DIRO
IFT 6150

TRAITEMENT D'IMAGES

TRANSFORMÉE DE FOURIER DISCRÈTE

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TF DISCRÈTE

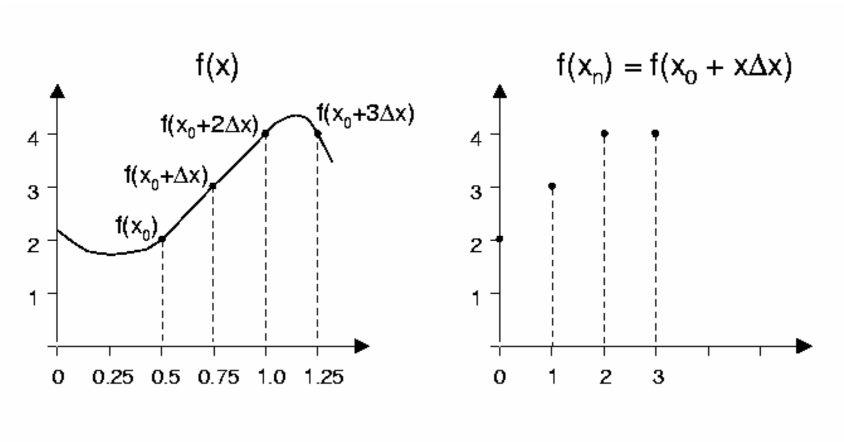
SOMMAIRE

TF DISCRÈTE
TF DISCRÈTE 1D (1)

CAS CONTINU (RAPPEL)

$$\boxed{\mathcal{F}[f(x)] = F(u) = \int_{-\infty}^{+\infty} f(x) \exp\left(-2\pi j u x\right) dx \quad u \in IR.}$$

CAS DISCRET



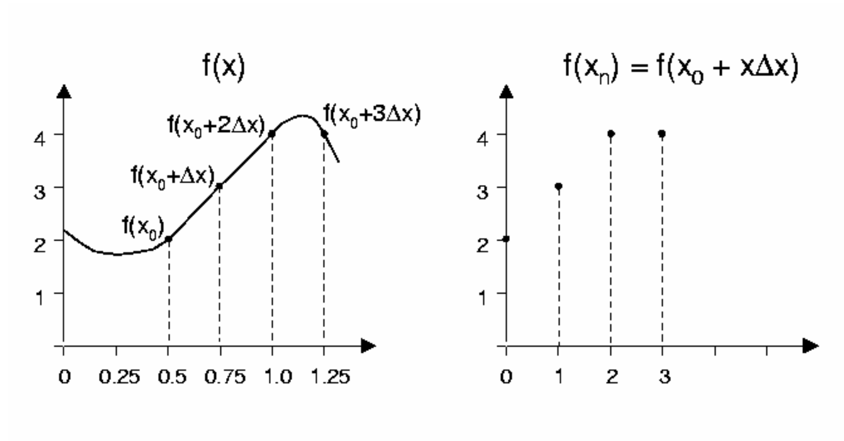
$$f(x) = f(x_0 + x\Delta x) \quad x = 0, 1, 2, \ldots N-1$$

TFD et TFD inverse

$$\begin{aligned} F(u) &= \frac{1}{N}\sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi j u x}{N}\right) \quad u = 0, 1, 2, \ldots, N-1 \\ f(x) &= \sum_{u=0}^{N-1} F(u) \exp\left(\frac{2\pi j u x}{N}\right) \quad x = 0, 1, 2, \ldots, N-1 \end{aligned}$$

$$\Delta u = \frac{1}{N\Delta x}$$

TF DISCRÈTE
TF DISCRÈTE 1D -EXEMPLE- (2)



$$\begin{aligned} F(0) &= 1/4 \sum_{x=0}^{3} f(x) \exp\left(\frac{-2\pi j 0 x}{4}\right) = 1/4 \sum_{x=0}^{3} f(x)\, 1 \\ &= 1/4\big(f(0)+f(1)+f(2)+f(3)\big) = 1/4(2+3+4+4) = 3.25 \\ F(1) &= 1/4 \sum_{x=0}^{3} f(x) \exp\left(\frac{-2\pi j x}{4}\right) \\ &= 1/4\big(2\exp(0) + 3\exp(-j\pi/2) + 4\exp(-j\pi) + 4\exp(-3j\pi/2)\big) \\ &= \frac{1}{4}(-2+j) \\ F(2) &= 1/4 \sum_{x=0}^{3} f(x) \exp\left(\frac{-4\pi j x}{4}\right) = \ldots = -1/4 \\ F(3) &= 1/4 \sum_{x=0}^{3} f(x) \exp\left(\frac{-6\pi j 0 x}{4}\right) = \ldots = 1/4(-2-j) \end{aligned}$$

Spectre d'amplitude

$|F(0)| = 3.25 \quad |F(1)| = \sqrt{5}/4 \quad |F(2)| = 1/4 \quad |F(3)| = \sqrt{5}/4$

TF DISCRÈTE
TF DISCRÈTE 2D (1)

TFD 2D et TFD 2D inverse

$$F(u,\nu) = \frac{1}{MN}\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} f(x,y)\,\text{exp}\left(-2\pi j\left(\frac{ux}{N}+\frac{\nu y}{M}\right)\right)$$
$$u = 0,1,2,\ldots,N-1$$
$$\nu = 0,1,2,\ldots,M-1$$

$$f(x,y) = \sum_{u=0}^{N-1}\sum_{\nu=0}^{M-1} F(u,\nu)\,\text{exp}\left(2\pi j\left(\frac{ux}{N}+\frac{\nu y}{M}\right)\right)$$
$$x = 0,1,2,\ldots,N-1$$
$$y = 0,1,2,\ldots,M-1$$

Le plus souvent, l'image à traiter est carrée ◀▶ $M = N$

PROPRIÉTÉS

- Les mêmes que ceux énoncé pour la TF continue
- Cyclique (périodique)

$$F(u,\nu) = F(u+N,\nu) = F(u,\nu+N) = F(u+N,\nu+N)$$

$$f(x,y) = f(x+N,y+N)$$

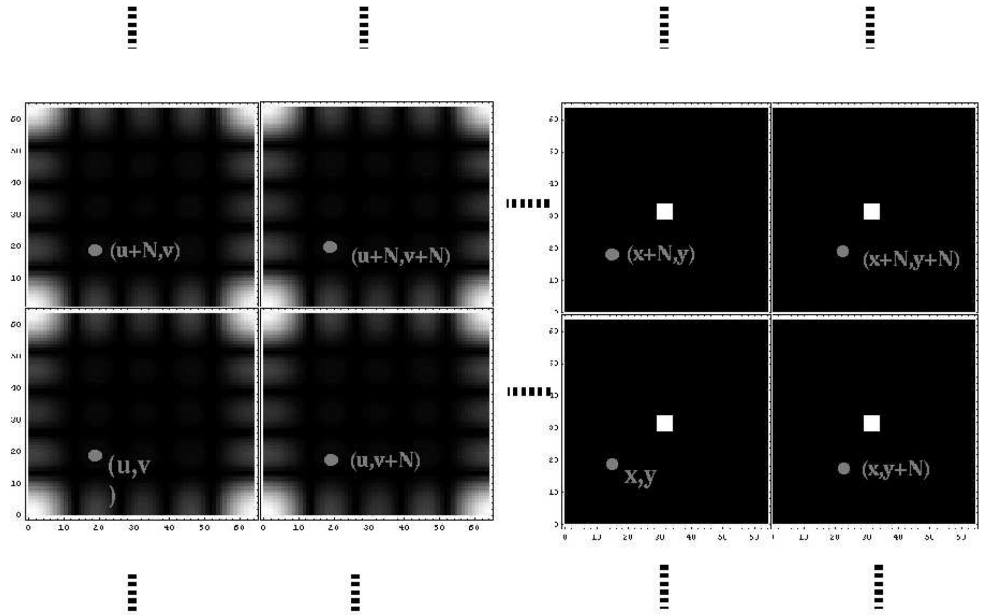
car

Signal périodique $-\mathcal{F}$ -> Spectre de raies

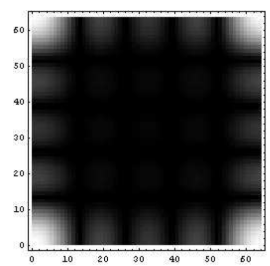
Signal échantillonné $-\mathcal{F}$ -> Spectre périodique

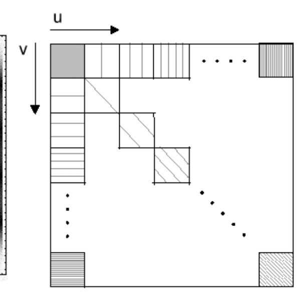
TF DISCRÈTE
TF DISCRÈTE 2D (4)

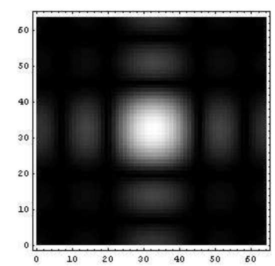
- Périodicité



- **Nota**





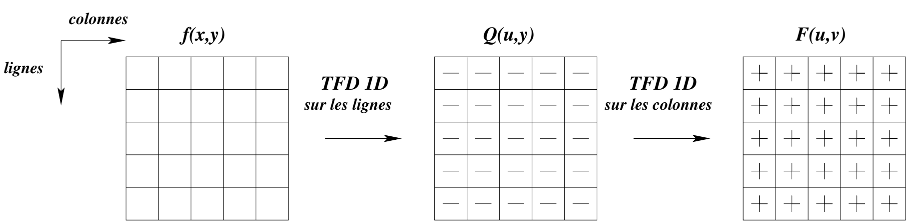


TF DISCRÈTE
TF DISCRÈTE 2D (5)

- Séparabilité

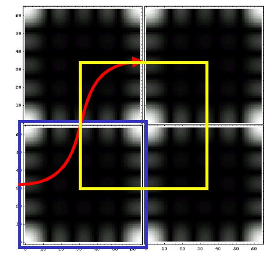
Pour une image carrée ◀▶ $M = N$

$$\begin{aligned}
F(u,\nu) &= \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\exp\left(-2\pi j\left(\frac{ux+\nu y}{N}\right)\right)\\
&= \frac{1}{N}\sum_{x=0}^{N-1}\left(\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp\left(-2\pi j\frac{\nu y}{N}\right)\right)\exp\left(-2\pi j\frac{ux}{N}\right)\\
&= \underbrace{\frac{1}{N}\sum_{x=0}^{N-1}\text{[TF 1D de la x ième colonne]}\exp\left(-2\pi j\frac{ux}{N}\right)}_{\text{TF 1D de la y ième ligne}}
\end{aligned}$$



TF DISCRÈTE
TF DISCRÈTE 2D (6)

Recalage Cyclique

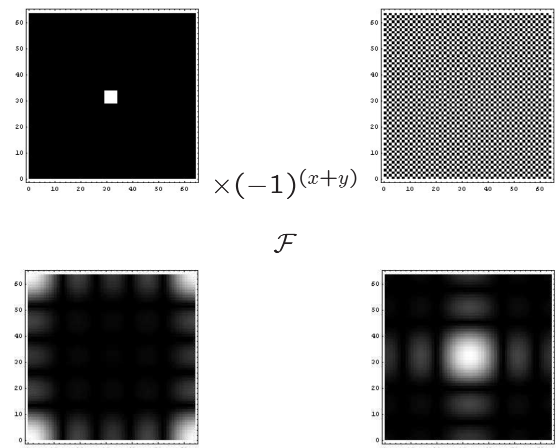


Translation -Rappel-

$$f(x-x_0, y-y_0) \quad \blacktriangleleft \mathcal{F} \blacktriangleright \quad F(u,\nu)\exp\left(\frac{-2\pi j(ux_0+\nu y_0)}{N}\right)$$

$$f(x,y)\exp\left(\frac{2\pi j(u_0x+\nu_0 y)}{N}\right) \quad \blacktriangleleft \mathcal{F} \blacktriangleright \quad F(u-u_0, \nu-\nu_0)$$

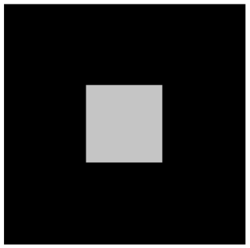
$$u_0=\nu_0=\frac{N}{2} \blacktriangleright \exp\left(\frac{2\pi j(u_0x+\nu_0y)}{N}\right)=\exp\left(\pi j(x+y)\right)=(-1)^{(x+y)}$$



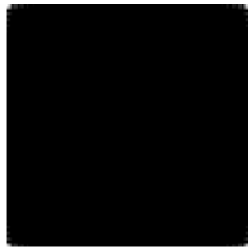
TF DISCRÈTE

TF DISCRÈTE 2D -VISUALISATION- (7)

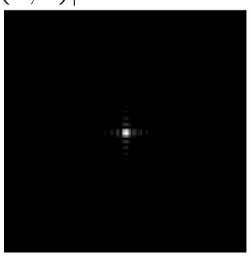
Image originale



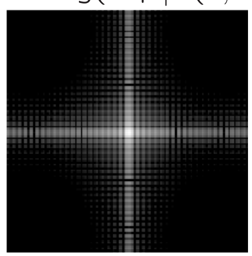
$|F(u,\nu)|$



$|F(0,0)|$ au centre



$k\log(1+|F(u,\nu)|)$



Nota

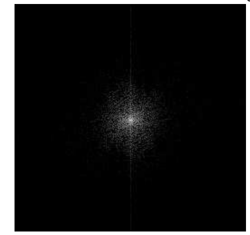
- On affiche généralement $|F(u,\nu)|$ ($F(u,\nu)$ complexe)

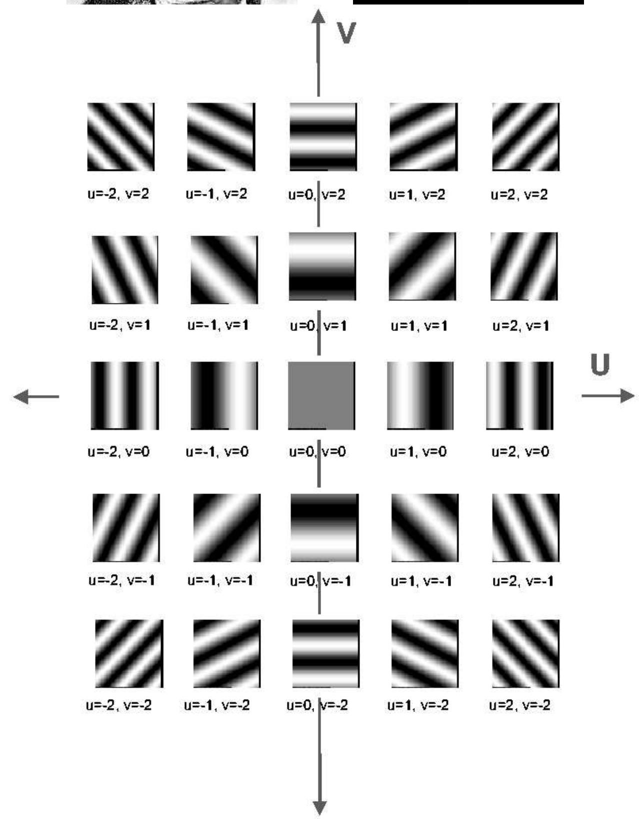
- Les fréquences élevées sont bcp plus faible que les fréquences plus basses. On affiche donc plutot $k\log(1+|F(u,\nu)|)$ (k est une constante de normalisation pour recaler les niveau de gris dans $[0,255]$)

- On met l'origine au centre de l'image en effectuant un décalage cyclique

TF DISCRÈTE

TF DISCRÈTE 2D -INTERPRÉTATION- (8)

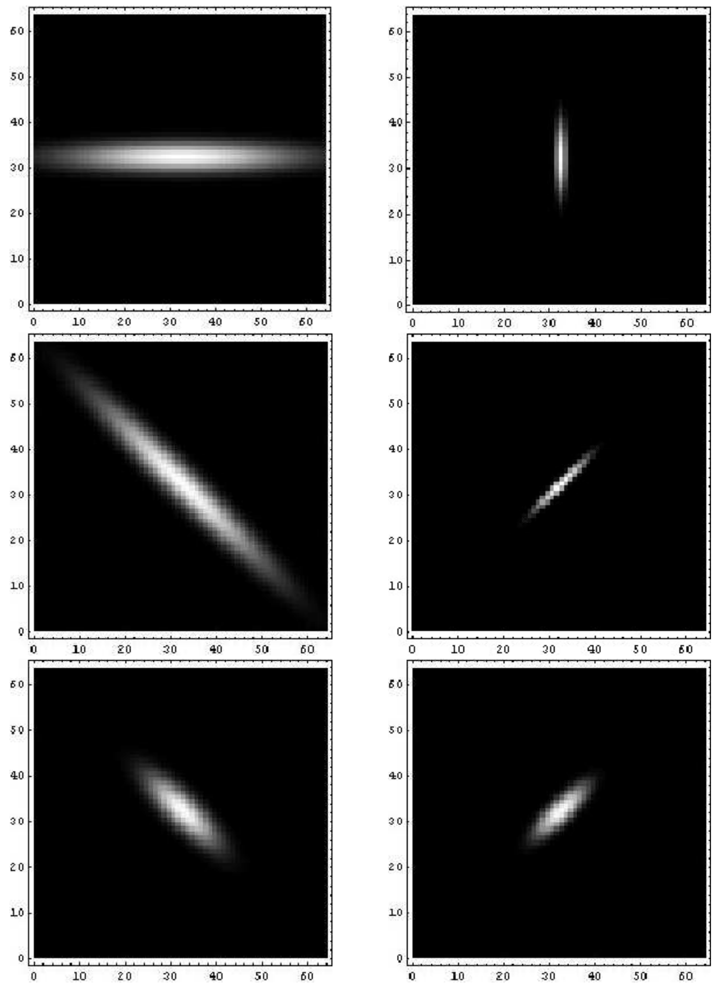




TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (9)

Image $\mathcal{F}$(image)

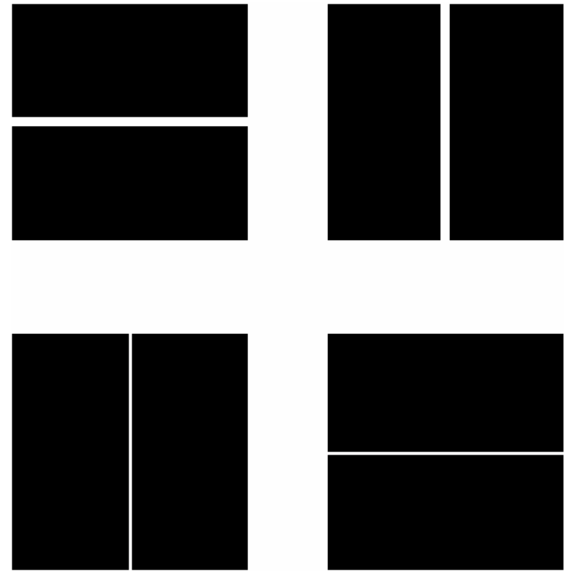


TF DISCRÈTE

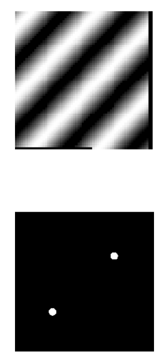
TF DISCRÈTE 2D -EXEMPLES- (10)

- Rotation

Image $\mathcal{F}$(image)



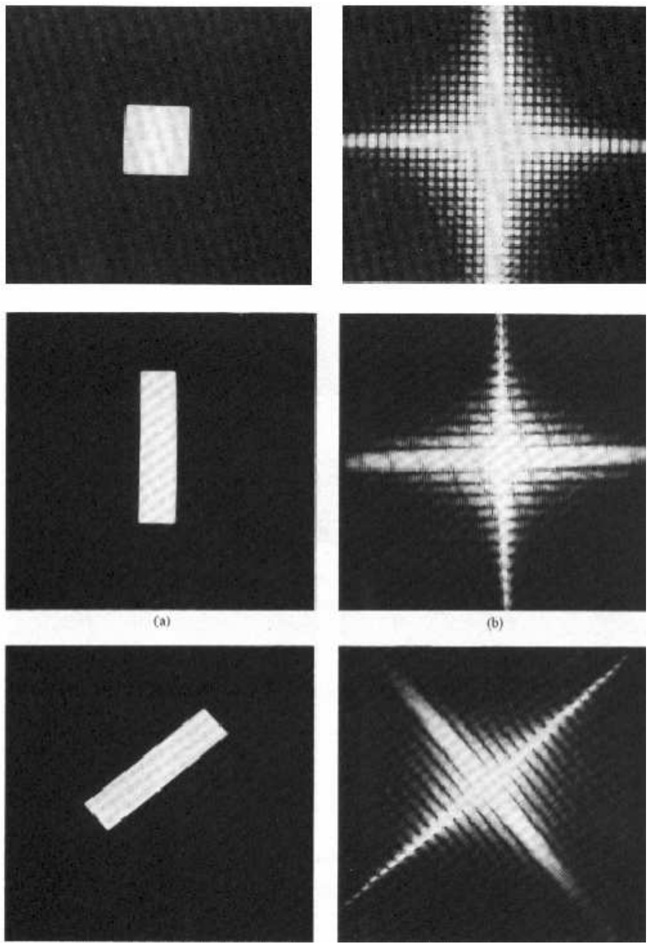
- Périodicité



TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (11)

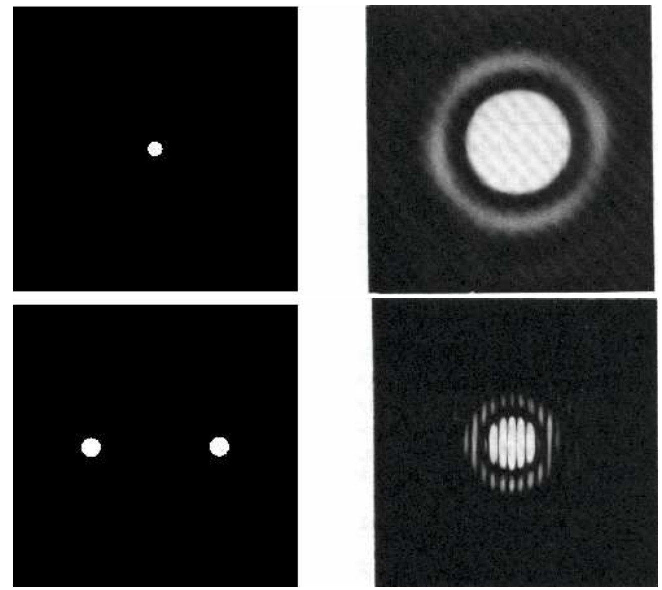
Image $\mathcal{F}$(image)



TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (12)

Image $\mathcal{F}$(image)



TF DISCRÈTE
CONVOLUTION DISCRÈTE 1D & 2D (1)

Théorème de convolution 2D

$$f(x,y) * g(x,y) \quad -\mathcal{F}-> \quad F(u,\nu) \cdot G(u,\nu)$$
$$f(x,y) \cdot g(x,y) \quad -\mathcal{F}-> \quad F(u,\nu) * G(u,\nu)$$

Convolution 2D -Version continue-

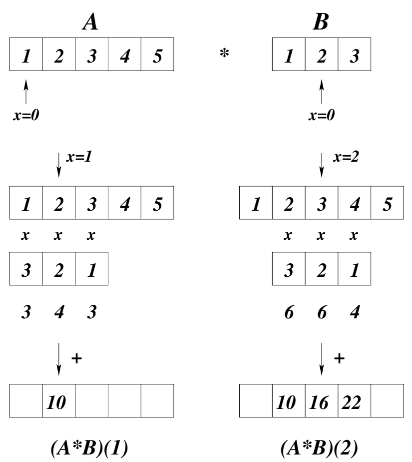
$$\boxed{(f*g)(x,y) = \int_0^\infty \int_0^\infty f(\alpha,\beta) g(x-\alpha, y-\beta)\, d\alpha\, d\beta}$$

Convolution 1D -Version discrète-

		A			*		*B*	

$$\boxed{(A*B)(x) = \sum_i A(i)B(x-i)}$$

-Exemple-



TF DISCRÈTE
CONVOLUTION DISCRÈTE 1D & 2D (2)

Convolution des bords

A) Ajout de zéro

0	0	1	2	3	4	5	0	0
		4	10	16	22	22		

B) Enroulement

4	5	1	2	3	4	5	1	2
		19	10	16	22	23		

C) Réflexion

3	2	1	2	3	4	5	4	3
		10	10	16	22	26		

D) Défaut

1	2	3	4	5
☒	10	16	22	☒

TF DISCRÈTE
CONVOLUTION DISCRÈTE 1D & 2D (3)

Convolution 2D -Version discrète-

A $*$ *B*

$$(A * B)(x, y) = \sum_i \sum_j A(i, j)B(x - i, y - j)$$

-Exemple-

10	5	20	20	20
10	5	20	20	20
10	5	20	20	20
10	5	20	20	20
10	5	20	20	20

$*$

-1	1
0	1

-10	5	-15	0	0
-10	15	-10	20	20
-10	15	-10	20	20
-10	15	-10	20	20
-10	15	-10	20	20

(Ajout de zéro)

-1	1
0	1

Original

1	-1
1	0

x reflexion

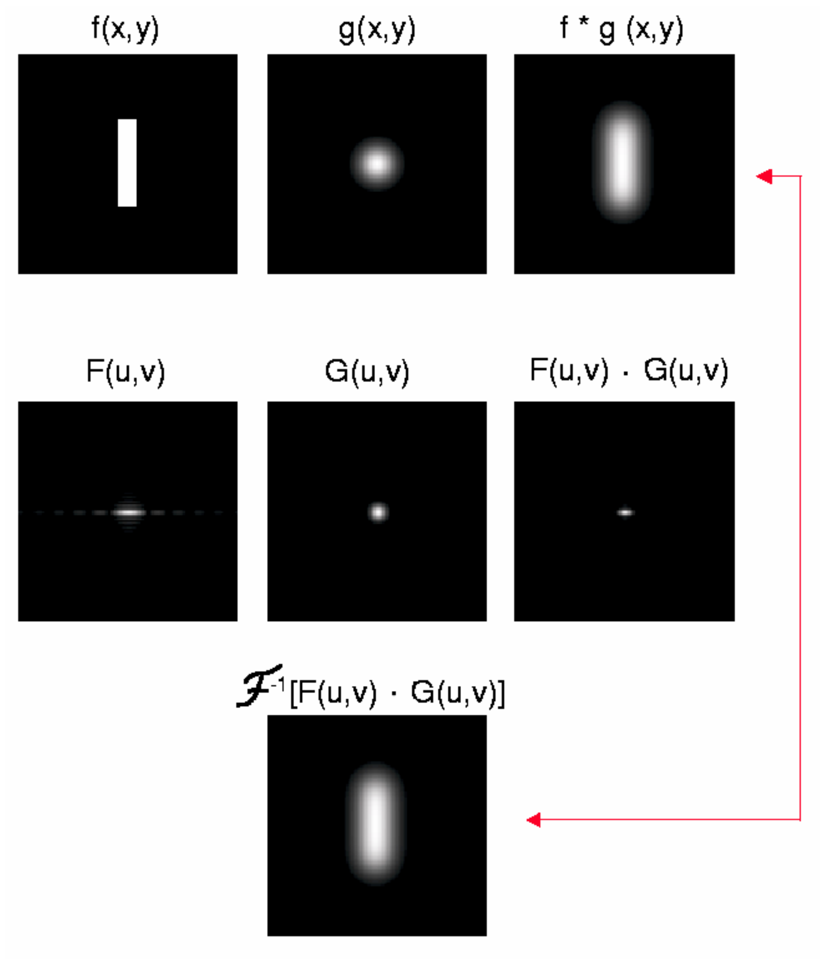
1	0
1	-1

y reflexion

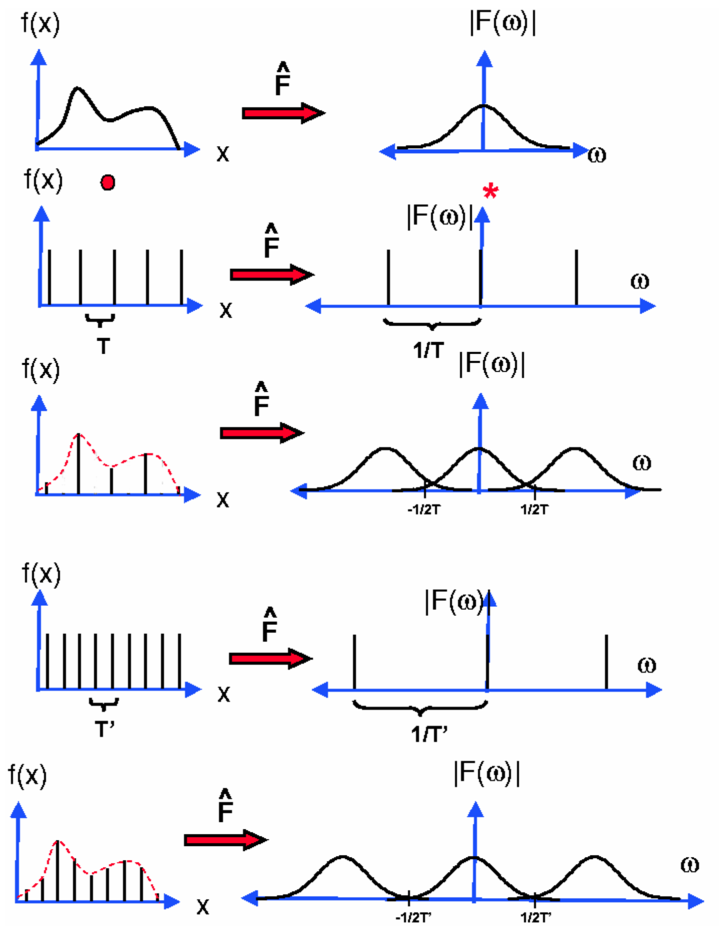
PROPRIÉTÉS

- Commutatif ▶ $f_1 * f_2 * f = f_2 * f_1 * f$
- Associatif ▶ $(f_1 * f_2) * f = f_1 * (f_2 * f)$
- Distributif ▶ $(f_1 + f_2) * f = f_1 * f + f_2 * f$
  ▶ $f * (f_1 + f_2) = f * f_1 + f * f_2$

TF DISCRÈTE
CONVOLUTION DISCRÈTE 1D & 2D (4)



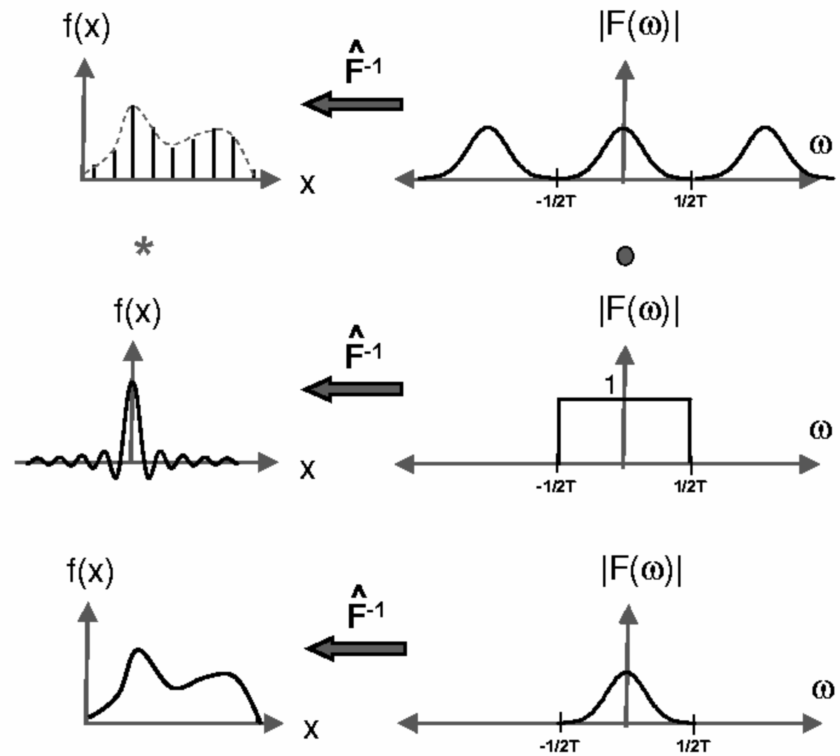
TF DISCRÈTE
THÉORÈME DE SCHANNON



$\boxed{T_e \leq \frac{1}{2\nu_0}}$

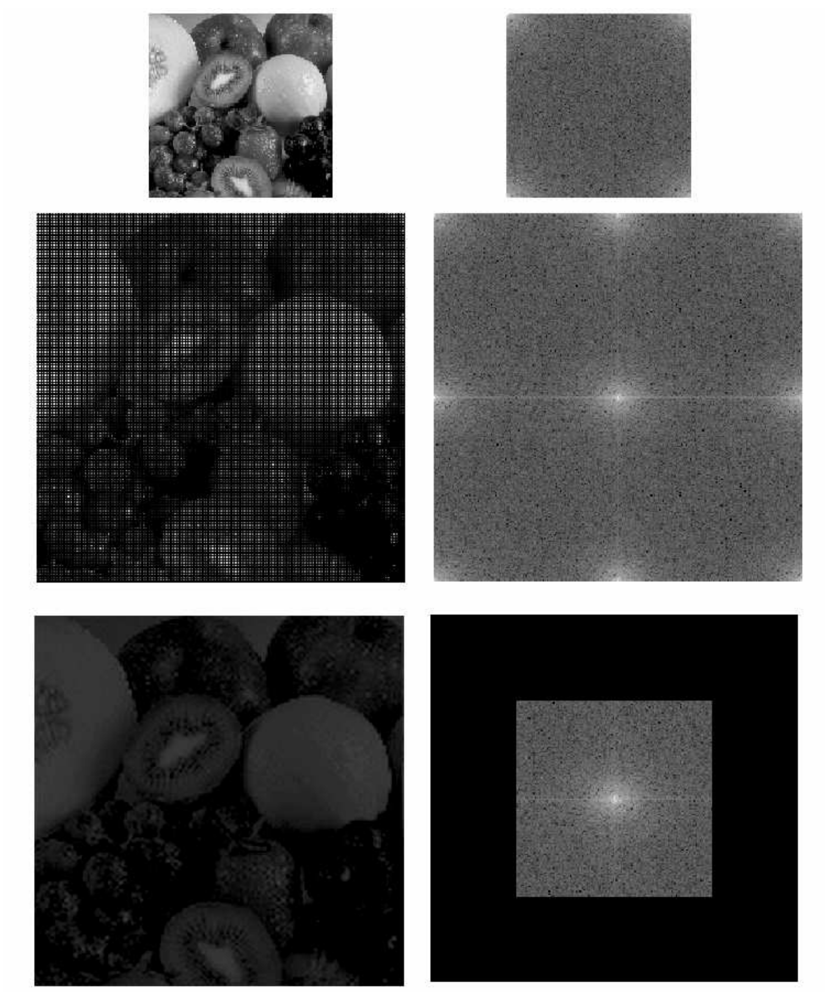
▶ sinon phénomène d'aliasing

TF DISCRÈTE
INTERPOLATION OPTIMALE (1)



▶ Possibilité de reconstruire totalement le signal 2D à partir de ses échantillons

TF DISCRÈTE
INTERPOLATION OPTIMALE (2)



TF DISCRÈTE
FFT (1)

$$\boxed{F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi jux}{N}\right) \qquad u = 0, 1, 2, \ldots, N-1}$$

$O(N^2)$ opérations

$$F(u) = \underbrace{\frac{1}{N}\sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi ju2x}{N}\right)}_{\text{x pair}} + \underbrace{\frac{1}{N}\sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi ju(2x+1)}{N}\right)}_{\text{x impair}}$$

$$= \frac{1}{2}\Bigg[\underbrace{\frac{1}{N/2}\sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi jux}{N/2}\right)}_{\text{TF de la partie paire}} + \exp\left(\frac{-2\pi ju}{N}\right) \underbrace{\frac{1}{N/2}\sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi jux}{N/2}\right)}_{\text{TF de la partie impaire}}\Bigg]$$

- TF [N éléments] $=$ 2 TF [$N/2$ éléments] $+$ $N/2$ multiplications et $N/2$ additions complexes (en fait $N/4$ pour les multiplications)
- $\boxed{Op(N) = 2Op(N/2) + N}$

$$F_N(u) = \frac{1}{2}\left[F_{N/2}^{\text{paire}}(u) + \exp\left(-\frac{2\pi ju}{N}\right) F_{N/2}^{\text{impaire}}(u)\right]$$

Pour $\quad u' = u + \dfrac{N}{2}$

$$\exp\left(-\frac{2\pi ju'}{N}\right) = \exp\left(-\frac{2\pi j(u+\frac{N}{2})}{N}\right) = \exp\left(-\frac{2\pi ju}{N}\right)\exp\left(-\pi j\right) = -\exp\left(-\frac{2\pi ju}{N}\right)$$

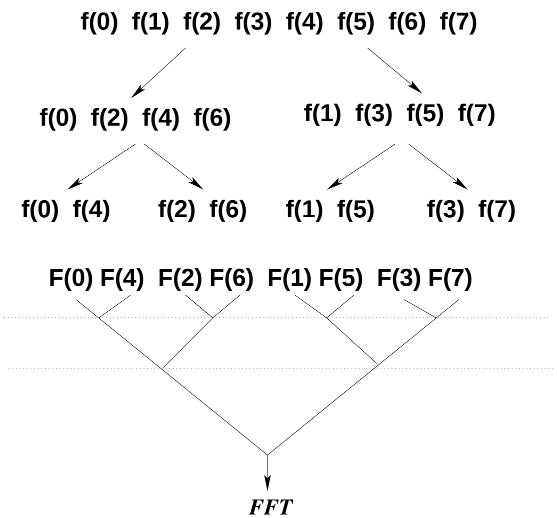
TF DISCRÈTE
FFT (2)

Ainsi

$$
\begin{aligned}
F_N(u) &= \frac{1}{2}\Big[F_{N/2}^{\text{paire}}(u) + \exp\big(-\frac{2\pi j u}{N}\big) F_{N/2}^{\text{impaire}}(u)\Big] \\
F_N(u+\frac{N}{2}) &= \frac{1}{2}\Big[F_{N/2}^{\text{paire}}(u) - \exp\big(-\frac{2\pi j u}{N}\big) F_{N/2}^{\text{impaire}}(u)\Big]
\end{aligned}
$$

Pour $\quad u = 0, 1, 2, \ldots, N/2-1$

1 multiplication est nécessaire pour deux termes



FFT 1D ▶ $\quad O(N \log N)$
FFT 2D ▶ $\quad O(N^2 \log N)$

TF DISCRÈTE
FFT (3)

Vérifions que $O(N \log N)$

(Par induction, sachant que $\text{Op}(N) = 2\,\text{Op}(N/2) + N$)

- Vrai pour $N = 1$? ▶ $\text{Op}(1) = 1 \log 1 = 0$ ▶ oui
- Vrai pour $N = 2$? ▶ $\text{Op}(2) = 2 \log 2 = 2$ ▶ oui
- Vrai pour $N \log N$?
  - $\text{Op}(N) = N \log N$?
  - $\text{Op}(N) = 2N/2 \log(2N/2)$
  - $\text{Op}(N) = 2N/2(\log(N/2) + \log 2)$
  - $\text{Op}(N) = 2N/2 \log N/2 + 2N/2 \log 2$
  - $\text{Op}(N) = 2\,\text{Op}(N/2) + N$

C.Q.F.D.

Rehaussement d'image