Theorem 4.3: Assume that the partitioning in (4.20) is made so that $\sigma_{\min}(\Sigma_1) > \sigma_{\max}(\Sigma_2)$. Then the subsystem (A_{11}, B_1, C) is reachable and observable.

Proof: From (4.22) it follows that

$$A_{11}^T \Sigma_1 A_{11} + A_{21}^T \Sigma_2 A_{21} - \Sigma_1 = -C_1^T C_1. \tag{4.27}$$

Assume that the subsystem (A_{11}, B_1, C_2) is not observable. Then there exists an eigenvector v with corresponding eigenvalue λ of A_{11}, such that

$$A_{11} v = \lambda v \tag{4.28}$$

$$C_1 v = 0 \tag{4.29}$$

$$\| v \| = 1. \tag{4.30}$$

Multiply (4.27) from the right by v and left by v^* to obtain

$$\left(1 - |\lambda|^2\right) v^* \Sigma_1 v = v^* A_{21}^T \Sigma_2 A_{21} v. \tag{4.31}$$

We have

$$v^* \Sigma_1 v \geqslant \sigma_{min}(\Sigma_1) \tag{4.32}$$

$$v^* A_{21}^T \Sigma_2 A_{21} v \leqslant \| A_{21} v \|^2 \sigma_{\max}(\Sigma_2). \tag{4.33}$$

Insertion of (4.32) and (4.33) into (4.31) gives

$$\left(1 - |\lambda|^2\right) \sigma_{\min}(\Sigma_1) \leqslant \| A_{21} v \|^2 \sigma_{\max}(\Sigma_2). \tag{4.34}$$

From Theorem 4.1 it follows that

$$\left\| \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} v \right\| \leqslant 1 \Leftrightarrow \| A_{11} v \|^2 + \| A_{21} v \|^2 \leqslant 1 \Leftrightarrow \| A_{21} v \|^2 \leqslant 1 - |\lambda|^2. \tag{4.35}$$

Insert (4.35) into (4.34) to get

$$\left(1 - |\lambda|^2\right) \sigma_{\min}(\Sigma_1) \leqslant \left(1 - |\lambda|^2\right) \sigma_{\max}(\Sigma_2). \tag{4.36}$$

We know that the subsystem (A_{11}, B_1, C_1) is asymptotically stable. Therefore

$$1 - |\lambda|^2 > 0$$

and

$$\sigma_{\min}(\Sigma_1) \leqslant \sigma_{\max}(\Sigma_2). \tag{4.37}$$

This contradicts the assumption of the theorem. Therefore, the subsystem is obvservable. Analogously it can be shown that it is reachable.

The following example shows that the other subsystem (A_{22}, B_2, C_2) is not necessarily controllable and observable.

Example 4.2: The system

$$x_{t+1} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} x_t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_t$$

$$y_t = (1 \quad 0) x_t$$

is asymptotically stable, reachable, observable, and balanced. The grammians are both equal to

$$\Sigma = \begin{pmatrix} \frac{16}{15} & 0 \\ 0 & \frac{4}{15} \end{pmatrix}.$$

The subsystem $(0, 1, 1)$, which corresponds to the large eigenvalue of Σ is controllable and observable in accordance to the theorem. The subsystem $(0, 0, 0)$, which corresponds to the smallest eigenvalue of Σ, is, however, neither reachable nor observable. Observe also that the balanced representation is unique, since Σ has distinct eigenvalues. Therefore, it is not possible to find an equivalent balanced representation, which is such that every subsystem is reachable and observable. It follows from Theorem 3.3 and Lemma 3.1 that this is always possible in the continuous time case.

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A Stochastic Control Approach to Group Preventive Replacement in a Multicomponent System

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***Abstract*—A group preventive replacement problem is formulated in continuous time for a multicomponent system having identical elements. The dynamic programming equation is obtained in the framework of the theory of optimal control of jump processes. For a discrete time version of the model, the numerical computation of optimal and suboptimal strategies of group preventive replacement are done. A monotonicity property of the Bellman functional (or cost-to-go function) is used to reduce the size of the computational problem. Some counterintuitive properties of the optimal strategy are apparent in the numerical results obtained.**

Notations

Throughout this paper $|A|$ denotes the cardinality of the set A. $\mathbf{1}$ is the indicator function

$$\mathbf{1}_{(\text{condition})} = \begin{cases} 1 & \text{if condition is true} \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbb{N}_0$ is the set $\mathbb{N} \cup \{0\}$, $\mathbb{R}_+^m$ is the positive orthant in $\mathbb{R}^m$. $\mathcal{P}(M)$ is the class of subsets of the set M.

I. Introduction

The aim of this paper is to formulate and solve a group preventive replacement (GPR) problem, considered in the framework of optimal stochastic control theory.

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A presentation of group preventive maintenance can be found in [1]. The GPR problem that we are considering occurs when a system consists of several elements of the same type which work under the same general conditions. All the elements have the same lifetime distribution. Only two actions are available for a single element: either replace it when it fails (emergency replacement) or replace the element before it fails (preventive replacement). The possibility to combine preventive replacements with emergency replacements in a multicomponent system can generate a substantial maintenance cost reduction if there is a large fixed cost associated with any intervention on the system, whatever be the number of elements replaced.

In [1] the GPR strategy is defined by a heuristic service rule which is clearly nonoptimal. In [2] a formulation of the GPR problem as a stochastic control problem has been proposed and a dynamic programming equation has been heuristically derived.

In Section II of the present paper, the dynamic programming equation of [2] is more rigorously obtained by using some general results of the control theory of jump processes.

In Section III a discrete time formulation of the problem is proposed and the approach for the numerical solution of this large scale problem is described.

In Section IV several numerical illustrations are fully developed to show the effect of the relative magnitudes of the fixed and the variable parts of the maintenance cost on the optimal GPR strategy. The optimal strategies are then compared with suboptimal strategies which are simpler to implement.

II. The Group Preventive Replacement Problem

We consider a system comprised of m identical elements working independently under the same conditions. At time $t_0 = 0$ every element is new; at time $T > 0$, the whole system is replaced by a new one. During the time interval $[0, T]$, if one element fails it has to be replaced immediately by a new one. This is called an emergency replacement (ER). Simultaneously with an ER the repairman can replace any number of working elements he wants. This is called the preventive replacement (PR) of a working element.

The cost of an intervention contains two parts: a fixed cost B is incurred whatever the number of elements replaced; a variable cost varies linearly with the number of elements which are replaced. Thus, the cost of an intervention where ν elements are replaced will be given by

$$B + \nu b, \qquad \text{for} \quad \nu \geqslant 1.$$

Knowing this cost structure and the failure rate function $l(r)$ for one element, where r is the age of the element, the problem is to find the optimal strategy for preventive replacement (optimal PR strategy).

We assume that ER's or PR's are performed instantaneously.

In order to establish rigorously the optimality conditions characterizing the optimal strategy, it will be convenient to formulate the problem in the framework of the theory of optimal control for jump processes. Recently, several authors [3]–[6] have obtained very general optimality conditions for the control of jump processes. The results obtained by Rishel are appropriate for the solution of the present problem.

Let us define the set

$$X = M \times \mathbb{R}_+^m$$

and consider a jump process $\boldsymbol{x} = (x(t);\ 0 \leqslant t \leqslant T)$ with value in X where $x(t) \triangleq (y(t), z(t))$, the component $y(t) \in M$ giving the identification number of the last element which has failed at or before t, while the vector $z(t) \in \mathbb{R}_+^m$ gives the age of each element just after the last intervention of the repairman.

The set U of possible actions describes the choices of emergency and preventive replacements that the repairman can do during an intervention. We shall describe an element u of U as a vector $(u^j)_{j \in M}$ where each component u^j is a subset of M which must contain j as an element.

$$j \in u^j \in \mathcal{P}(M).$$

When element j fails, all elements contained in u^j are replaced. Thus, u is already defined as a policy telling the repairman which working element he has to replace preventively, knowing that an emergency replacement is necessary for the element j.

For all $t \in [0, T]$, let $n(t)$ be the number of jumps during $[0, t]$. Define the jump times $\tau_1, \tau_2, \cdots, \tau_n, \cdots$ by letting $\tau_0 = 0$ and τ_n be the time of the nth jump if $n(T) \geqslant n$ or $\tau_n = T$ if $n(T) < n$. Let

$$x_n \triangleq x(\tau_n). \tag{1}$$

Denote by X_n the random sequence

$$X_n \triangleq (x_0, \tau_1, x_1, \tau_2, \cdots, \tau_n, x_n). \tag{2}$$

Rishel [6] has shown that the controlled jump process is determined if one has given the family of conditional jump rate functions

$$q(t \mid X_n, u) \triangleq \lim_{dt \to 0^+} \frac{P[t \leqslant \tau_{n+1} < t + dt \mid X_n, u]}{dt\, P[t \leqslant \tau_{n+1} \mid X_n, u]}, \qquad t \geqslant \tau_n,\ u \in U \tag{3}$$

and the family of conditional state jump probabilities

$$\pi(A \mid X_n, \tau_{n+1}, u) \triangleq P[x_{n+1} \in A \mid X_n, \tau_{n+1}, u], \qquad A \subset X,\ u \in U. \tag{4}$$

A control is a functional $u(t, X_n)$ on the past of the process, with value in U. We notice that, given the information X_n, the action $u(t, X_n)$ has no effect on the jump rate at t.

Let us define

$$y_n \triangleq y(\tau_n), \tag{5}$$

$$z_n = (z_n^j)_{j \in M} \triangleq z(\tau_n). \tag{6}$$

Thus, following the assumption of independence and given the failure rate function $l(r)$ for one element, the jump rate at time t is given by

$$q(t \mid X_n) = \sum_{j \in M} l(z_n^j + t - \tau_n). \tag{7}$$

Once a jump has occurred, the state jump probabilities now depend on u.

Given z_n, τ_{n+1}, and u, let us define the vector $\xi(j, u) \in \mathbb{R}^m$ by

$$\begin{aligned} &\xi^i(j, u) = 0 \qquad \text{if} \quad i \in u_j \\ &\xi^i(j, u) = z_n^i + \tau_{n-1} - \tau_n \qquad \text{if} \quad i \notin u_j. \end{aligned} \tag{8}$$

Then we have

$$P[x_{n-1} = (j, \xi(j, u)) \mid X_n, \tau_{n+1}, u] = \frac{l(z_n^j + \tau_{n+1} - \tau_n)}{\sum_{i=1}^{m} l(z_n^i + \tau_{n+1} - \tau_n)} \qquad \forall j \in M. \tag{9}$$

For all subsets A of $M \times \mathbb{R}^m$ not containing an element $(j, \xi(j, u))$ the probability (6) is equal to zero. Thus, at any time t, only m state jumps are possible, given an action u and an age vector z_n at $\tau_n < t$.

Equations (7)–(9) define a controlled jump process associated with the GPR problem. A cost functional $\varphi(T, X_n)$, giving the amount paid at the terminal time T if $n(T) = n$, is defined by

$$\varphi(T, X_n) \triangleq \sum_{k=1}^{n} \left(B + b \sum_{j \in M} \mathbf{1}_{(z_k^j = 0)} \right) e^{-\rho t_k} \tag{10}$$

since, at each jump k, the fixed cost B is incurred and the unit cost b is multiplied by the number of elements replaced ($\Sigma_{j \in M} \mathbf{1}_{(z_k^j = 0)}$). ρ is the discount rate.

The control problem is to pick the control $u(t, X_n)$ in the admissible class so that

$$E[\varphi(T, X_{n(T)})] \tag{11}$$

is minimized.

The optimal control thus obtained will produce an optimal GPR strategy.

Notice that according to (7) and (9) we can write

$$q(t, \tau_n, z_n) = q(t \mid X_n)$$

and

$$P[x_{n-1}=(j,\xi(j,u))|\tau_n,z_n,\tau_{n-1},u] = P[x_{n+1}=(j,\xi(j,u))|X_n,\tau_{n+1},u].$$

Thus, the control can be restricted to a functional $u(t,\tau_n,z_n)$.

Using the dynamic programming optimality conditions given by Rishel [6], we can say that an optimal control $u^*(t,\tau_n,z_n)$ is such that there exists a function $V(t,\tau_n,z_n)$ absolutely continuous in t such that

$$V(T,\tau_n,z_n)\equiv 0 \tag{12}$$

$$-\frac{\partial}{\partial t}V(t,\tau_n,z_n)=\min_{u\in U}\left[\sum_{j=1}^{m} l\left(z_n^j+t-\tau_n\right)\cdot\left\{e^{-\rho t}\left(B+b|u_j|\right)+V(t,t,\xi(j,u))-V(t,\tau_n,z_n)\right\}\right] \tag{13}$$

where the optimal GPR strategy $u^*(t,\tau_n,z_n)$ solves the right-hand side minimization. The dynamic programming equation (13) also yields a sufficient condition for optimality.

When looking for a solution of (12) and (13) we could take advantage of the fact that the age at time t of the element j is given by

$$r_j=z_n^j+t-\tau_n,\qquad j\in M. \tag{14}$$

Thus, defining the vector $r=(r_j)_{j\in M}$, we could restrict the function V to the class of functions satisfying

$$V(t,\tau_n,z_n)\equiv W(t,r) \tag{15}$$

whenever τ_n, z_n satisfy (14) and where $W(t,r)$ is a given function.

Clearly, the following relation will thus hold:

$$\frac{\partial V}{\partial t}=\frac{\partial W}{\partial t}+\sum_{j\in M}\frac{\partial W}{\partial r_j}. \tag{16}$$

With this restricted class of function the sufficient conditions (12) and (13) become

$$W(T,r)\equiv 0 \tag{17}$$

$$-\frac{\partial W}{\partial t}(t,r)=\sum_{j\in M}\frac{\partial W(t,r)}{\partial r_j}+\sum_{j=1}^{m} l(r_j)\min_{u_j\in U_j}\left\{\left(B+b\ u_j|\right)e^{-\rho t}-W\left(t,\gamma(r,u_j)\right)-W(t,r)\right\} \tag{18}$$

where $\gamma(r,u_j)$ is the vector defined by

$$\gamma(r,u_j)\triangleq\left(\gamma^i(r,u_j)\right)_{i\in M}\in\mathbb{R}^m \qquad \text{with}\begin{cases}\gamma^i(r,u_j)=0 & \text{if } i\in u_j\\ \gamma^i(r,u_j)=r_i & \text{if } i\notin u_j.\end{cases} \tag{19}$$

Going from (12), (13) to (17), (18) is going from optimality conditions which involve a single differential equation in t to a partial differential equation.

It is not clear which system will be easier to solve numerically.

As the (y,r) process is Markovian, the optimality conditions (17), (18) will have a discrete time counterpart related to the theory of the control of Markov chains. This is the approach followed in the next section.

III. A Discrete Time Formulation

We consider now the case where the system is observed at discrete sampled times. Assume that the time interval $[0,T]$ is divided into n subintervals of length Δt. Thus, the sampled times of observation are

$$t_0=0,\quad t_1=\Delta t,\cdots,t_\sigma=\sigma\Delta t,\cdots,T=n\Delta t.$$

We assume that ER's and PR's are possible only at sampled times. Denote by t_σ^+ the time t_σ immediately after the replacements (if any).

The replacement decision at time t_σ, if any, will generate an instantaneous state transition. More precisely, at a sampled time t_σ, the state of the system is a vector $r=(r_j)_{j\in M}\in(\mathbb{N}\cup\{d\})^m$. If $r_j=d$, this means that the element j is observed to be in a failure state at time t_σ. If $r_j\in\mathbb{N}$, then the element j has the age $r_j\Delta t$.

At time t_σ^-, the state of the system is a vector $\xi=(\xi_j)_{j\in M}\in(\mathbb{N}_0)^m$ whose component ξ_j is given by

$$\xi_j=\begin{cases}0 & \text{if the element } j \text{ has been replaced at } t_\sigma\\ r_j & \text{otherwise.}\end{cases}$$

We assume again that an element which has been observed to be in a state of failure d at time t_σ must be replaced immediately.

Thus, the control action at a sampled time t_σ is the subset R of elements replaced. This set must contain, as a subset, the set H of failed elements.

If an element j is in state $\xi_j\in\mathbb{N}_0$ at time t_σ^+, then the conditional probability that it will fail during the next time period of length Δt given that it has not failed during $(t_\sigma-\xi_j\Delta t,t_\sigma]$ is given by

$$\Delta Q(\xi_j)=1-\exp\left[-\int_0^{\Delta t} l\left(\xi_j\Delta t+\zeta\right)d\zeta\right]=l\left(\xi_j\Delta t\right)\Delta t-o(\Delta t) \tag{20}$$

where $o(\Delta t)$ is such that $\lim_{\Delta t\to 0}o(\Delta t)/\Delta t=0$.

From now on, we assume an increasing failure rate (IFR), and so is $\Delta Q(\xi_j)$, as a function of ξ_j. (IFR is a standard assumption in reliability theory and it corresponds to adverse aging. See [9], [1].)

Given the state vector ξ at t_σ^+, the conditional probability that the subset $G\subset M$ will contain all the elements which will fail during $(t_\sigma,t_{\sigma+1}]$ and only these is given by

$$P(G|\xi)=\prod_{i\in G}\Delta Q(\xi_i)\prod_{j\in M-G}\left(1-\Delta Q(\xi_j)\right). \tag{21}$$

We shall look for the solution of the GPR problem formulated on this discrete sampled time set and use this solution as an approximation to the exact solution of the continuous time GPR problem.

In a discrete time setting, there is no theoretical difficulty to assume that a preventive replacement can be made even if no failure has occurred (in a continuous time setting this would have led to an impulse control problem). It could be shown, however, that with the chosen cost structure, preventive replacements are never made in the absence of failures [13].

Let us assume a discount factor $\beta\in[0,1]$ per time period Δt and define the cost-to-go function $J_k(r)$ as the total minimal expected cost given that we are at time

$$t_\sigma=T-k\Delta t \tag{22}$$

and that the system is observed to be in state r.

We will use the notation e_i for the ith unit vector $(0,\cdots,1,\cdots,0)\in\mathbb{R}^m$, and write de_i for the vector whose ith component is equal to d, the other components being zeros.

By a standard dynamic programming approach (cf. [7], [8]), we obtain the following equations defining the functions J_k, $k=0,\cdots,m$:

$$Z_0\equiv J_0\equiv 0 \tag{23}$$

and for $k=1,\cdots,m$

$$\begin{cases}J_k(r)=\min_{\{R\ H\subset R\subset M\}}\left\{B+b\ R|+Z_k\left(\sum_{i\in M-R}r_ie_i\right)\right\} & \text{if } H\neq\phi\\ J_k(r)=Z_k(r) & \text{if } H=\phi\end{cases} \tag{24}$$

where

$$H\triangleq\{j|r_j=d\}$$

is the set of failed elements at time $t_\sigma=T-k\Delta t$ and

$$Z_k(r) \triangleq \beta \sum_{G \subset M} P(G|r) J_{k-1}\left(\sum_{j \in G} de_j + \sum_{j \in M-G} (r_j+1)e_j \right). \quad (25)$$

The equations (23)–(25) are clearly the discrete time counterpart of (17)–(19).

At time T, the age of each element can take $n+1$ possible values (assuming that $r=0$ at time t_0) and so there are $(n+1)^m$ possible states of the system.

We can define an order on $\mathbb{N} \cup \{d\}$ by setting $d > k \ \forall k \in \mathbb{N}$. As the elements are identical, it is possible to reduce the number of possible states of the system by considering that the components of r are always ranked by decreasing order of magnitude:

$$r_1 \geqslant r_2 \geqslant \cdots \geqslant r_m. \quad (26)$$

This is equivalent to saying that the name of an element is no longer recorded. The number of possible states is now the number of combinations with repetitions of m elements among $n+1$, which is given by

$$K_{n+1}^m = C_{n+m}^m = \frac{(n+m)!}{m!\,n!}. \quad (27)$$

In practice, there may be an age which can be attained by an element only with a very small probability. In that case, to reduce further the number of possible states, we may approximate $\Delta Q(r_j)$ by a function of the form

$$p(r_j) = \begin{cases} \Delta Q(r_j) & \forall r_j \leqslant a \\ \Delta Q(a) & \forall r_j > a \end{cases} \quad (28)$$

where $a \leqslant n$. Hence, an element of age greater than a is equivalent to an element of age a.

This reduces the number of possible states of the system to

$$K_{a+1}^m = C_{a+m}^m = \frac{(m+a)!}{m!\,a!}. \quad (29)$$

Equation (27) then becomes

$$P(G|r) = \prod_{i \in G} p(r_i) \cdot \prod_{j \in M-G} (1 - p(r_j)). \quad (30)$$

In another way, it is possible to limit the life of any element to $a+1$ periods by taking

$$p(a) = 1. \quad (31)$$

It means that any element of age $a+1$ is replaced immediately.

In (24), for a given H, we have $2^{m-|H|}$ possible candidates for R. This number can be very large if there are many elements. However, since we assume that $\Delta Q(x)$ is not decreasing in x, it seems intuitively clear that the elements preventively replaced, if any, will always be the oldest elements. A proof of this assertion can be found in [13].

Hence, choosing R reduces to choosing a number η, $0 \leqslant \eta \leqslant m - H$ and replacing the η oldest elements. The number of candidates for R is thus reduced to $m-1-|H|$.

Notice that at a sampled time t_σ, $\sigma \geqslant 1$, no element can be of age 0. The set of all possible states of the system at that time is thus

$$R = \left\{ r \in \{1,2,\cdots,a,d\}^m \,|\, r_1 \geqslant r_2 \geqslant \cdots \geqslant r_m \right\}. \quad (32)$$

At t_σ^-, $\sigma \geqslant 0$, the set of all possible states of the system is

$$R_0 = \left\{ r \in \{0,1,\cdots,a\}^m \,|\, r_1 \geqslant r_2 \geqslant \cdots \geqslant r_m \right\}. \quad (33)$$

In the light of the preceding remarks, the dynamic programming algorithm can be reformulated as follows:

1) Set, $\forall r \in R$:

$$J_0(r) = 0. \quad (34)$$

2) For $k = 1, \cdots, n$:

$\forall \xi \in R_0$, set

$$Z_k(\xi) = \beta \sum_{G \subset M} P(G|\xi) J_{k-1}\left(\operatorname{ord}\left(\sum_{J \in G} de_j + \sum_{J \in M-G} (\xi_j+1)e_j \right) \right) \quad (35)$$

$\forall r \in R \cup R_0$, set

$$J_k(r) = \begin{cases} Z_k(r) & \text{if } r \in X_0 \\ B + \min\limits_{p \leqslant \theta \leqslant m} \left\{ \theta b + Z_k\left(\operatorname{ord}\left(\sum\limits_{i=\theta-1-p}^{m-p} r_i e_i \right) \right) \right\} & \text{otherwise} \end{cases} \quad (36)$$

where

$$p \triangleq \max\{i \,|\, r_i = d\} \quad (37)$$

represents the number of failed elements and $\operatorname{ord}(s)$ is the vector $s \in \mathbb{R}^m$ whose components have been rearranged by decreasing order.

Let $\theta_k(r)$ be the value of θ for which the minimum is attained in (36). It represents the optimal number of elements to replace when we are in state x at time $T - k\Delta t$.

Another reduction of the computation time is possible by using a result proved in [13] which states that if r and $\tilde{r}$ are in R, $\tilde{r} \geqslant r$ and $\theta_k(r) = m$, then $\theta_k(\tilde{r}) = m$.

For $m = 1$, that result is the well-known control-limit rule (see [7, Theorem 6.9]). But for $m > 1$ and $\theta_k(r) < m$, $\tilde{r} \geqslant r$ does not imply $\theta_k(\tilde{r}) \geqslant \theta_k(r)$. A counterexample will be given in Section IV.

However, one could restrict beforehand the class of admissible strategies to monotone strategies (i.e., $\theta_k(r)$ is nondecreasing with respect to r) and thus look for suboptimality instead of optimality. In Section IV, this approach is discussed.

Consider finally the case where Δt is fixed and $T = \infty$. Let $J(r)$ be the expected discounted cost over an infinite horizon when $r \in R \cup R_0$ is the state of the system. Assume $\beta < 1$. From Proposition 1 on p. 227 of [8], we have, $\forall r \in R \cup R_0$:

$$J(r) = \lim_{k \to \infty} J_k(r) \leqslant \frac{B + bm}{1-\beta} < \infty. \quad (38)$$

According to Proposition 2 on p. 229 of [8], there exists a stationary optimal strategy $\{u^*(r)\}_{x \in X}$ telling which elements to replace when the system is observed to be in state r. Moreover, from (38) and (34)–(37) it could be shown (see [13]) that $J(r)$ is a nondecreasing function of each component r_j and that the optimal stationary control law will still have all the properties stated above for the finite horizon optimal control law.

According to Proposition 4 on page 237 of [8], the function J_* obtained by the following algorithm is such that

$$\max_{r \in R} |J_*(r) - J(r)| \leqslant \epsilon.$$

Algorithm:

1) Set $k = 0$ and define J_0 as in (34).

2) Repeat n times: $k := k-1$ and calculate $\{Z_k(r)\}_{r \in R_0}$, $\{J_k(r)\}_{r \in R \cup R_0}$, and $\{\theta_k(r)\}_{r \in R}$ as in (35)–(36).

3) Set

$$\gamma_1 = \frac{\beta}{1-\beta} \min_{r \in R_0} (J_k(r) - J_{k-1}(r))$$

$$\gamma_2 = \frac{\beta}{1-\beta} \max_{r \in R_0} (J_k(r) - J_{k-1}(r)).$$

4) If $\gamma_2 - \gamma_1 > \epsilon$ return to step 2).

5) Set

$$J_*(r) = J_k(r) - \frac{\gamma_1 \top \gamma_2}{2} \qquad \forall r \in R_0$$

$$\theta_*(r) = \theta_k(r) \qquad \forall r \in R.$$

Stop.

IV. Numerical Illustrations

In this section various examples are worked out. The analysis has been restricted to the infinite time horizon case.

A. Example 1

Consider $m = 6$ elements, $\beta = 0.95$, a fixed cost $B = 8.0$, and a unit cost of $b = 6.0$ for each replaced element. The probability $p(t)$ that an element of age t will fail during the next period is given in Table I.

We thus have $a = 4$. After 50 iterations (CPU time = 131 s on a cyber 173), we obtain the solution depicted in Table II, where the maximum error on the expected cost-to-go is less than $\epsilon = 0.01$. For each possible observed state of the system, Table II indicates the best action to do. It is already known that when there is no failure, we do no PR. So, these states without failed elements do not have to appear in the table. The expected discounted cost-to-go starting with a new system is $J_*(0) = 274.49$.

Notice that in state (33111*) we replace only the failed element, but that in state (31111*) we replace two elements! Hence, $s \geqslant r$ does *not* imply that $\theta_*(s) \geqslant \theta_*(r)$.

This counterintuitive result can be explained as follows: when the system is in state (31111*), replacing two elements leads to a state where all elements are not old, whereas when the system is in state (33111*), the same replacement decision would leave one element at age 3 and, thus, a rather high probability of failure. To lower significantly the probability of failure in that latter state, we must replace three elements. But this is too expensive; hence, the best decision is to replace none of the elements of age 3 and wait for the next sampled time.

B. Comparison with Suboptimal Solutions

One simple suboptimal strategy is the following:

1) Do no PR's when all elements are operative.

2) When there is at least one failure, replace only the failed elements.

This strategy will be called NOPR (no preventive replacements).

Another suboptimal strategy consists of the following:

1) Do no PR's when all elements are operative.

2) When there is at least one failure, replace all failed elements and all those elements whose ages are greater or equal to a certain threshold a_*, which is independent of the observed state of the system.

The number a_* is chosen in the set $\{1, 2, \cdots, a+1\}$. Such a strategy will be called a FAT strategy (fixed age threshold). Notice that a FAT strategy is a particular monotone strategy.

Adopting the NOPR strategy corresponds to taking $a_* = a + 1$. Hence, there exists a number $a_* \in \{1, \cdots, a+1\}$ such that starting with a new system and using the corresponding FAT strategy will yield an expected discounted cost-to-go not greater than the one obtained when using the NOPR strategy.

The following example is an illustration of what can be the percentage of increase of the expected cost when using NOPR or FAT instead of the optimal strategy.

C. Example 2

Consider a system of $m = 6$ elements, each having a Gamma (4, 1) lifetime distribution. Thus, $\forall t \in \mathbb{N}$

$$\Delta Q(t) = \frac{F(t-1) - F(t)}{1 - F(t)}$$

where

$$F(t) = \frac{1}{3!} \int_0^t z^3 e^{-z}\, dz.$$

TABLE I
Definition of the Probabilities of Failure

t	p(t)
0	.05
1	.10
2	.20
3	.40
≥ 4	.90

The failure rate $\Delta Q(t)$ can be approximated reasonably (taking $a = 7$) by

$$p(t) = \begin{cases} \Delta Q(t) & t \leqslant 7 \\ \Delta Q(7) & t > 7. \end{cases}$$

We then obtain a definition of the probabilities of failure (see Table III).

For $\beta = 0.90$, $b = 1.0$ and for different values of B, the expected discounted cost-to-go starting with a new system ($J_*(0)$) has been computed using 1) the algorithm of Section III, which gives the optimal strategy, 2) NOPR, and 3) FAT with different values of a_*. Table IV gives the results of this computation.

Notice that

1) The cost reduction in doing PR's increases as the ratio B/b of the fixed cost over the unit cost increases.

2) The optimal value of a_* for the strategy decreases as B/b increases.

3) The cost-to-go when using the optimal FAT strategy is in general not very much higher than when using the optimal strategy (1.3 percent increase in the worst case).

This numerical illustration suggests that, in a real-life large-scale problem, a FAT suboptimal PR strategy could be a good substitute to the optimal closed loop strategy. This could, however, be true only for systems observed over an infinite time horizon. When the time horizon is finite the age threshold should at least depend on time.

For this example, the values of $J_*(0)$ for the suboptimal strategies have been computed by using the dynamic programming algorithm of Section III, in which the strategy was fixed [modifying (36)]. Hence, the computational times were almost the same. However, for a very large scale problem, it is clear that the optimal value of a_* for a FAT strategy can be found by using a linear search combined with simulation, whereas due to the curse of dimensionality, the algorithm of Section II would be impossible to use.

For $t > 7$, $p(t)$ has been approximated by $p(7) = 0.482$. Let us see what happens now, for the case $B = 3$, if $p(7)$ is raised to 0.6 or 0.8.

We can see, in Table V, that the optimal value of a_* changes with $p(7)$. However, the value of $J_*(0)$ does not change very much. Thus, the approximation of $\Delta Q(t)$ by $p(t)$ seems reasonable.

V. Conclusion

Multicomponent systems are commonplace in practice, whereas the theory of optimal maintenance is mainly directed towards single component systems (cf. the recent survey of [9]). The results obtained in previous sections show that an optimal GPR strategy can be devised through the use of modern stochastic control theory in the case where all components have the same lifetime distribution.

The problem considered in this paper is related to the problem considered by Vergin in [10]. However, the methodology we have used permitted us to solve a much larger problem than the one considered in [10].

Two extensions of this work can be considered:

1) The actions available for maintenance of one element could refer to a larger variety of repair and overhaul activities. This would necessitate a more detailed description of the state of wear of an element than the one given by the age of the element. A first attempt to generalize the model in this direction can be found in [2].

2) The elements could have distinct lifetime characteristics. This is the case for many systems like trucks [11], aircraft engines, etc. There is no

TABLE II
SOLUTION FOR THE NUMERICAL EXAMPLE

Observed ages of the elements (* indicates failure) r	Number of elements to replace preventively $\theta_*(r)-\lvert H\rvert$	Total cost of the intervention $B+b\theta_*(r)$	Description of the system after the intervention ξ	Expected Discounted cost to go $J_*(\xi)$
* * * * * *		44.00	0 0 0 0 0 0	274.49
1 * * * * *		38.00	1 0 0 0 0 0	276.85
2 * * * * *		38.00	2 0 0 0 0 0	279.47
3 * * * * *	1	44.00	0 0 0 0 0 0	274.49
4 * * * * *	1	44.00	0 0 0 0 0 0	274.49
1 1 * * * *		32.00	1 1 0 0 0 0	278.99
2 1 * * * *		32.00	2 1 0 0 0 0	281.56
3 1 * * * *	1	38.00	1 0 0 0 0 0	276.85
4 1 * * * *	1	38.00	1 0 0 0 0 0	276.85
2 2 * * * *		32.00	2 2 0 0 0 0	283.59
3 2 * * * *	1	38.00	2 0 0 0 0 0	279.47
4 2 * * * *	1	38.00	2 0 0 0 0 0	279.47
3 3 * * * *		32.00	3 3 0 0 0 0	286.35
4 3 * * * *	2	44.00	0 0 0 0 0 0	274.49
4 4 * * * *	2	44.00	0 0 0 0 0 0	274.49
1 1 1 * * *		26.00	1 1 1 0 0 0	280.92
2 1 1 * * *		26.00	2 1 1 0 0 0	283.42
3 1 1 * * *	1	32.00	1 1 0 0 0 0	278.99
4 1 1 * * *	1	32.00	1 1 0 0 0 0	278.99
2 2 1 * * *		26.00	2 2 1 0 0 0	285.50
3 2 1 * * *	1	32.00	2 1 0 0 0 0	281.56
4 2 1 * * *	1	32.00	2 1 0 0 0 0	281.56
3 3 1 * * *		26.00	3 3 1 0 0 0	288.52
4 3 1 * * *	2	38.00	1 0 0 0 0 0	276.85
4 4 1 * * *	2	38.00	1 0 0 0 0 0	276.85
2 2 2 * * *		26.00	2 2 2 0 0 0	287.08
3 2 2 * * *		26.00	3 2 2 0 0 0	288.96
4 2 2 * * *	1	32.00	2 2 0 0 0 0	283.59
3 3 2 * * *		26.00	3 3 2 0 0 0	290.03
4 3 2 * * *	2	38.00	2 0 0 0 0 0	279.47
4 4 2 * * *	2	38.00	2 0 0 0 0 0	279.47
3 3 3 * * *		26.00	3 3 3 0 0 0	290.58
4 3 3 * * *		26.00	4 3 3 0 0 0	291.89
4 4 3 * * *		26.00	4 4 3 0 0 0	292.11
4 4 4 * * *		26.00	4 4 4 0 0 0	292.14
1 1 1 1 * *		20.00	1 1 1 1 0 0	282.62
2 1 1 1 * *		20.00	2 1 1 1 0 0	285.03
3 1 1 1 * *	1	26.00	1 1 1 0 0 0	280.92
4 1 1 1 * *	1	26.00	1 1 1 0 0 0	280.92
2 2 1 1 * *		20.00	2 2 1 1 0 0	287.09
3 2 1 1 * *		20.00	3 2 1 1 0 0	289.26
4 2 1 1 * *	1	26.00	2 1 1 0 0 0	283.42
3 3 1 1 * *		20.00	3 3 1 1 0 0	290.43
4 3 1 1 * *	2	32.00	1 1 0 0 0 0	278.99
4 4 1 1 * *	2	32.00	1 1 0 0 0 0	278.99
2 2 2 1 * *		20.00	2 2 2 1 0 0	288.79
3 2 2 1 * *		20.00	3 2 2 1 0 0	290.85
4 2 2 1 * *	1	26.00	2 2 1 0 0 0	285.50
3 3 2 1 * *		20.00	3 3 2 1 0 0	292.18

Observed ages of the elements (* indicates failure) r	Number of elements to replace preventively $\theta_*(r)-\lvert H\rvert$	Total cost of the intervention $B+b\theta_*(r)$	Description of the system after the intervention ξ	Expected Discountes cost to go $J_*(\xi)$
4 3 2 1 * *	2	32.00	2 1 0 0 0 0	281.56
4 4 2 1 * *	2	32.00	2 1 0 0 0 0	281.56
3 3 3 1 * *		20.00	3 3 3 1 0 0	292.83
4 3 3 1 * *		20.00	4 3 3 1 0 0	294.21
4 4 3 1 * *		20.00	4 4 3 1 0 0	294.44
4 4 4 1 * *		20.00	4 4 4 1 0 0	294.48
2 2 2 2 * *		20.00	2 2 2 2 0 0	290.06
3 2 2 2 * *		20.00	3 2 2 2 0 0	291.93
4 2 2 2 * *	1	26.00	2 2 2 0 0 0	287.08
3 3 2 2 * *		20.00	3 3 2 2 0 0	293.20
4 3 2 2 * *	1	26.00	3 2 2 0 0 0	288.96
4 4 2 2 * *		20.00	4 4 2 2 0 0	295.39
3 3 3 2 * *		20.00	3 3 3 2 0 0	293.95
4 3 3 2 * *		20.00	4 3 3 2 0 0	295.38
4 4 3 2 * *		20.00	4 4 3 2 0 0	295.72
4 4 4 2 * *		20.00	4 4 4 2 0 0	295.79
3 3 3 3 * *		20.00	3 3 3 3 0 0	294.41
4 3 3 3 * *		20.00	4 3 3 3 0 0	295.49
4 4 3 3 * *		20.00	4 4 3 3 0 0	295.74
4 4 4 3 * *		20.00	4 4 4 3 0 0	295.80
4 4 4 4 * *		20.00	4 4 4 4 0 0	295.81
1 1 1 1 1 *		14.00	1 1 1 1 1 0	284.11
2 1 1 1 1 *		14.00	2 1 1 1 1 0	286.39
3 1 1 1 1 *	1	20.00	1 1 1 1 0 0	282.62
4 1 1 1 1 *	1	20.00	1 1 1 1 0 0	282.62
2 2 1 1 1 *		14.00	2 2 1 1 1 0	288.36
3 2 1 1 1 *		14.00	3 2 1 1 1 0	290.63
4 2 1 1 1 *	1	20.00	2 1 1 1 0 0	285.03
3 3 1 1 1 *		14.00	3 3 1 1 1 0	292.04
4 3 1 1 1 *	2	26.00	1 1 1 0 0 0	280.92
4 4 1 1 1 *	2	26.00	1 1 1 0 0 0	280.92
2 2 2 1 1 *		14.00	2 2 2 1 1 0	290.03
3 2 2 1 1 *		14.00	3 2 2 1 1 0	292.21
4 2 2 1 1 *	1	20.00	2 2 1 1 0 0	287.09
3 3 2 1 1 *		14.00	3 3 2 1 1 0	293.76
4 3 2 1 1 *	1	20.00	3 2 1 1 0 0	289.26
4 4 2 1 1 *	2	26.00	2 1 1 0 0 0	283.42
3 3 3 1 1 *		14.00	3 3 3 1 1 0	294.65
4 3 3 1 1 *		14.00	4 3 3 1 1 0	296.14
4 4 3 1 1 *		14.00	4 4 3 1 1 0	296.43
4 4 4 1 1 *		14.00	4 4 4 1 1 0	296.49
2 2 2 2 1 *		14.00	2 2 2 2 1 0	291.38
3 2 2 2 1 *		14.00	3 2 2 2 1 0	293.38
4 2 2 2 1 *	1	20.00	2 2 2 1 0 0	288.79
3 3 2 2 1 *		14.00	3 3 2 2 1 0	294.88
4 3 2 2 1 *		14.00	4 3 2 2 1 0	296.79
4 4 2 2 1 *		14.00	4 4 2 2 1 0	297.27
3 3 3 2 1 *		14.00	3 3 3 2 1 0	295.88
4 3 3 2 1 *		14.00	4 3 3 2 1 0	297.48
4 4 3 2 1 *		14.00	4 4 3 2 1 0	298.01
4 4 4 2 1 *		14.00	4 4 4 2 1 0	298.15

Observed ages of the elements (* indicates failure) r	Number of elements to replace preventively $\theta_*(r)-\lvert H\rvert$	Total cost of the intervention $B+b\theta_*(r)$	Description of the system after the intervention ξ	Expected Discounted cost to go $J_*(\xi)$
3 3 3 3 1 *		14.00	3 3 3 3 1 0	296.44
4 3 3 3 1 *		14.00	4 3 3 3 1 0	297.67
4 4 3 3 1 *		14.00	4 4 3 3 1 0	298.06
4 4 4 3 1 *		14.00	4 4 4 3 1 0	298.16
4 4 4 4 1 *		14.00	4 4 4 4 1 0	298.18
2 2 2 2 2 *		14.00	2 2 2 2 2 0	292.42
3 2 2 2 2 *		14.00	3 2 2 2 2 0	294.20
4 2 2 2 2 *	1	20.00	2 2 2 2 0 0	290.06
3 3 2 2 2 *		14.00	3 3 2 2 2 0	295.54
4 3 2 2 2 *		14.00	4 3 2 2 2 0	297.41
4 4 2 2 2 *		14.00	4 4 2 2 2 0	298.00
3 3 3 2 2 *		14.00	3 3 3 2 2 0	296.49
4 3 3 2 2 *		14.00	4 3 3 2 2 0	298.09
4 4 3 2 2 *		14.00	4 4 3 2 2 0	298.75
4 4 4 2 2 *		14.00	4 4 4 2 2 0	298.99
3 3 3 3 2 *		14.00	3 3 3 3 2 0	297.12
4 3 3 3 2 *		14.00	4 3 3 3 2 0	298.43
4 4 3 3 2 *		14.00	4 4 3 3 2 0	298.98
4 4 4 3 2 *		14.00	4 4 4 3 2 0	299.19
4 4 4 4 2 *		14.00	4 4 4 4 2 0	299.26
3 3 3 3 3 *		14.00	3 3 3 3 3 0	297.56
4 3 3 3 3 *		14.00	4 3 3 3 3 0	298.62
4 4 3 3 3 *		14.00	4 4 3 3 3 0	299.06
4 4 4 3 3 *		14.00	4 4 4 3 3 0	299.22
4 4 4 4 3 *		14.00	4 4 4 4 3 0	299.27
4 4 4 4 4 *		14.00	4 4 4 4 4 0	299.28

difficulty in obtaining the dynamic programming equation for the characterization of optimal GPR strategies in the case where the components are not identical [12]. However, the size of the state set could rapidly become much too large to allow for a direct adaptation of the numerical procedure presented in this paper to obtain an optimal solution. Numerical examples suggest that a "reasonable" or near-optimal solution can be found in the class of FAT type policies defined in Section IV. To restrict the search for a best policy to such a class of simpler policies can be the best way to obtain a practical solution to a very large scale problem.

A realistic maintenance problem will, in general, involve a multicomponent system with nonidentical elements and with a complex description of the state of wear of each element as well as of the set of available maintenance activities. Such problems offer stimulating challenges to stochastic control theorists.

TABLE III
DEFINITION OF THE PROBABILITIES OF FAILURE

t	p(t)
0	.019
1	.126
2	.245
3	.330
4	.389
5	.429
6	.459
≥ 7	.482

TABLE IV
COMPARISON OF THE EXPECTED TOTAL DISCOUNTED COST-TO-GO FOR A NEW SYSTEM FOR THREE KINDS OF STRATEGIES

	Optimal Strategy	N O P R		F A T		
B	$J_*(0)$	$J_*(0)$	% of increase	a_*	$J_*(0)$	% of increase
1.0	16.693	16.693	0.00%	8	16.693	0.00%
2.0	22.907	22.921	0.06%	8 7	22.921 23.025	0.06%
3.0	28.772	29.149	1.31%	8 7 6 5 4	29.149 29.17 29.18 29.20 29.27	1.31%
4.0	33.830	35.38	4.58%	8 4 3 2	35.38 34.7 34.21 34.90	1.12%
5.0	38.296	41.61	8.65%	8 3 2 1	41.61 38.84 38.627 39.41	0.86%
10.0	57.189	72.75	27.21%	8 2 1	72.75 57.253 57.322	0.11%

TABLE V
VARIATION OF $J_*(0)$ WHEN $p(7)$ GOES FROM 0.482 TO 0.6 OR 0.8

	Optimal Strategy		F A T		
p(7)	$J_*(0)$	% of increase	a_*	$J_*(0)$	% of increase
.482	28.772		8 7 6	29.149 29.17 29.18	
.6	28.779	< .03%	8 7 6	29.25 29.18 29.18	< 0.38% < 0.06% < 0.03%
.8	28.784	< .05%	8 7 6	29.36 29.20 29.18	< 0.75% < 0.14% < 0.03%

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A Lagrange Approach to Set-Theoretic Control Synthesis

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***Abstract*—A Lagrange approach to solving the nonlinear constrained optimization problem arising in the set-theoretic control problem is described. By introducing matrix Lagrange multipliers, the problem is reduced to that of solving a set of nonlinear simultaneous matrix equations, one of which is the familiar matrix Riccati equation frequently encountered in linear-quadratic control theory. The structural similarities and differences between set-theoretic and linear-quadratic control methods are identified. The results obtained from the set-theoretic control approach are compared with those obtained from the linear-quadratic control approach.**

I. INTRODUCTION

A set-theoretic approach to solving constrained control problems has evolved from Schweppe's work on "unknown-but-bounded" representation of uncertainty [1], [2], a modeling technique which assumes no statistics for the uncertainty, and the only information that is known about the uncertainty is its bound. Later work on target reachability problems—Delfour and Mitter [3], Bertsekas and Rhodes [4], Glover and Schweppe [5], and Sira-Ramirez [6]—has enhanced the development of the set-theoretic concept into a practical control system design tool. In fact, a set-theoretic control synthesis algorithm for linear systems has been developed, implemented in a computer program, and tested [7].

The set-theoretic control problem for a linear system is as follows. Consider the system described in state-space form by the continuous time model

$$\dot{x} = Ax + Bu + G\omega \tag{1}$$

$$y = Hx \tag{2}$$

where $x = n$-dimensional state vector, $u = r$-dimensional control vector, $y = m$-dimensional output vector, $\omega =$ a single input disturbance (scalar), and $A, B, G, H =$ matrices of appropriate dimensions.

The input disturbance ω is modeled as an unknown-but-bounded quantity represented by $|\omega| \leq Q^{1/2}$, or equivalently

$$\omega \in \Omega_\omega = \{\omega : \omega' Q^{-1} \omega \leq 1\} \tag{3}$$

where $Q^{1/2}$ is the bound on the amplitude of the disturbance.

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