficult [27]. When the points are constructed with the wrong weights,
 90 the QMC method can be quite ineffective in general. These drawbacks have been
 91 addressed very partially in recent papers. In [12], the authors introduces a construc-
 92 tion algorithm that does not require the knowledge of α . It uses a CBC construction
 93 algorithm with a figure of merit that assumes $\alpha = 1$, and for each coordinate it also
 94 constructs the generating vector one binary digit at a time. The method provides a
 95 convergence rate of $\mathcal{O}(N^{-\alpha+\epsilon})$. In [4], the authors study the stability of rank-1 lattice
 96 rules and polynomial lattice rules to a (limited) misspecification of α and the weights,
 97 for product and POD weights. They obtain worst-case error bounds for function spaces
 98 determined by parameters α' and $\gamma' = \{\gamma'_u\}$ when the rules are constructed using
 99 parameters α and $\gamma = \{\gamma_u\}$ instead, under certain conditions on those weights. These
 100 results are interesting but they do not completely eliminate the need to specify the
 101 weights.

102 The method studied in this paper requires no knowledge at all on α and the
 103 weights γ_u . No value needs to be specified for any of these parameters. The algorithm
 104 is inspired by recent work from Pan and Owen [39], and works as follows. For a
 105 fixed odd integer $r > 0$, we draw r generating vectors independently and uniformly
 106 from the set of all admissible generating vectors. For each of them, we compute the
 107 corresponding QMC approximation $Q_{P_{N,s}}(f)$, then we take the median $M(f)$ of these
 108 r approximations as our final estimate of $I_s(f)$. Since the method does not require
 109 the explicit construction of a good point set, we call it a *construction-free median*
 110 *QMC rule*.

111 Our main contribution is to prove that for \mathcal{F} representing either a weighted
 112 Korobov or weighted Sobolev space determined by parameters α and γ , the error
 113 $|M(f) - I_s(f)|$ obeys the following type of probabilistic bound: For any $\epsilon > 0$ and
 114 $0 < \rho < 1$, there is a constant $c_1 = c_1(\alpha, \gamma, \epsilon) > 0$ (which depends on α , the γ_u 's and
 115 ϵ) such that

$$116 \quad \mathbb{P} \left[\sup_{f \in \mathcal{F}; \|f\| \leq 1} |M(f) - I_s(f)| \leq \frac{c_1(\alpha, \gamma, \epsilon)}{(\rho N)^{\alpha-\epsilon}} \right] \geq 1 - \rho^{(r+1)/2}/4.$$

117 In other words, the worst-case error of the median estimator is bounded by a quantity
 118 that decreases almost at the best possible rate of $\mathcal{O}(N^{-\alpha})$, with a probability that
 119 converges to 1 exponentially fast as a function of r . That is, we have a simple method
 120 that provides essentially the best possible convergence rate with very high probability,
 121 without requiring any knowledge of α and the weights. The key reason why this is
 122 possible is that the vast majority of the choices of generating vectors turn out to be
 123 quite good and give a QMC approximation which is quite close to $I_s(f)$. Only a small
 124 minority give a large error. For the vector giving the median value to be in that small
 125 minority, there must be at least $(r+1)/2$ generating vectors in the sample of size r
 126 that belong to this small minority, and the probability that this happens decreases
 127 towards zero exponentially in r .

128 The remainder is organized as follows. In Section 2, we recall some basic facts
 129 on lattice rules for Korobov spaces, and we prove our main result for the median
 130 estimator in this setting. In Section 3, we do the same for high-order polynomial

131 lattice rules in Sobolev spaces. In Section 4, we report numerical experiments to
 132 support our theoretical findings.

133 2. Lattice rules for Korobov spaces.

134 **2.1. Definitions.** Lattice point sets are well suited for performing numerical
 135 integration of smooth periodic functions. A rank-1 lattice point set is defined as
 136 follows:

137 **DEFINITION 2.1** (rank-1 lattice point set). *Let $N \geq 2$ be the number of points*
 138 *and $\mathbf{z} = (z_1, \dots, z_s) \in \{1, \dots, N-1\}^s$. The rank-1 lattice point set defined by N and*
 139 *the generating vector \mathbf{z} is*

$$140 \quad P_{N,s,\mathbf{z}} = \left\{ \left(\left\{ \frac{nz_1}{N} \right\}, \dots, \left\{ \frac{nz_s}{N} \right\} \right) \in [0, 1)^s \mid n = 0, 1, \dots, N-1 \right\},$$

141 where $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of a real $x \geq 0$. The QMC algorithm
 142 using $P_{N,s,\mathbf{z}}$ as a point set is called the rank-1 lattice rule with generating vector \mathbf{z} .

143 Let $f : [0, 1)^s \rightarrow \mathbb{R}$ be periodic with an absolutely convergent Fourier series

$$144 \quad f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^s} \hat{f}(\mathbf{k}) \exp(2\pi i \mathbf{k} \cdot \mathbf{x}),$$

145 where the dot product \cdot denotes the usual inner product of two vectors on the Euclid-
 146 ean space \mathbb{R}^s and $\hat{f}(\mathbf{k})$ denotes the \mathbf{k} -th Fourier coefficient of f :

$$147 \quad \hat{f}(\mathbf{k}) := \int_{[0,1)^s} f(\mathbf{x}) \exp(-2\pi i \mathbf{k} \cdot \mathbf{x}) \, d\mathbf{x}.$$

148 Note that $\hat{f}(\mathbf{0})$ coincides with the integral $I_s(f)$. As a class of periodic functions, we
 149 consider the following *weighted Korobov space*.

150 **DEFINITION 2.2** (weighted Korobov space). *Let $\alpha > 1/2$ and $\boldsymbol{\gamma} = \{\gamma_u\}_{u \subseteq \{1, \dots, s\}}$*
 151 *be a set of positive weights with $\gamma_\emptyset = 1$. For a non-empty subset $u \subseteq \{1, \dots, s\}$ and a*
 152 *vector $\mathbf{k}_u \in (\mathbb{Z} \setminus \{0\})^{|u|}$, we denote by $(\mathbf{k}_u, \mathbf{0})$ the vector $\mathbf{h} \in \mathbb{Z}^s$ such that $h_j = k_j$ if*
 153 *$j \in u$ and $h_j = 0$ otherwise, and define*

$$154 \quad r_{\alpha,\boldsymbol{\gamma}}(\mathbf{k}_u, \mathbf{0}) := \gamma_u \prod_{j \in u} \frac{1}{|k_j|^\alpha},$$

156 and set $r_{\alpha,\boldsymbol{\gamma}}(\mathbf{0}) = 1$. The weighted Korobov space, denoted by $\mathcal{F}_{s,\alpha,\boldsymbol{\gamma}}^{\text{kor}}$, is a reproducing
 157 kernel Hilbert space with reproducing kernel

$$158 \quad K_{s,\alpha,\boldsymbol{\gamma}}^{\text{kor}}(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{k} \in \mathbb{Z}^s} (r_{\alpha,\boldsymbol{\gamma}}(\mathbf{k}))^2 \exp(2\pi i \mathbf{k} \cdot (\mathbf{x} - \mathbf{y})),$$

159 and inner product

$$160 \quad \langle f, g \rangle_{s,\alpha,\boldsymbol{\gamma}}^{\text{kor}} = \sum_{\mathbf{k} \in \mathbb{Z}^s} \frac{\hat{f}(\mathbf{k}) \overline{\hat{g}(\mathbf{k})}}{(r_{\alpha,\boldsymbol{\gamma}}(\mathbf{k}))^2}.$$

161 We denote the induced norm by $\|f\|_{s,\alpha,\boldsymbol{\gamma}}^{\text{kor}} := \sqrt{\langle f, f \rangle_{s,\alpha,\boldsymbol{\gamma}}^{\text{kor}}}$.

162 One wishes to have a good generating vector \mathbf{z} such that the worst-case error of
 163 the corresponding lattice rule for $\mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}$, defined by

$$164 \quad e^{\text{wor}}(Q_{P_{N,s,\mathbf{z}}}; \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}) := \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} |Q_{P_{N,s,\mathbf{z}}}(f) - I_s(f)|,$$

165 is small. No good explicit construction scheme for such a \mathbf{z} is known for $s \geq 3$, so that
 166 we usually resort to a computer search algorithm as mentioned earlier. By restricting
 167 each z_j to be in the set

$$168 \quad \mathbb{U}_N := \{1 \leq z \leq N-1 \mid \gcd(z, N) = 1\},$$

169 we ensure that each projection of $P_{N,s,\mathbf{z}}$ on a single coordinate contains the N dis-
 170 tinct values $\{0, 1/N, \dots, (N-1)/N\}$ (no **superposed** points). The CBC construction
 171 algorithm for a good generating vector \mathbf{z} starts with $z_1 = 1$, then for $j = 2, \dots, s$ it
 172 searches for the best component z_j from the set \mathbb{U}_N while keeping the earlier compo-
 173 nents z_1, \dots, z_{j-1} unchanged.

174 For our median QMC rank-1 lattice rule for weighted Korobov spaces, we select
 175 an independent random sample $\mathbf{z}_1, \dots, \mathbf{z}_r$ from the set \mathbb{U}_N^s , and we approximate $I_s(f)$
 176 by the median

$$177 \quad M_{N,s,r}(f) := \text{median}(Q_{P_{N,s,\mathbf{z}_1}}(f), \dots, Q_{P_{N,s,\mathbf{z}_r}}(f)).$$

178 *Note that, for given $\mathbf{z}_1, \dots, \mathbf{z}_r$, the index ℓ for which \mathbf{z}_ℓ gives the median $M_{N,s,r}(f)$*
 179 *generally depends on f .* The worst-case error in this case is the random variable

$$180 \quad (2.1) \quad e^{\text{wor}}(M_{N,s,r}; \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}) := \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} |M_{N,s,r}(f) - I_s(f)|.$$

181 In this random expression, we assume that $\mathbf{z}_1, \dots, \mathbf{z}_r$ are first picked randomly, then
 182 f is taken as the worst-case function for the median, for these given $\mathbf{z}_1, \dots, \mathbf{z}_r$.

183 **2.2. Our main results on lattice rules for Korobov spaces.** To prove our
 184 main result, we need a few more definitions.

185 **DEFINITION 2.3** (dual lattice). *For $N \geq 2$ and $\mathbf{z} \in \mathbb{U}_N^s$, the set*

$$186 \quad P_{N,s,\mathbf{z}}^\perp := \{\mathbf{k} \in \mathbb{Z}^s \mid \mathbf{k} \cdot \mathbf{z} \equiv 0 \pmod{N}\}$$

187 *is called the dual lattice of the rank-1 lattice point set $P_{N,s,\mathbf{z}}$.*

188 The following character property of the rank-1 lattice rule is well-known, see for
 189 instance [5, Lemmas 4.2 and 4.3].

190 **LEMMA 2.4** (character property). *For $N \geq 2$, $\mathbf{z} \in \mathbb{U}_N^s$ and $\mathbf{k} \in \mathbb{Z}^s$, we have*

$$191 \quad \frac{1}{N} \sum_{\mathbf{x} \in P_{N,s,\mathbf{z}}} \exp(2\pi i \mathbf{k} \cdot \mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{k} \in P_{N,s,\mathbf{z}}^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

192 As our first main result, we prove a probabilistic upper bound on the worst-case
 193 error of our median rank-1 lattice rule for weighted Korobov spaces.

194 THEOREM 2.5. Let $N \geq 2$ be an integer, $r > 0$ be an odd integer, and $\mathbf{z}_1, \dots, \mathbf{z}_r$
 195 be chosen independently and randomly from the set \mathbb{U}_N^s (with replacement). Then,
 196 for any $\alpha > 1/2$ and γ , the worst-case error of the median rule obeys the following
 197 bound:

$$198 \quad e^{\text{wor}}(M_{N,s,r}; \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}) \leq \inf_{1/(2\alpha) < \lambda < 1} \left(\frac{1}{\eta\varphi(N)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} (2\zeta(2\alpha\lambda))^{|u|} \right)^{1/(2\lambda)}$$

199 with a probability of at least

$$200 \quad 1 - \binom{r}{(r+1)/2} \eta^{(r+1)/2},$$

201 for any $0 < \eta < 1$, where φ and ζ denote the Euler totient function and the Riemann
 202 zeta function, respectively.

203 We note that the result for $r = 1$, i.e., the case without taking the median, can
 204 be found, for instance, in [11, Theorem 2], and has been used together with a random
 205 choice of N in [20] to prove an improved rate of convergence of the randomized error.

206 The following inequality on medians is a key ingredient in the proof of the theorem.
 207 Although it can be regarded as a special case of Jensen's inequality on medians proven
 208 in [32], we give a short direct proof to make the paper more self-contained.

209 LEMMA 2.6. For any odd integer r and real numbers a_1, \dots, a_r , it holds that

$$210 \quad |\text{median}(a_1, \dots, a_r)| \leq \text{median}(|a_1|, \dots, |a_r|).$$

211 *Proof.* Let $a_m = \text{median}(a_1, \dots, a_r)$. If $a_m \geq 0$, then $|a_m| = a_m \leq \text{median}(|a_1|,$
 212 $\dots, |a_r|)$. If $a_m < 0$, there is at least $(r-1)/2$ other a_ℓ 's for which $a_\ell \leq a_m < 0$, so
 213 $|a_\ell| \geq |a_m| > 0$. Then, $|\text{median}(a_1, \dots, a_r)| = |a_m| \leq \text{median}(|a_1|, \dots, |a_r|)$. \square

214 We now prove our main result.

215 *Proof of Theorem 2.5.* Since any $f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}$ has an absolutely convergent Fourier
 216 series, by applying Lemma 2.4, Lemma 2.6 and the Cauchy-Schwarz inequality, it
 217 holds for given $\mathbf{z}_1, \dots, \mathbf{z}_r$ that

$$\begin{aligned} 218 \quad & e^{\text{wor}}(M_{N,s,r}; \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}) \\ 219 \quad &= \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \left| \text{median}_{1 \leq \ell \leq r} \frac{1}{N} \sum_{\mathbf{x} \in P_{N,s,\mathbf{z}_\ell}} f(\mathbf{x}) - I(f) \right| \\ 220 \quad &= \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \left| \text{median}_{1 \leq \ell \leq r} \frac{1}{N} \sum_{\mathbf{x} \in P_{N,s,\mathbf{z}_\ell}} \sum_{\mathbf{k} \in \mathbb{Z}^s} \hat{f}(\mathbf{k}) \exp(2\pi i \mathbf{k} \cdot \mathbf{x}) - \hat{f}(\mathbf{0}) \right| \\ 221 \quad &= \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \left| \text{median}_{1 \leq \ell \leq r} \sum_{\mathbf{k} \in P_{N,s,\mathbf{z}_\ell}^\perp \setminus \{\mathbf{0}\}} \hat{f}(\mathbf{k}) \right| \\ 222 \quad &\leq \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \text{median}_{1 \leq \ell \leq r} \sum_{\mathbf{k} \in P_{N,s,\mathbf{z}_\ell}^\perp \setminus \{\mathbf{0}\}} |\hat{f}(\mathbf{k})| \end{aligned}$$

$$\begin{aligned}
223 \quad &= \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \text{median}_{1 \leq \ell \leq r} \sum_{\mathbf{k} \in P_{N,s,z_\ell}^\perp \setminus \{\mathbf{0}\}} \frac{|\hat{f}(\mathbf{k})|}{r_{\alpha,\gamma}(\mathbf{k})} r_{\alpha,\gamma}(\mathbf{k}) \\
224 \quad &\leq \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \text{median}_{1 \leq \ell \leq r} \left(\sum_{\mathbf{k} \in P_{N,s,z_\ell}^\perp \setminus \{\mathbf{0}\}} \frac{|\hat{f}(\mathbf{k})|^2}{(r_{\alpha,\gamma}(\mathbf{k}))^2} \right)^{1/2} \\
225 \quad &\quad \times \left(\sum_{\mathbf{k} \in P_{N,s,z_\ell}^\perp \setminus \{\mathbf{0}\}} (r_{\alpha,\gamma}(\mathbf{k}))^2 \right)^{1/2} \\
226 \quad &\leq \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \left(\sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}} \frac{|\hat{f}(\mathbf{k})|^2}{(r_{\alpha,\gamma}(\mathbf{k}))^2} \right)^{1/2} \text{median}_{1 \leq \ell \leq r} \mathcal{S}_{\alpha,\gamma}(z_\ell) \\
227 \quad (2.2) \quad &\leq \text{median}_{1 \leq \ell \leq r} \mathcal{S}_{\alpha,\gamma}(z_\ell), \\
228 \quad &
\end{aligned}$$

229 in which

$$230 \quad \mathcal{S}_{\alpha,\gamma}(z) := \left(\sum_{\mathbf{k} \in P_{N,s,z}^\perp \setminus \{\mathbf{0}\}} (r_{\alpha,\gamma}(\mathbf{k}))^2 \right)^{1/2}.$$

231 For $1/(2\alpha) < \lambda \leq 1$, by using the subadditivity

$$232 \quad (2.3) \quad \left(\sum_i a_i \right)^\lambda \leq \sum_i a_i^\lambda, \\
233 \quad$$

234 which holds for non-negative reals $a_1, a_2, \dots > 0$, see [5, Theorem 2.2], and noting
235 that the cardinality of \mathbb{U}_N is equal to $\varphi(N)$, we have

$$\begin{aligned}
236 \quad &\frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} (\mathcal{S}_{\alpha,\gamma}(z))^{2\lambda} \\
237 \quad &= \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} \left(\sum_{\mathbf{k} \in P_{N,s,z}^\perp \setminus \{\mathbf{0}\}} (r_{\alpha,\gamma}(\mathbf{k}))^2 \right)^\lambda \\
238 \quad &\leq \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} \sum_{\mathbf{k} \in P_{N,s,z}^\perp \setminus \{\mathbf{0}\}} (r_{\alpha,\gamma}(\mathbf{k}))^{2\lambda} \\
239 \quad &= \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}} (r_{\alpha,\gamma}(\mathbf{k}))^{2\lambda} \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} \frac{1}{N} \sum_{\mathbf{x} \in P_{N,s,z}} \exp(2\pi i \mathbf{k} \cdot \mathbf{x}) \\
240 \quad &= \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}} (r_{\alpha,\gamma}(\mathbf{k}))^{2\lambda} \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i n \mathbf{k} \cdot \mathbf{z}/N) \\
241 \quad &= \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} \sum_{\mathbf{k}_u \in (\mathbb{Z} \setminus \{0\})^{|u|}} \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{j \in u} \frac{\exp(2\pi i n k_j z_j / N)}{|k_j|^{2\alpha\lambda}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \frac{\gamma_u^{2\lambda}}{(\varphi(N))^{|u|}} \sum_{\mathbf{z}_u \in \mathbb{U}_N^{|u|}} \sum_{\mathbf{k}_u \in (\mathbb{Z} \setminus \{0\})^{|u|}} \prod_{j \in u} \frac{\exp(2\pi i n k_j z_j / N)}{|k_j|^{2\alpha\lambda}} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} (T_{2\alpha\lambda}(n, N))^{|u|},
\end{aligned}$$

where we write

$$T_{2\alpha\lambda}(n, N) := \frac{1}{\varphi(N)} \sum_{z \in \mathbb{U}_N} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\exp(2\pi i n k z / N)}{|k|^{2\alpha\lambda}}.$$

Since it follows from [22, Lemmas 2.1 & 2.2] that, for any positive integer d

$$\frac{1}{N} \sum_{n=0}^{N-1} (T_{2\alpha\lambda}(n, N))^d \leq \frac{(2\zeta(2\alpha\lambda))^d}{\varphi(N)},$$

we obtain

$$\begin{aligned}
\frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} (\mathcal{S}_{\alpha, \gamma}(\mathbf{z}))^{2\lambda} &\leq \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} \frac{1}{N} \sum_{n=0}^{N-1} (T_{2\alpha\lambda}(n, N))^{|u|} \\
&\leq \frac{1}{\varphi(N)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} (2\zeta(2\alpha\lambda))^{|u|}.
\end{aligned}$$

This gives an upper bound on the average of $(\mathcal{S}_{\alpha, \gamma}(\mathbf{z}))^{2\lambda}$ over all of the admissible $\mathbf{z} \in \mathbb{U}_N^s$, which holds for any $1/(2\alpha) < \lambda \leq 1$.

Then, Markov's inequality ensures that for any $0 < \eta < 1$, the probability of having

$$\mathcal{S}_{\alpha, \gamma}(\mathbf{z}) > \inf_{1/(2\alpha) < \lambda < 1} \left(\frac{1}{\eta \varphi(N)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} (2\zeta(2\alpha\lambda))^{|u|} \right)^{1/(2\lambda)} =: B(\alpha, \gamma)$$

is at most η for a random choice of $\mathbf{z} \in \mathbb{U}_N^s$. For the median estimator $M_{N, s, r}$ to be larger than this bound $B(\alpha, \gamma)$, we must have $\mathcal{S}_{\alpha, \gamma}(\mathbf{z}_\ell) > B(\alpha, \gamma)$ for at least $(r+1)/2$ vectors among $\mathbf{z}_1, \dots, \mathbf{z}_r$. **Taking the union bound on possible sets of $(r+1)/2$ vectors with $\mathcal{S}_{\alpha, \gamma}(\mathbf{z}_\ell) > B(\alpha, \gamma)$,** the probability that this happens is bounded above by

$$\binom{r}{(r+1)/2} \eta^{(r+1)/2}.$$

Combining this with the bound shown in (2.2) completes the proof. \square

Remark 2.7. One can easily prove by induction on k that $\binom{2k-1}{k} < 4^{k-1}$ for $k \geq 2$. Indeed, this is true for $k = 2$, and for $k \geq 2$, one has

$$\binom{2k+1}{k+1} = \frac{2(2k+1)}{(k+1)} \binom{2k-1}{k} < 4 \binom{2k-1}{k} < 4^k.$$

Then, for any odd $r \geq 3$, we have

$$(2.4) \quad \binom{r}{(r+1)/2} \eta^{(r+1)/2} < (4\eta)^{(r+1)/2} / 4.$$

Thus, the probability given in Theorem 2.5 must be larger than $1 - (4\eta)^{(r+1)/2} / 4$, which converges to 1 exponentially fast as a function of r for $0 < \eta < 1/4$.

271 By taking $1/(2\lambda) = \alpha - \epsilon$ and $\rho = 4\eta$ and using the previous remark, we obtain
 272 the following corollary as a simplified version of Theorem 2.5.

273 **COROLLARY 2.8.** *For any odd $r \geq 3$, $\epsilon > 0$, and $0 < \rho < 1$, there is a constant*
 274 *$c_1 = c_1(\alpha, \gamma, \epsilon) > 0$ (which depends on α , the γ_u 's and ϵ) such that*

$$275 \quad \mathbb{P} \left[e^{\text{wor}}(M_{N,s,r}; \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}) \leq \frac{c_1(\alpha, \gamma, \epsilon)}{(\rho N)^{\alpha-\epsilon}} \right] \geq 1 - \rho^{(r+1)/2}/4.$$

276 *Proof.* For prime N , we have $\varphi(N) = N - 1$ and the corollary follows from
 277 Theorem 2.5 and the bound (2.4) in Remark 2.7. For a general N , we know from [41,
 278 Theorem 15] that

$$279 \quad \frac{1}{\varphi(N)} \leq \frac{1}{N} \left[e^C \log \log N + \frac{2.50637}{\log \log N} \right]$$

280 for any $N \geq 3$, where $C = 0.57721 \dots$ is the Euler's constant. From this, the proba-
 281 bilistic bound follows. \square

282 *Remark 2.9.* Under some additional conditions on the weights γ , this bound de-
 283 pends only polynomially on the dimension s , and can even be independent on the
 284 dimension s . We refer to [11, Theorem 3] for the case of general weights and [11,
 285 Theorem 4] for the case of product weights.

286 *Remark 2.10.* As mentioned in Section 1, our median rank-1 lattice rule is moti-
 287 vated by the observation that most of the possible generating vectors $\mathbf{z} \in \mathbb{U}_N^s$ are
 288 a good choice, but the remaining ones are bad. To show this, let us first point out that
 289 $\mathcal{S}_{\alpha,\gamma}(\mathbf{z})$ coincides with the worst-case error of the rank-1 lattice rule with the given
 290 \mathbf{z} , see [7, Theorem 5.12]. As already seen in the proof of Theorem 2.5, a proportion
 291 of the generating vectors \mathbf{z} which satisfy the bound of order $N^{-\alpha+\epsilon}$, i.e.,

$$292 \quad \mathcal{S}_{\alpha,\gamma}(\mathbf{z}) \leq \inf_{1/(2\alpha) < \lambda < 1} \left(\frac{1}{\eta \varphi(N)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} (2\zeta(2\alpha\lambda))^{|u|} \right)^{1/(2\lambda)},$$

293 is greater than or equal to $1 - \eta$, for any $0 < \eta < 1$. On the other hand, the averaging
 294 argument in the proof of Theorem 2.5 with $\lambda = 1$ gives

$$295 \quad \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} (\mathcal{S}_{\alpha,\gamma}(\mathbf{z}))^2 \leq \frac{1}{\varphi(N)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^2 (2\zeta(2\alpha))^{|u|}.$$

296 This implies that, for each N , there exists a small, distinct set of “bad” generating
 297 vectors whose $\mathcal{S}_{\alpha,\gamma}(\mathbf{z})$ values are quite large so that the average of the *squared* worst-
 298 case error over all the possible generating vectors is merely of order N^{-1} . Such bad
 299 vectors may include those with all components being the same.

300 *Remark 2.11.* Theorem 2.5 (or its corollary) gives a probabilistic error bound on
 301 the worst-case error (2.1), together with a lower bound on the probability that this
 302 error bound holds. The exact value of this probability depends on the probability
 303 distribution of $Q_{P_{N,s,\mathbf{z}}}(f)$ when \mathbf{z} is drawn uniformly from \mathbb{U}_N^s , and on the choices
 304 of r and of the other parameters in the error bound. To get some insight on how
 305 it behaves, we will simplify the setting slightly and look at a one-side error bound
 306 for a fixed f : we want to estimate the probability that the median $M_{N,s,r}(f)$ does
 307 not exceed some arbitrary constant y larger than the mean $I_s(f)$. Suppose that this

308 y is the q -quantile y_q of the distribution of $Q_{P_{N,s,\mathbf{z}}}(f)$ for some $q \in (3/4, 1)$, i.e.,
 309 $q = \mathbb{P}[Q_{P_{N,s,\mathbf{z}}}(f) \leq y_q]$. Then the median $M_{N,s,r}(f)$ is larger than y_q if and only if at
 310 least $(r+1)/2$ values are larger than y_q , and the probability that this happens is

$$311 \quad (2.5) \quad p_+(r, q) = \sum_{i=(r+1)/2}^r \binom{r}{i} (1-q)^i q^{r-i}.$$

313 Figure 1 plots $\log_{10} p_+(r, q)$ as a function of $r \in \{3, 5, \dots, 49\}$ for $q = 0.5$, $q = 0.75$ and
 314 $q = 0.9$, respectively. We see that $p_+(r, 0.5) = 0.5$ for any r and that $p_+(r, q) \approx 10^{-\gamma r}$
 315 where $\gamma \approx 0.071$ for $q = 0.75$ and $\gamma \approx 0.231$ for $q = 0.9$. These plots provide some
 316 insight on the choice of r . In particular, for fixed $q > 0.5$, doubling r squares the
 317 probability $p_+(r, q)$. Suppose for example that we want $p_+(r, q) \leq 10^{-4}$, to have a
 318 reasonable assurance that $M_{N,s,r}(f) \leq y_q$. The plot shows that the minimal value of
 319 r for this is about $r = 13$ for $q = 0.9$, and about $r = 49$ for $q = 0.75$. For a given
 320 $f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}$ and fixed N , a larger q means a larger y_q , but for a fixed q we can reduce
 321 y_q and bring it close to $I_s(f)$ by increasing N . From Theorem 2.5 with $r = 1$ and
 322 $\eta = 1 - q < 1/4$ (or its corollary with $\rho = 4(1 - q) < 1$), we have that $|y_q - I_s(f)|$
 323 is $\mathcal{O}(N^{-\alpha+\epsilon})$. In summary, for a fixed $q > 3/4$, we can decrease the error bound
 324 by increasing N and increase the probability that the bound holds by increasing r .
 325 We can also increase both q and N in a way y_q remains about the same; then the
 326 same $p_+(r, q)$ can be obtained with a reduced r . What we just said is for the upper
 327 bound $M_{N,s,r}(f) \leq y_q$, but essentially the same discussion can be made concerning
 328 the assurance that $M_{N,s,r}(f) > y_{1-q}$. In applications, the values of y_q and y_{1-q} are
 329 unknown, but our reasoning suggests that a moderate value of r , say no more than
 330 25, should be sufficient in practice, together with a large N (as large as the computing
 331 budget allows). The results of our numerical experiments support this.

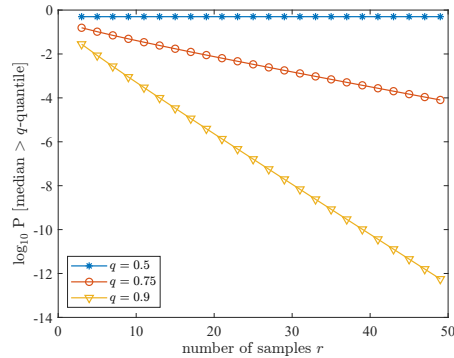


FIG. 1. Probability $p_+(r, q)$ (on a \log_{10} scale) as a function of r for $q = 0.5$ (blue), $q = 0.75$ (red) and $q = 0.9$ (yellow).

332 *Remark 2.12.* It is known that rank-1 lattice rules also work for non-periodic
 333 functions by applying the tent transformation

$$334 \quad \pi(x) = 1 - |2x - 1|$$

335 component-wise to every point in the set $P_{N,s,\mathbf{z}}$ [18, 8, 2, 16]. The same probabilistic
 336 upper bound, shown in Theorem 2.5, holds for the worst-case error of the median rule

337 built up of the tent-transformed rank-1 lattice rules in the so-called *weighted half-*
 338 *period cosine spaces* with any parameter α and weights γ . As shown in [8, Lemma 1],
 339 the half-period cosine space coincides with an unanchored Sobolev space with smooth-
 340 ness 1 when $\alpha = 1$.

341 **3. High-order polynomial lattice rules for Sobolev spaces.** We now con-
 342 sider high-order polynomial lattice point sets as defined in [9]. These point sets are
 343 well suited for performing numerical integration of smooth non-periodic functions.
 344 In what follows let b be a prime, and \mathbb{F}_b be the finite field of order b , which we
 345 identify with the set $\{0, 1, \dots, b-1\}$. Let \mathbb{N} be the set of positive integers and
 346 $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $k \in \mathbb{N}_0$ having the b -adic finite expansion $k = \kappa_0 + \kappa_1 b + \dots$, we
 347 write $k(x) = \kappa_0 + \kappa_1 x + \dots \in \mathbb{F}_b[x]$. With these ingredients, we have the following
 348 definition from [9].

349 **DEFINITION 3.1** (high-order polynomial lattice point sets). *Let $m, n \in \mathbb{N}$ with*
 350 *$m \leq n$, $p \in \mathbb{F}_b[x]$ with $\deg(p) = n$ and $\mathbf{q} = (q_1, \dots, q_s) \in (\mathbb{F}_b[x])^s$ with $\deg(q_j) < n$.*
 351 *The high-order polynomial lattice point set defined by m, n, p and \mathbf{q} consists of $N = b^m$*
 352 *points and is given by*

$$353 \quad P_{m,n,s,p,\mathbf{q}} = \left\{ \left(\nu_n \left(\frac{h(x)q_1(x)}{p(x)} \right), \dots, \nu_n \left(\frac{h(x)q_s(x)}{p(x)} \right) \right) \in [0, 1]^s \mid h = 0, \dots, b^m - 1 \right\},$$

354 where $\nu_n : \mathbb{F}_b((x^{-1})) \rightarrow [0, 1)$ is defined by

$$355 \quad \nu_n \left(\sum_{i=w}^{\infty} \frac{a_i}{x^i} \right) := \sum_{i=\max(1,w)}^n \frac{a_i}{b^i}.$$

356 *The QMC algorithm using $P_{m,n,s,p,\mathbf{q}}$ as a point set is called the high-order polynomial*
 357 *lattice rule with modulus p and generating vector \mathbf{q} . The order of this rule is defined*
 358 *as $\lfloor n/m \rfloor$.*

359 Typically, n will be a multiple of m . When $n = m$, this gives the digital net
 360 construction introduced in [34] and called polynomial lattice rule in [10]. Note that [25,
 361 30] introduced the term “polynomial lattice rule” with a slightly different definition,
 362 in which the coordinates of the points have an infinite periodic expansion and the
 363 modulus p has degree m . The construction in Definition 3.1 essentially builds a
 364 polynomial lattice point set with b^n points and uses only the first b^m points.

365 Instead of the weighted Korobov space $\mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}$, we consider the following Sobolev-
 366 type Banach space as our target space for high-order polynomial lattice rules.

367 **DEFINITION 3.2** (weighted Sobolev space). *Let $\alpha \in \mathbb{N}$, $\alpha \geq 2$, $1 \leq q \leq \infty$ and*
 368 *let $\gamma = \{\gamma_u\}_{u \subseteq \{1, \dots, s\}}$ be a set of positive weights with $\gamma_\emptyset = 1$. The weighted Sobolev*
 369 *space, denoted by $\mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}}$, is a Banach space consisting of non-periodic (in the sense*
 370 *of not necessarily periodic) smooth functions with the norm*

$$371 \quad \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} :=$$

$$372 \quad \sup_{u \subseteq \{1, \dots, s\}} \gamma_u^{-1} \left(\sum_{v \subseteq u} \sum_{\tau_{u \setminus v} \in \{1, \dots, \alpha\}^{|u \setminus v|}} \int_{[0,1]^{|v|}} \left| \int_{[0,1]^{s-|v|}} f^{(\tau_{u \setminus v}, \alpha_v, \mathbf{0})}(\mathbf{x}) \, d\mathbf{x}_{-v} \right|^q \, d\mathbf{x}_v \right)^{1/q},$$

373 where $(\tau_{u \setminus v}, \alpha_v, \mathbf{0})$ denotes the vector $\mathbf{h} \in \mathbb{N}_0^s$ such that $h_j = \tau_j$ if $j \in u \setminus v$, $h_j = \alpha$
 374 if $j \in v$, and $h_j = 0$ otherwise, and $f^{(\tau_{u \setminus v}, \alpha_v, \mathbf{0})}(\mathbf{x})$ denotes the mixed derivative of
 375 order $(\tau_{u \setminus v}, \alpha_v, \mathbf{0})$ of f . Moreover, we write $\mathbf{x}_v = (x_j)_{j \in v}$ and $\mathbf{x}_{-v} = (x_j)_{j \in \{1, \dots, s\} \setminus v}$.

376 This Sobolev space was introduced by [6] in the context of partial differential
 377 equations with random coefficients. (The original function space in [6] contains the
 378 additional parameter $r \in [1, \infty]$, and the definition of the norm has been corrected in
 379 <https://arxiv.org/abs/1309.4624>. In this paper we choose $r = \infty$, which makes the
 380 norm smallest over $r \in [1, \infty]$.) The parameter α determines the differentiability of
 381 the non-periodic functions. As for the rank-1 lattice rules for the weighted Korobov
 382 spaces, it is desirable to have good modulus p and generating vector \mathbf{q} such that the
 383 worst-case error of the corresponding high-order polynomial lattice rule for $\mathcal{F}_{s,\alpha,\gamma}^{\text{sob}}$ is
 384 small. Originally in [6], interlaced polynomial lattice rules [13, 15] were used instead
 385 of high-order polynomial lattice rules, and it was shown that the worst-case error
 386 bound of order $N^{-\alpha+\epsilon}$ with arbitrarily small $\epsilon > 0$ can be achieved by the CBC
 387 algorithm applied to interlaced polynomial lattice rules. The major advantage of
 388 interlaced polynomial lattice rules over high-order polynomial lattice rules lies in the
 389 construction cost for the CBC algorithm: for the product weights, constructing an
 390 interlaced rule requires $\mathcal{O}(\alpha s N \log N)$ operations with $\mathcal{O}(N)$ memory [13], whereas
 391 constructing a high-order rule require $\mathcal{O}(s N^\alpha \log N)$ operations with $\mathcal{O}(N^\alpha)$ memory
 392 [1].

393 However, interlaced polynomial lattice rules are not necessarily a better choice
 394 than high-order polynomial lattice rules. To construct an interlaced polynomial lat-
 395 tice rule, which relies on the digit interlacing method due to Dick [3], we must select
 396 an integer interlacing factor d , the construction cost increases linearly with d , and
 397 the resulting rule cannot exploit the smoothness of functions beyond d . This means
 398 that if $\alpha > d$, the worst-case error bound is only of order $N^{-d+\epsilon}$. High-order poly-
 399 nomial lattice rules do not explicitly require such a factor. We only need to specify
 400 the maximum precision n of the points. This n can be set as large as possible, for in-
 401 stance, with $b = 2$, we can take $n = 53$ for the double-precision floating-point format.
 402 This way, high-order polynomial lattice rules can be possibly made adaptive to the
 403 smoothness α of functions, addressing the drawback of interlaced polynomial lattice
 404 rules. Since we do not apply CBC in this paper, we prefer high-order rules over the
 405 interlaced ones.

406 In what follows, we assume that the polynomial p is irreducible and we write

$$407 \quad G_n := \{q \in \mathbb{F}_b[x] \mid q \neq 0 \text{ and } \deg(q) < n\}.$$

408 Analogously to the rank-1 lattice case in Section 2, we consider the following median
 409 high-order polynomial lattice rule for weighted Sobolev spaces. For an odd integer r ,
 410 we draw $\mathbf{q}_1, \dots, \mathbf{q}_r$ randomly and independently from the set G_n^s , and we approximate
 411 $I_s(f)$ by

$$412 \quad M_{m,n,s,p,r}(f) := \text{median} (Q_{P_{m,n,s,p,\mathbf{q}_1}}(f), \dots, Q_{P_{m,n,s,p,\mathbf{q}_r}}(f)).$$

413 The worst-case error is the random variable

$$414 \quad e^{\text{wor}}(M_{m,n,s,p,r}; \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}}) := \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}} \\ \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \leq 1}} |M_{m,n,s,p,r}(f) - I_s(f)|.$$

415 3.1. Main results for polynomial lattice point sets in Sobolev spaces.

416 We first need a few definitions and lemmas.

417 **DEFINITION 3.3** (dual polynomial lattice). *Let $m, n \in \mathbb{N}$ with $m \leq n$, $p \in \mathbb{F}_b[x]$
 418 with $\deg(p) = n$ and $\mathbf{q} \in G_n^s$. For $k \in \mathbb{N}_0$ with the b -adic finite expansion $k =$*

419 $\kappa_0 + \kappa_1 b + \dots$, we define

$$420 \quad \text{tr}_n(k) = \sum_{i=0}^{n-1} \kappa_i x^i \in G_n \cup \{0\}.$$

421 *This operator is applied component-wise to a vector. Then the set*

$$422 \quad P_{m,n,s,p,q}^\perp = \{\mathbf{k} \in \mathbb{N}_0^s \mid \text{tr}_n(\mathbf{k}) \cdot \mathbf{q} \equiv a \pmod{p} \text{ with } \deg(a) < n - m\},$$

423 *is called the dual net of the high-order polynomial lattice point set $P_{m,n,s,p,q}$.*

424 **DEFINITION 3.4** (Walsh functions). *Let us write $\omega_b := \exp(2\pi i/b)$. For $k \in \mathbb{N}_0$,*
 425 *we denote the b -adic expansion of k by $k = \kappa_0 + \kappa_1 b + \dots$. The k -th Walsh function*
 426 *$\text{wal}_k: [0, 1) \rightarrow \mathbb{C}$ is defined by*

$$427 \quad \text{wal}_k(x) := \omega_b^{\kappa_0 \xi_1 + \kappa_1 \xi_2 + \dots},$$

428 *where the b -adic expansion of $x \in [0, 1)$ is denoted by $x = \xi_1/b + \xi_2/b^2 + \dots$, which is*
 429 *understood to be unique in the sense that infinitely many of the ξ_i are different from*
 430 *$b - 1$.*

431 *For $s \geq 2$ and $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$, the s -dimensional \mathbf{k} -th Walsh function*
 432 *$\text{wal}_{\mathbf{k}}: [0, 1)^s \rightarrow \mathbb{C}$ is defined by*

$$433 \quad \text{wal}_{\mathbf{k}}(\mathbf{x}) := \prod_{j=1}^s \text{wal}_{k_j}(x_j).$$

434 It is well-known that the system of Walsh functions is a complete orthogonal system in
 435 $L_2([0, 1)^s)$, see [10, Appendix A]. The following character property of the high-order
 436 polynomial lattice point set is analogous to what is stated in Lemma 2.4.

437 **LEMMA 3.5** (character property). *For $m, n \in \mathbb{N}$ with $m \leq n$, $p \in \mathbb{F}_b[x]$ with*
 438 *$\deg(p) = n$ and $\mathbf{q} \in G_n^s$, we have*

$$439 \quad \frac{1}{b^m} \sum_{\mathbf{x} \in P_{m,n,s,p,q}} \text{wal}_{\mathbf{k}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{k} \in P_{m,n,s,p,q}^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

440 For any $f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}}$, we have the following absolutely convergent Walsh series

$$441 \quad f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{N}_0^s} \tilde{f}(\mathbf{k}) \text{wal}_{\mathbf{k}}(\mathbf{x}),$$

442 where $\tilde{f}(\mathbf{k})$ denotes the \mathbf{k} -th Walsh coefficient of f :

$$443 \quad \tilde{f}(\mathbf{k}) := \int_{[0,1)^s} f(\mathbf{x}) \overline{\text{wal}_{\mathbf{k}}(\mathbf{x})} \, d\mathbf{x}.$$

444 Note that $\tilde{f}(\mathbf{0})$ coincides with the integral $I_s(f)$. The following result on the decay of
 445 Walsh coefficients for $f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}}$ was shown in [6, Theorem 3.5].

446 **LEMMA 3.6** (decay of Walsh coefficients). *Let $\alpha \in \mathbb{N}$, $\alpha \geq 2$, $1 \leq q \leq \infty$ and*
 447 *$\gamma = \{\gamma_u\}_{u \subseteq \{1, \dots, s\}}$ be a set of positive weights with $\gamma_\emptyset = 1$. For any $f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}}$, a*
 448 *non-empty subset $u \subseteq \{1, \dots, s\}$ and $\mathbf{k}_u \in \mathbb{N}^{|u|}$, it holds that*

$$449 \quad |\tilde{f}(\mathbf{k}_u, \mathbf{0})| \leq \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \gamma_u C_\alpha^{|u|} b^{-\mu_\alpha(\mathbf{k}_u)},$$

450 where

$$451 \quad C_\alpha := \left(1 + \frac{1}{b} + \frac{1}{b(b+1)}\right)^{\alpha-2} \left(3 + \frac{2}{b} + \frac{2b+1}{b-1}\right) \max\left(\frac{2}{(2 \sin \frac{\pi}{b})^\alpha}, \max_{1 \leq \tau < \alpha} \frac{1}{(2 \sin \frac{\pi}{b})^\tau}\right)$$

452 and $\mu_\alpha(\mathbf{k}_u) := \sum_{j \in u} \mu_\alpha(k_j)$, with

$$453 \quad \mu_\alpha(k) = \sum_{i=1}^{\min(\alpha, c)} a_i$$

454 for $k \in \mathbb{N}$ whose b -adic expansion is given by $k = \kappa_1 b^{a_1-1} + \kappa_2 b^{a_2-1} + \dots + \kappa_c b^{a_c-1}$
 455 such that $c \geq 1$, $\kappa_1, \dots, \kappa_c \in \{1, \dots, b-1\}$ and $a_1 > \dots > a_c$.

456 As the second main result of this paper, we show a probabilistic upper bound
 457 on the worst-case error of our median high-order polynomial lattice rule for weighted
 458 Sobolev spaces.

459 **THEOREM 3.7.** *Let $m, n \in \mathbb{N}$ with $m \leq n$, $p \in \mathbb{F}_b[x]$ be irreducible with $\deg(p) =$
 460 n , r be odd and $\mathbf{q}_1, \dots, \mathbf{q}_r$ be chosen independently and randomly from the set G_n^s .
 461 Then, for any integer $\alpha \geq 2$ and γ , the worst-case error is bounded above by*

$$462 \quad e^{\text{wor}}(M_{m,n,s,p,r}; \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}}) \leq \inf_{1/\alpha < \lambda < 1} \left(\frac{2}{\eta(b^{\min(m,\lambda n)} - 1)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^\lambda C_\alpha^{\lambda|u|} A_{\alpha,\lambda}^{|u|} \right)^{1/\lambda}$$

463 with a probability of at least

$$464 \quad 1 - \binom{r}{(r+1)/2} \eta^{(r+1)/2},$$

465 for any $0 < \eta < 1$, where we write

$$466 \quad (3.1) \quad A_{\alpha,\lambda} = \sum_{\tau=1}^{\alpha-1} \prod_{i=1}^{\tau} \frac{b-1}{b^{\lambda i} - 1} + \frac{b^{\lambda \alpha} - 1}{b^{\lambda \alpha} - b} \prod_{i=1}^{\alpha} \frac{b-1}{b^{\lambda i} - 1}.$$

468 *Proof.* Throughout this proof, we write

$$469 \quad \tilde{r}_{\alpha,\gamma}(\mathbf{k}_u, \mathbf{0}) = \gamma_u C_\alpha^{|u|} b^{-\mu_\alpha(\mathbf{k}_u)},$$

470 for a non-empty subset $u \subseteq \{1, \dots, s\}$ and $\mathbf{k}_u \in \mathbb{N}^{|u|}$. Since any $f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}}$ has
 471 an absolutely convergent Walsh series, by applying Lemma 3.5, Lemma 2.6, Hölder's
 472 inequality and Lemma 3.6 in this order, it holds that

$$473 \quad e^{\text{wor}}(M_{m,n,s,p,r}; \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}})$$

$$474 \quad = \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}} \\ \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \leq 1}} \left| \text{median}_{1 \leq \ell \leq r} \frac{1}{b^m} \sum_{\mathbf{x} \in P_{m,n,s,p,q_\ell}} f(\mathbf{x}) - I(f) \right|$$

$$475 \quad = \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}} \\ \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \leq 1}} \left| \text{median}_{1 \leq \ell \leq r} \frac{1}{b^m} \sum_{\mathbf{x} \in P_{m,n,s,p,q_\ell}} \sum_{\mathbf{k} \in \mathbb{N}_0^s} \tilde{f}(\mathbf{k}) \text{wal}_{\mathbf{k}}(\mathbf{x}) - \tilde{f}(\mathbf{0}) \right|$$

$$\begin{aligned}
476 \quad &= \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}} \\ \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \leq 1}} \left| \text{median}_{1 \leq \ell \leq r} \sum_{\mathbf{k} \in P_{m,n,s,p,q_\ell}^\perp \setminus \{\mathbf{0}\}} \tilde{f}(\mathbf{k}) \right| \\
477 \quad &\leq \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}} \\ \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \leq 1}} \text{median}_{1 \leq \ell \leq r} \sum_{\mathbf{k} \in P_{m,n,s,p,q_\ell}^\perp \setminus \{\mathbf{0}\}} \frac{|\tilde{f}(\mathbf{k})|}{\tilde{r}_{\alpha,\gamma}(\mathbf{k})} \tilde{r}_{\alpha,\gamma}(\mathbf{k}) \\
478 \quad &\leq \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}} \\ \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \leq 1}} \text{median}_{1 \leq \ell \leq r} \left(\sup_{\mathbf{k} \in P_{m,n,s,p,q_\ell}^\perp \setminus \{\mathbf{0}\}} \frac{|\tilde{f}(\mathbf{k})|}{\tilde{r}_{\alpha,\gamma}(\mathbf{k})} \right) \left(\sum_{\mathbf{k} \in P_{m,n,s,p,q_\ell}^\perp \setminus \{\mathbf{0}\}} \tilde{r}_{\alpha,\gamma}(\mathbf{k}) \right) \\
479 \quad &\leq \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}} \\ \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \leq 1}} \left(\sup_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} \frac{|\tilde{f}(\mathbf{k})|}{\tilde{r}_{\alpha,\gamma}(\mathbf{k})} \right) \text{median}_{1 \leq \ell \leq r} \tilde{\mathcal{S}}_{\alpha,\gamma,p}(\mathbf{q}_\ell) \\
480 \quad (3.2) \quad &\leq \text{median}_{1 \leq \ell \leq r} \tilde{\mathcal{S}}_{\alpha,\gamma,p}(\mathbf{q}_\ell), \\
481 \quad &
\end{aligned}$$

482 where

$$483 \quad \tilde{\mathcal{S}}_{\alpha,\gamma,p}(\mathbf{q}) = \sum_{\mathbf{k} \in P_{m,n,s,p,q}^\perp \setminus \{\mathbf{0}\}} \tilde{r}_{\alpha,\gamma}(\mathbf{k}).$$

484 For $1/(\alpha) < \lambda \leq 1$, by using the subadditivity (2.3), we have

$$\begin{aligned}
485 \quad &\frac{1}{|G_n|^s} \sum_{\mathbf{q} \in G_n^s} (\tilde{\mathcal{S}}_{\alpha,\gamma,p}(\mathbf{q}))^\lambda = \frac{1}{(b^n - 1)^s} \sum_{\mathbf{q} \in G_n^s} \left(\sum_{\mathbf{k} \in P_{m,n,s,p,q}^\perp \setminus \{\mathbf{0}\}} \tilde{r}_{\alpha,\gamma}(\mathbf{k}) \right)^\lambda \\
486 \quad &\leq \frac{1}{(b^n - 1)^s} \sum_{\mathbf{q} \in G_n^s} \sum_{\mathbf{k} \in P_{m,n,s,p,q}^\perp \setminus \{\mathbf{0}\}} (\tilde{r}_{\alpha,\gamma}(\mathbf{k}))^\lambda \\
487 \quad &= \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} (\tilde{r}_{\alpha,\gamma}(\mathbf{k}))^\lambda \frac{1}{(b^n - 1)^s} \sum_{\substack{\mathbf{q} \in G_n^s \\ \text{tr}_n(\mathbf{k}) \cdot \mathbf{q} \equiv a \pmod{p} \\ \deg(a) < n-m}} 1. \\
488 \quad &
\end{aligned}$$

489 If $p \mid \text{tr}_n(\mathbf{k})$, the condition $\text{tr}_n(\mathbf{k}) \cdot \mathbf{q} \equiv a \pmod{p}$ trivially holds with $a = 0$ for all
490 $\mathbf{q} \in G_n^s$. Otherwise if $p \nmid \text{tr}_n(\mathbf{k})$, i.e., if there exists a non-empty subset $u \subseteq \{1, \dots, s\}$
491 such that $p \nmid \text{tr}_n(k_j)$ for all $j \in u$ and $p \mid \text{tr}_n(k_j)$ for $j \notin u$, the condition $\text{tr}_n(\mathbf{k}) \cdot \mathbf{q} \equiv a$
492 \pmod{p} is equivalent to $\text{tr}_n(\mathbf{k}_u) \cdot \mathbf{q}_u \equiv a \pmod{p}$, which itself is equivalent to

$$493 \quad \text{tr}_n(k_j)q_j \equiv a - \text{tr}_n(\mathbf{k}_{u \setminus \{j\}}) \cdot \mathbf{q}_{u \setminus \{j\}} \pmod{p},$$

494 for any $j \in u$. As we have $p \nmid \text{tr}_n(k_j)$ and we assume that p is irreducible, there
495 exists at most one $q_j \in G_n$ which satisfies the above equality for each $a \in \mathbb{F}_b[x]$ with
496 $\deg(a) < n - m$ and $\mathbf{q}_{u \setminus \{j\}} \in G_n^{|u|-1}$. Therefore, the number of $\mathbf{q} \in G_n^s$ which satisfy
497 $p \nmid \text{tr}_n(\mathbf{k})$ and $\text{tr}_n(\mathbf{k}) \cdot \mathbf{q} \equiv a \pmod{p}$ with $\deg(a) < n - m$ is bounded above by
498 the product of the number of possible choices for $a \in \mathbb{F}_b[x]$, which is b^{n-m} , and the
499 number of possible choices for $\mathbf{q}_{\{1, \dots, s\} \setminus \{j\}}$, which is $(b^n - 1)^{s-1}$. Thus it follows that

$$500 \quad \frac{1}{|G_n|^s} \sum_{\mathbf{q} \in G_n^s} (\tilde{\mathcal{S}}_{\alpha,\gamma,p}(\mathbf{q}))^\lambda$$

$$\begin{aligned}
501 & \leq \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\} \\ p|\text{tr}_n(\mathbf{k})}} (\tilde{r}_{\alpha, \gamma}(\mathbf{k}))^\lambda + \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\} \\ p \nmid \text{tr}_n(\mathbf{k})}} (\tilde{r}_{\alpha, \gamma}(\mathbf{k}))^\lambda \frac{b^{n-m}(b^n-1)^{s-1}}{(b^n-1)^s} \\
502 & \leq \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} (\tilde{r}_{\alpha, \gamma}(b^n \mathbf{k}))^\lambda + \frac{1}{b^m-1} \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} (\tilde{r}_{\alpha, \gamma}(\mathbf{k}))^\lambda \\
503 & = \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \sum_{\mathbf{k}_u \in \mathbb{N}^{|u|}} (\tilde{r}_{\alpha, \gamma}(b^n \mathbf{k}_u, \mathbf{0}))^\lambda + \frac{1}{b^m-1} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \sum_{\mathbf{k}_u \in \mathbb{N}^{|u|}} (\tilde{r}_{\alpha, \gamma}(\mathbf{k}_u, \mathbf{0}))^\lambda \\
504 & = \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^\lambda C_\alpha^{\lambda|u|} \left(\sum_{k \in \mathbb{N}} b^{-\lambda \mu_\alpha(b^n k)} \right)^{|u|} \\
505 & \quad + \frac{1}{b^m-1} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^\lambda C_\alpha^{\lambda|u|} \left(\sum_{k \in \mathbb{N}} b^{-\lambda \mu_\alpha(k)} \right)^{|u|} \\
506 & \leq \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^\lambda \frac{C_\alpha^{\lambda|u|} A_{\alpha, \lambda}^{|u|}}{b^{\lambda n|u|}} + \frac{1}{b^m-1} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^\lambda C_\alpha^{\lambda|u|} A_{\alpha, \lambda}^{|u|} \\
507 & \leq \frac{2}{b^{\min(m, \lambda n)} - 1} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^\lambda C_\alpha^{\lambda|u|} A_{\alpha, \lambda}^{|u|}, \\
508 &
\end{aligned}$$

509 where we have used the results of [14, Lemma 7] on the sums of $b^{-\lambda \mu_\alpha(k)}$ and $b^{-\lambda \mu_\alpha(b^n k)}$
510 in the third inequality, which involve $A_{\alpha, \lambda}$ given in (3.1). This gives a bound on the
511 average of $(\tilde{\mathcal{S}}_{\alpha, \gamma, p}(\mathbf{q}))^\lambda$ which holds for any $1/\alpha < \lambda \leq 1$.

512 Then, Markov's inequality ensures that, for any $0 < \eta < 1$, the event

$$513 \quad \tilde{\mathcal{S}}_{\alpha, \gamma, p}(\mathbf{q}) > \inf_{1/\alpha < \lambda < 1} \left(\frac{2}{\eta(b^{\min(m, \lambda n)} - 1)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^\lambda C_\alpha^{\lambda|u|} A_{\alpha, \lambda}^{|u|} \right)^{1/\lambda} =: \tilde{B}(\alpha, \gamma)$$

514 happens with a probability of at most η under a random choice of $\mathbf{q} \in G_n^s$. For
515 the median estimator $M_{m, n, s, p, r}$ to be larger than this bound $\tilde{B}(\alpha, \gamma)$, we must have
516 $\tilde{\mathcal{S}}_{\alpha, \gamma}(\mathbf{q}_\ell) > \tilde{B}(\alpha, \gamma)$ for at least $(r+1)/2$ vectors among $\mathbf{q}_1, \dots, \mathbf{q}_r$. The probability
517 that this happens is bounded above by

$$518 \quad \binom{r}{(r+1)/2} \eta^{(r+1)/2}.$$

519 Combining this with the bound shown in (3.2) completes the proof. \square

520 As pointed out in [6, Section 3.1], $\alpha \geq 2$ is required to ensure the convergence of
521 the infinite sum

$$522 \quad \sum_{\mathbf{k} \in P_{m, n, s, p, q}^\perp \setminus \{\mathbf{0}\}} b^{-\mu_\alpha(\mathbf{k})},$$

523 for any irreducible p and $\mathbf{q} \in G_n^s$. Thus, the case $\alpha = 1$ is not covered by our result.

524 Using Remark 2.7, we obtain the following corollary:

525 **COROLLARY 3.8.** *Let $\alpha \geq 2$, γ be a set of weights, and $n \geq \alpha m = \alpha \log_b N$. Then*
526 *for any odd $r \geq 3$, $\epsilon > 0$ and $0 < \rho < 1$, there is a constant $c_1 = c_1(\alpha, \gamma, \epsilon) > 0$ (which*
527 *depends on α , the γ_u 's and ϵ) such that*

$$528 \quad \mathbb{P} \left[e^{\text{wor}}(M_{m, n, s, p, r}; \mathcal{F}_{s, \alpha, \gamma}^{\text{sob}}) \leq \frac{c_1(\alpha, \gamma, \epsilon)}{(\rho N)^{\alpha - \epsilon}} \right] \geq 1 - \rho^{(r+1)/2} / 4.$$

529 *Proof.* Take $1/\lambda = \alpha - \epsilon$. Under the assumption on n , we have $n \geq \alpha m > m/\lambda$
 530 and then $(b^{\min(m, \lambda n)})^{-1/\lambda} = b^{-\min(m/\lambda, n)} = N^{-1/\lambda} = N^{-\alpha+\epsilon}$. Then the result follows
 531 from Theorem 3.7 and the bound (2.4) in Remark 2.7. \square

532 Thus, provided that we take n large enough, we get a convergence rate of almost
 533 $\mathcal{O}(N^{-\alpha})$ (with high probability) for any $\alpha \geq 2$. In other words, our median high-
 534 order polynomial lattice rule exploits the smoothness of functions adaptively. Note
 535 that Remark 2.11 also applies here.

536 **4. Numerical experiments.** We conclude this paper with numerical experi-
 537 ments both for rank-1 lattice rules and high-order polynomial lattice rules. The goal
 538 is to illustrate how the worst-case error for the median rule truly behaves on some
 539 concrete examples. In particular, we want to illustrate the fact that most of the
 540 possible generating vectors are a good choice, while a small minority are bad.

541 4.1. Lattice rules for periodic functions.

542 **EXAMPLE 4.1.** For our first example, we consider a weighted Korobov space with
 543 integer smoothness parameter $\alpha \geq 1$ and product weights $\gamma_u = \prod_{j \in u} \gamma_j$. The worst-
 544 case error of the rank-1 lattice rule with generating vector \mathbf{z} for that space has the
 545 explicit form

$$546 \quad \mathcal{S}_{\alpha, \gamma}(\mathbf{z}) = \left(-1 + \frac{1}{N} \sum_{\mathbf{x} \in P_{N, s, \mathbf{z}}} \prod_{j=1}^s \left[1 + \gamma_j^2 \frac{(-1)^{\alpha+1} (2\pi)^{2\alpha}}{(2\alpha)!} B_{2\alpha}(x_j) \right] \right)^{1/2},$$

547 where $B_{2\alpha}$ denotes the Bernoulli polynomial of degree 2α ; see [27] and [7, Section 5].
 548 In this artificial simple case, we know the exact optimal weights that must be taken in
 549 a CBC search for \mathbf{z} , so we can compare the median estimator with the best possible
 550 case of a CBC search.

551 We take two primes $N = 251$ and $N = 2039$, both for $s = 50$ dimensions, with
 552 $\alpha = 2$ and $\gamma_j = 1/j^3$. For each of those N , we drew 10^5 generating vectors \mathbf{z}
 553 randomly and uniformly from $\{1, \dots, N-1\}^s$, and computed $\mathcal{S}_{\alpha, \gamma}(\mathbf{z})$ for each. The
 554 left panels of Figure 2 show a histogram of the 10^5 realizations of $\log_2 \mathcal{S}_{\alpha, \gamma}(\mathbf{z})$ for each
 555 of these two cases. Each histogram provides a good estimate of the true distribution
 556 of $\log_2 \mathcal{S}_{\alpha, \gamma}(\mathbf{z})$, which is a discrete distribution because \mathbf{z} is drawn from a finite set.
 557 Interestingly, the distributions are very asymmetric and are far from smooth on the
 558 right side: some rectangles are very high while others are zero in the same area. The
 559 largest observed values are -2.4353 for $N = 251$ and -2.4967 for $N = 2039$. We can
 560 estimate from this data the q -quantiles y_q of the distribution of $\mathcal{S}_{\alpha, \gamma}(\mathbf{z})$, similar to
 561 those of the distribution of $Q_{P_{N, s, \mathbf{z}}}(f)$ in Remark 2.11. For $q = 0.75$, the corresponding
 562 empirical q -quantiles are -8.3907 for $N = 251$ and -12.0306 for $N = 2039$, while,
 563 for $q = 0.9$, they are -7.0975 for $N = 251$ and -10.3101 for $N = 2039$. These
 564 quantiles are much less than the worst observed values. These empirical results agree
 565 with the fact that only a very small proportion of the vectors \mathbf{z} are bad. Suppose we
 566 draw r random realizations of \mathbf{z} and want the median of the r corresponding values
 567 of $\mathcal{S}_{\alpha, \gamma}(\mathbf{z})$ to be larger than $y = 10^{-3} \approx 2^{-10}$ with a probability smaller than 10^{-4} .
 568 For $N = 2039$ this y equals y_q for $q \approx 0.9$, and Figure 1 shows that we can achieve
 569 approximately the target probability of 10^{-4} with $r = 13$. For a larger N , the required
 570 r is even smaller. Note that for $N = 251$, $y_p = 2^{-10}$ corresponds to some $q < 0.5$, for
 571 which the target probability of 10^{-4} cannot be achieved even for a very large r , as
 572 shown in Figure 1.

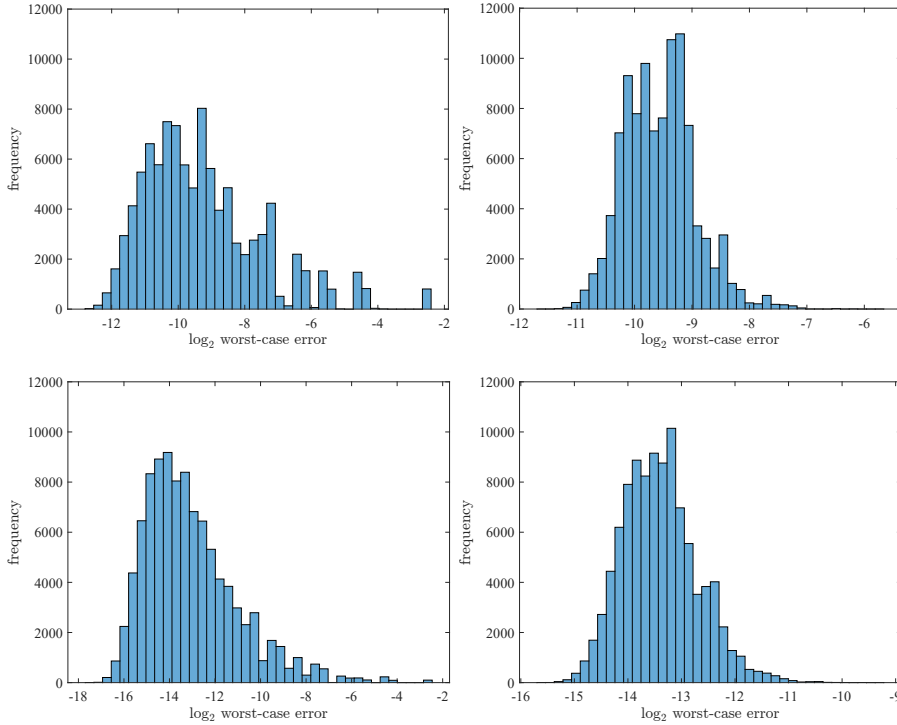


FIG. 2. Histograms of the \log_2 of the worst-case error $\mathcal{S}_{\alpha,\gamma}(\mathbf{z})$ with $\alpha = 2$ and $\gamma_j = 1/j^3$ for rank-1 lattice rules with randomly chosen generating vectors with $N = 251$ (upper panels) and $N = 2039$ (lower panels). The left panels are for a single choice ($r = 1$), while for the right panels we take the median of the worst-case error for rank-1 lattice rules with $r = 11$ randomly chosen generating vectors.

573 For the remainder of our experiments reported in this paper, we took $r = 11$.
 574 The right panels of Figure 2 show histograms of 10^5 independent realizations of
 575 $\log_2[\text{median}(\mathcal{S}_{\alpha,\gamma}(\mathbf{z}_1), \dots, \mathcal{S}_{\alpha,\gamma}(\mathbf{z}_r))]$ for randomly chosen $\mathbf{z}_1, \dots, \mathbf{z}_r$ with $r = 11$,
 576 corresponding to the cases $N = 251$ and $N = 2039$. We see that the distributions
 577 have much less variance and are more symmetric than for a single random \mathbf{z} , con-
 578 firming the fact that taking the median successfully filters (adaptively) the bad vector
 579 generators. Recall that the standard deviation of the empirical median as a function
 580 of r generally decreases as $\mathcal{O}(r^{-1/2})$. That is, increasing r decreases the noise rather
 581 slowly. For the following examples, we made additional experiments with $r = 31$
 582 to see if it would make the error plots less noisy, and we did not see much visible
 583 difference.

584 EXAMPLE 4.2. For our second example, we perform a numerical integration of
 585 the smooth, periodic function

$$586 \quad f_{\beta,\omega}^{\text{per}}(\mathbf{x}) = \prod_{j=1}^s [1 + \omega_j (g_{\beta}(x_j) - 1)],$$

587 with parameters β and ω_j , where the univariate function $g_{\beta} : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$588 \quad g_{\beta}(x) = (2\beta + 1) \binom{2\beta}{\beta} x^{\beta} (1-x)^{\beta}.$$

589 **Note that $I_s(f_{\beta,\omega}^{\text{per}}) = 1$. The function g_β has been used for periodization of non-**
 590 **periodic functions, and our test function $f_{\beta,\omega}^{\text{per}}$ belongs to the Korobov space with**
 591 **$\alpha = \beta$ when β is a positive integer, see [7, Section 5.10]. In what follows, we take**
 592 **$s = 50$ and consider the four cases that correspond to $\beta = 2$ or $\beta = 5$, and $\omega_j = 1/j^{\beta+1}$**
 593 **or $\omega_j = 1/(s - j + 1)^{\beta+1}$. We compare our median lattice rule with $r = 11$, a**
 594 **QMC rule using non-randomized Sobol' points provided by MATLAB, and the rank-**
 595 **1 lattice rule with generating vector constructed by the fast CBC algorithm with**
 596 **$\mathcal{S}_{\alpha,\gamma}(\mathbf{z})$ as a criterion, with $\alpha = 2$ and the product weights $\gamma_j = 1/j^3$. These weights**
 597 **are not optimal, but they are a good heuristic choice when $\omega_j = 1/j^{\beta+1}$. When**
 598 **$\omega_j = 1/(s - j + 1)^{\beta+1}$, on the other hand, the weights decrease in the opposite**
 599 **direction as they should: they are very large for the unimportant coordinates and**
 600 **small for the important ones. We do this to show how badly the CBC construction**
 601 **method can work when we have the wrong weights, whereas the median estimator**
 602 **does not need any knowledge about the weights to perform well. We choose N to be**
 603 **a power of 2 for Sobol' points and to be a prime close to a power of 2 for lattice point**
 604 **sets.**

605 The results for the four cases are shown in the corresponding panels of Figure 3.
 606 Both our median lattice rule and the rank-1 lattice rule constructed by the CBC
 607 algorithm can exploit the periodicity of the integrand and achieve a higher-order
 608 rate of convergence than $\mathcal{O}(1/N)$. With a good choice of the weights in the CBC
 609 algorithm, the resulting rank-1 lattice rule performs better than our median lattice
 610 rule, as shown in the left panels. However, as the right panels clearly depict, if
 611 the relative importance of each of individual variables is not correctly specified, the
 612 performance of the rank-1 lattice rule with the CBC algorithm can deteriorate and
 613 even become inferior to the QMC rule using the Sobol' points when N is not large. In
 614 contrast, our median lattice rule performs quite stably regardless of smoothness and
 615 weights.

616 To show that our median lattice rule performs well for functions of non-product
 617 forms, let us consider the additional test functions given by

$$618 \quad f_\beta^{\text{per,cyc}}(\mathbf{x}) = \frac{1}{5} \sum_{\ell=1}^5 \prod_{j=1}^{s/5} g_\beta(x_{j+s(\ell-1)/5}) \quad \text{and} \quad f_\beta^{\text{per,mod}}(\mathbf{x}) = \frac{1}{5} \sum_{\ell=1}^5 \prod_{j=1}^{s/5} g_\beta(x_{\ell+5(j-1)}),$$

619 respectively, with $\beta = 5$ and $s = 20$. We have that $I_s(f_\beta^{\text{per,cyc}}) = I_s(f_\beta^{\text{per,mod}}) = 1$ and
 620 these two integrands belong to the Korobov space with $\alpha = \beta$. The results for the two
 621 integrands are shown in the corresponding panels of Figure 4. For large N , both our
 622 median lattice rule and the rank-1 lattice rule constructed by the CBC algorithm are
 623 superior to the QMC rule using the Sobol' points. Although the difference between
 624 $f_\beta^{\text{per,cyc}}$ and $f_\beta^{\text{per,mod}}$ lies only in the ordering of variables, the convergence behavior
 625 of the rank-1 lattice rule constructed by the CBC algorithm is not consistent for these
 626 functions and a strange zig-zag pattern shows up for $f_\beta^{\text{per,mod}}$. On the contrary, our
 627 median lattice rule is not subject to the difference between the ordering of variables
 628 and performs almost equivalently.

629 **4.2. High-order polynomial lattice rules for non-periodic functions.**
 630 Our next examples concern high-order polynomial lattice rules. Here we fix the pre-
 631 cision to $n = 52$ and always use the primitive polynomial $p(x) = x^{52} + x^3 + 1$, found
 632 in [17], as the modulus of the polynomial lattice point sets.

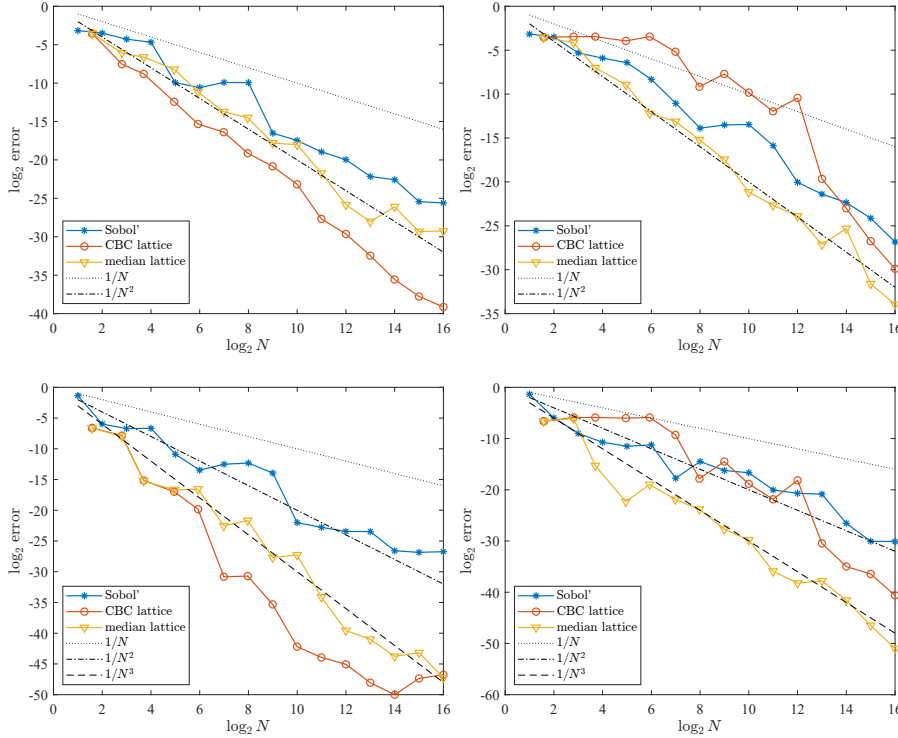


FIG. 3. Comparison of the integration error by our median lattice rule (yellow), QMC rule using Sobol' points (blue), and rank-1 lattice rule with the fast CBC algorithm (orange). The results are shown for the test function $f_{\beta,\omega}^{\text{per}}$ with the choices $\beta = 2$ and $\omega_j = 1/j^3$ (upper left), $\beta = 2$ and $\omega_j = 1/(s-j+1)^3$ (upper right), $\beta = 5$ and $\omega_j = 1/j^6$ (lower left), and $\beta = 5$ and $\omega_j = 1/(s-j+1)^6$ (lower right).

633 EXAMPLE 4.3. We first consider the two following one-dimensional test functions:

634
$$f_1^{\text{nonper}}(x) = x^3(1/4 + \log x) \quad \text{and} \quad f_2^{\text{nonper}}(x) = xe^{x/4}.$$

635 We can see that the third derivative of f_1^{nonper} is in $L_q([0, 1])$ for any $1 \leq q < \infty$,
 636 whereas the fourth derivative is not in $L_1([0, 1])$, implying that $f_1^{\text{nonper}} \in \mathcal{F}_{1,3,\gamma,q}^{\text{sob}}$ but
 637 $f_1^{\text{nonper}} \notin \mathcal{F}_{1,4,\gamma,1}^{\text{sob}}$. Thus f_1^{nonper} has a finite smoothness. On the other hand, f_2^{nonper}
 638 is obviously infinitely differentiable, so that $f_2^{\text{nonper}} \in \mathcal{F}_{1,\alpha,\gamma,q}^{\text{sob}}$ for any $\alpha \geq 2$ and
 639 $1 \leq q \leq \infty$. Note that $I_1(f_1^{\text{nonper}}) = 0$ and $I_1(f_2^{\text{nonper}}) = 16 - 12e^{1/4}$. We compare
 640 our median high-order polynomial lattice rule with $r = 11$ and QMC rules using order
 641 2 and order 3 Sobol' points constructed by the interlacing procedure of [3], with the
 642 direction numbers provided in MATLAB. To construct a Sobol' point set of order
 643 d by interlacing, we first construct a ds -dimensional Sobol point set with 2^m points
 644 (with $s = 1$ in this case) and then apply the digit interlacing procedure defined in [3]
 645 to obtain the digits of the s -dimensional points. This procedure extracts the first m
 646 digits of the ds -dimensional points and reorders them in a special way to obtain the
 647 first dm digits of the s -dimensional points.

648 The results for the two one-dimensional functions are shown in Figure 5, respec-
 649 tively. As we can see from the result for f_1^{nonper} , the QMC rule using order 2 Sobol'
 650 points cannot fully exploit the smoothness of the function and the error decays at the

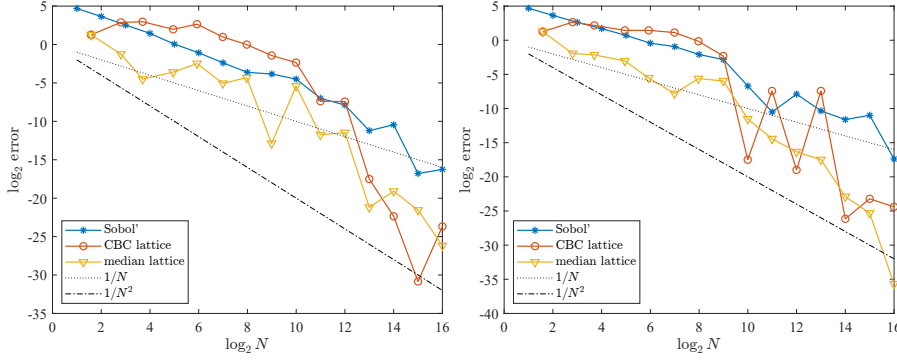


FIG. 4. Comparison of the integration error by our median lattice rule (yellow), QMC rule using Sobol' points (blue), and rank-1 lattice rule with the fast CBC algorithm (orange). The results are shown for the test functions $f_{\beta}^{\text{per,cyc}}$ (left) and $f_{\beta}^{\text{per,mod}}$ (right) with the choices $\beta = 5$ and $s = 20$.

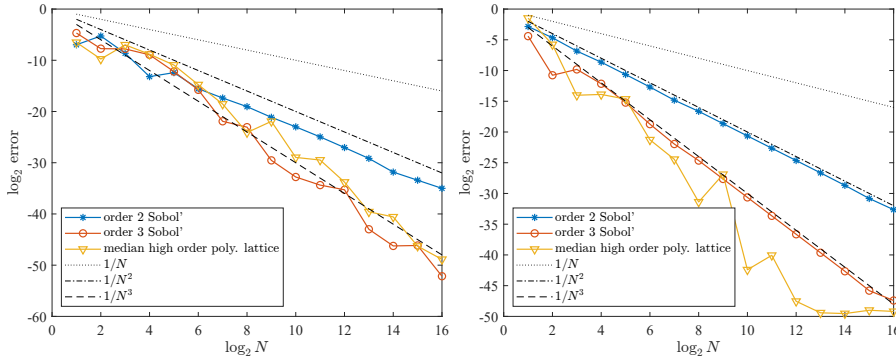


FIG. 5. Comparison of the one-dimensional integration error by our median high-order polynomial lattice rule (yellow), QMC rules using order 2 Sobol' points (blue) and order 3 Sobol' points (orange). The results are shown for the test functions f_1^{nonper} (left) and f_2^{nonper} (right), respectively.

651 rate of N^{-2} . On the other hand, the QMC rule using order 3 Sobol' points and our
 652 median high-order polynomial lattice rule can exploit the smoothness and achieves
 653 the convergence rate of N^{-3} . For the infinitely differentiable function f_2^{nonper} , the
 654 plot suggests that our median high-order polynomial lattice rule may converge even
 655 faster than N^{-3} . These numerical results show the major advantage of our proposed
 656 rule in terms of adaptivity in smoothness.

657 EXAMPLE 4.4. Finally, we consider the two multivariate non-periodic test func-
 658 tions

$$659 \quad f_{3,\omega}^{\text{nonper}}(\mathbf{x}) = \exp\left(-\sum_{j=1}^s \omega_j x_j\right) \quad \text{and} \quad f_{3,\omega,\text{flip}}^{\text{nonper}}(\mathbf{x}) = \exp\left(-\sum_{j=1}^s \omega_j x_{s-j+1}\right),$$

660 with $s = 10$ and $\omega_j = 1/(4j^4)$. It is obvious that both $f_{3,\omega}^{\text{nonper}}$ and $f_{3,\omega,\text{flip}}^{\text{nonper}}$ are infinitely
 661 differentiable and belong to $\mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}}$ with arbitrary $\alpha \geq 2$ and $1 \leq q \leq \infty$. Note that
 662 $f_{3,\omega,\text{flip}}^{\text{nonper}}$ is defined by reordering the variables of $f_{3,\omega}^{\text{nonper}}$ so that x_j is replaced by

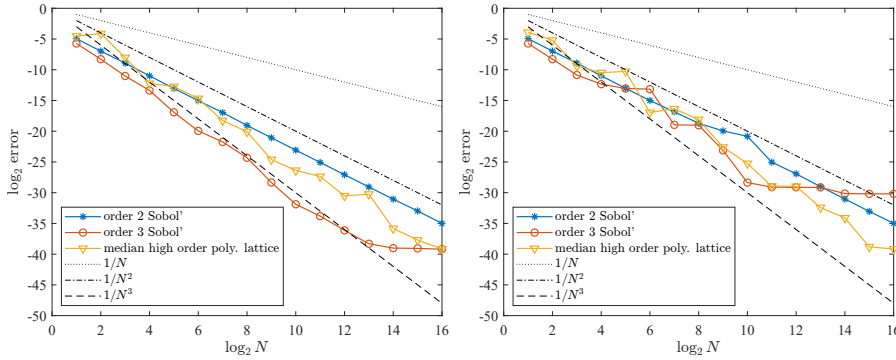


FIG. 6. Comparison of the multi-dimensional integration error by our median high-order polynomial lattice rule (yellow), QMC rules using order 2 Sobol' points (blue) and order 3 Sobol' points (orange). The results are shown for the test functions $f_{3,\omega}^{\text{nonper}}$ (left) and $f_{3,\omega,\text{flip}}^{\text{nonper}}$ (right) with the choice $\omega_j = 1/(4j^4)$.

663 x_{s-j+1} , and that we have

$$664 \quad I_s(f_{3,\omega}^{\text{nonper}}) = I_s(f_{3,\omega,\text{flip}}^{\text{nonper}}) = \prod_{j=1}^s \frac{1 - \exp(-\omega_j)}{\omega_j}.$$

665 The variables are ordered by decreasing order of importance in the first function, and
 666 by increasing order in the second one. As our median high-order polynomial lattice
 667 rule, based on random choices of generating vectors, does not care about the ordering
 668 of variables, it should perform the same for $f_{3,\omega}^{\text{nonper}}$ and $f_{3,\omega,\text{flip}}^{\text{nonper}}$.

669 The results are shown in Figure 6. Here again, we compare our median high-
 670 order polynomial lattice rule with $r = 11$ and QMC rules using order 2 and order 3
 671 interlaced Sobol' points. For the function $f_{3,\omega}^{\text{nonper}}$, our median high-order polynomial
 672 lattice rule can exploit the smoothness better than the QMC rule using order 2 Sobol'
 673 points. The QMC rule using order 3 Sobol' points exploits the smoothness of the
 674 integrand best and the error decays at the rate of N^{-3} and outperforms our median
 675 high-order polynomial lattice rule approximately by a constant factor for small N ,
 676 but this rate breaks down at around $\log_2 N = 13$ and our median rule catches up at
 677 $\log_2 N = 16$.

678 For $f_{3,\omega,\text{flip}}^{\text{nonper}}$, the situation changes. Our median high-order polynomial lattice
 679 rule is now comparable to the QMC rule using order 3 Sobol' points when N is small,
 680 and performs better for larger N . The error decays approximately at the rate of
 681 $N^{-2.5}$, which the QMC rule using order 2 Sobol' points cannot attain. The slowdown
 682 of the convergence for the QMC rule using order 3 Sobol' points might be due to
 683 the misspecification of important variables. Although x_s, x_{s-1}, \dots are the order of
 684 the relatively important variables for $f_{3,\omega,\text{flip}}^{\text{nonper}}$, we use the later coordinates of order 3
 685 Sobol' points, whose lower-dimensional projections are not well-distributed compared
 686 to the earlier coordinates. In this sense, the median high-order polynomial lattice rule
 687 is more robust and adaptive to the integrand at hand.

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