



## **TRAITEMENT D'IMAGES**

### **TRANSFORMÉE DE FOURIER DISCRÈTE**

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# TF DISCRÈTE

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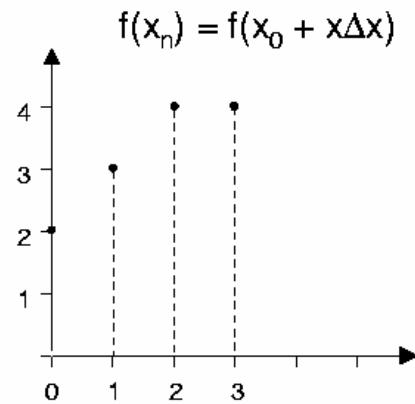
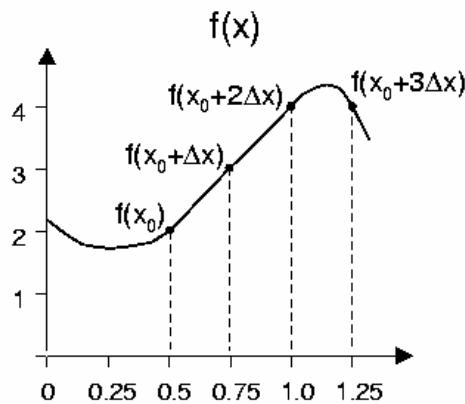
# TF DISCRÈTE

## TF DISCRÈTE 1D (1)

### CAS CONTINU (RAPPEL)

$$\mathcal{F}[f(x)] = F(u) = \int_{-\infty}^{+\infty} f(x) \exp(-2\pi j u x) dx \quad u \in IR.$$

### CAS DISCRET



$$f(x) = f(x_0 + x\Delta x) \quad x = 0, 1, 2, \dots, N-1$$

### TFD et TFD inverse

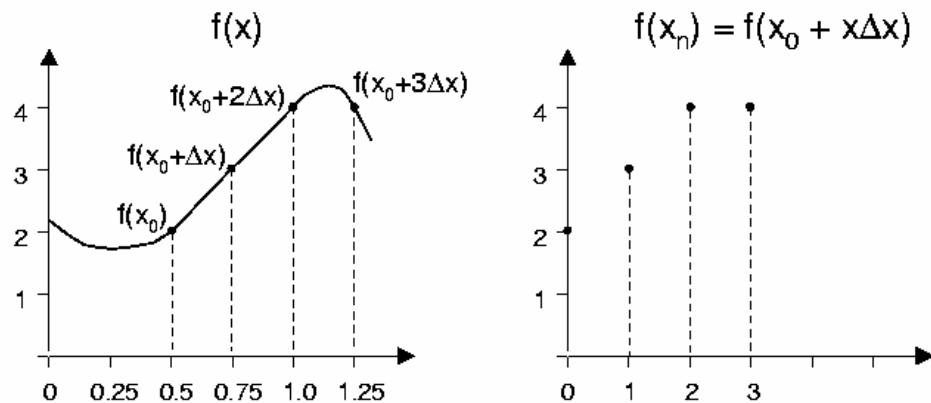
$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi j u x}{N}\right) \quad u = 0, 1, 2, \dots, N-1$$

$$f(x) = \sum_{u=0}^{N-1} F(u) \exp\left(\frac{2\pi j u x}{N}\right) \quad x = 0, 1, 2, \dots, N-1$$

$$\Delta u = \frac{1}{N \Delta x}$$

## TF DISCRÈTE

### TF DISCRÈTE 1D -EXEMPLE- (2)



$$\begin{aligned}
 F(0) &= 1/4 \sum_{x=0}^3 f(x) \exp\left(\frac{-2\pi j 0 x}{4}\right) = 1/4 \sum_{x=0}^3 f(x) 1 \\
 &= 1/4(f(0) + f(1) + f(2) + f(3)) = 1/4(2 + 3 + 4 + 4) = 3.25 \\
 F(1) &= 1/4 \sum_{x=0}^3 f(x) \exp\left(\frac{-2\pi j x}{4}\right) \\
 &= 1/4(2 \exp(0) + 3 \exp(-j\pi/2) + 4 \exp(-j\pi) + 4 \exp(-3j\pi/2)) \\
 &= \frac{1}{4}(-2 + j) \\
 F(2) &= 1/4 \sum_{x=0}^3 f(x) \exp\left(\frac{-4\pi j x}{4}\right) = \dots = -1/4 \\
 F(3) &= 1/4 \sum_{x=0}^3 f(x) \exp\left(\frac{-6\pi j 0 x}{4}\right) = \dots = 1/4(-2 - j)
 \end{aligned}$$

#### Spectre d'amplitude

$$|F(0)| = 3.25 \quad |F(1)| = \sqrt{5}/4 \quad |F(2)| = 1/4 \quad |F(3)| = \sqrt{5}/4$$

## TF DISCRÈTE

### TF DISCRÈTE 2D (1)

#### TFD 2D et TFD 2D inverse

$$F(u, \nu) = \frac{1}{MN} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x, y) \exp \left( -2\pi j \left( \frac{ux}{N} + \frac{\nu y}{M} \right) \right)$$

$u = 0, 1, 2, \dots, N - 1$   
 $\nu = 0, 1, 2, \dots, M - 1$

$$f(x, y) = \sum_{u=0}^{N-1} \sum_{\nu=0}^{M-1} F(u, \nu) \exp \left( 2\pi j \left( \frac{ux}{N} + \frac{\nu y}{M} \right) \right)$$

$x = 0, 1, 2, \dots, N - 1$   
 $y = 0, 1, 2, \dots, M - 1$

Le plus souvent, l'image à traiter est carrée  $\blacktriangleleft \blacktriangleright M = N$

#### PROPRIÉTÉS

- Les mêmes que ceux énoncé pour la TF continue
- **Cyclique (périodique)**

$$F(u, \nu) = F(u + N, \nu) = F(u, \nu + N) = F(u + N, \nu + N)$$

$$f(x, y) = f(x + N, y + N)$$

car

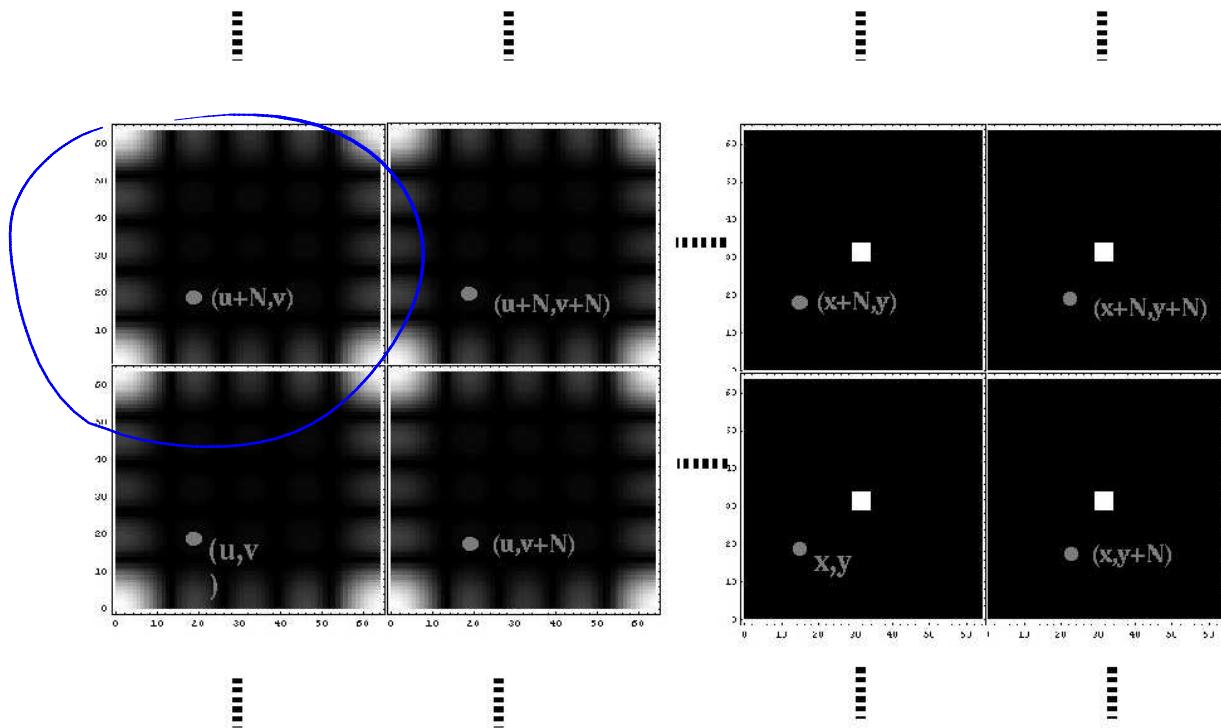
Signal périodique  $\mathcal{F} \rightarrow$  Spectre de raies

Signal échantillonné  $\mathcal{F} \rightarrow$  Spectre périodique

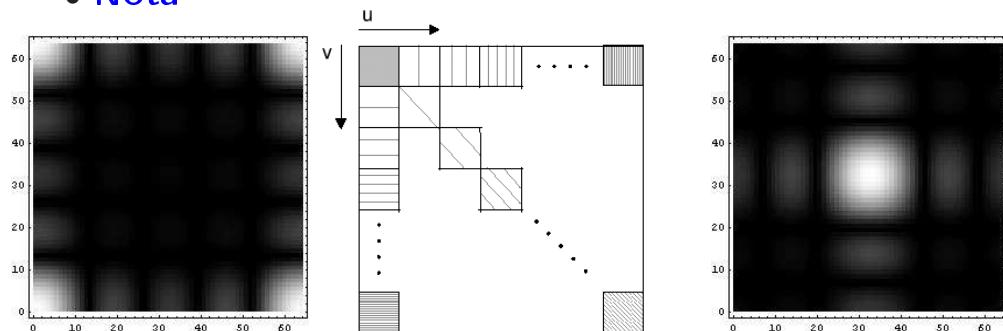
## TF DISCRÈTE

### TF DISCRÈTE 2D (4)

- Périodicité



- Nota



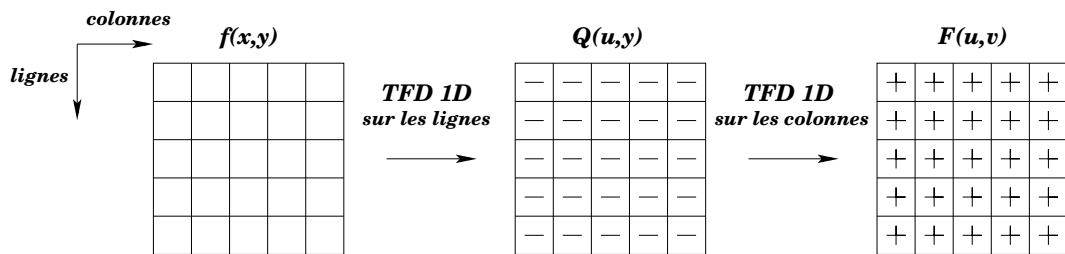
## TF DISCRÈTE

### TF DISCRÈTE 2D (5)

- Séparabilité

Pour une image carrée  $\blacktriangleleft \blacktriangleright M = N$

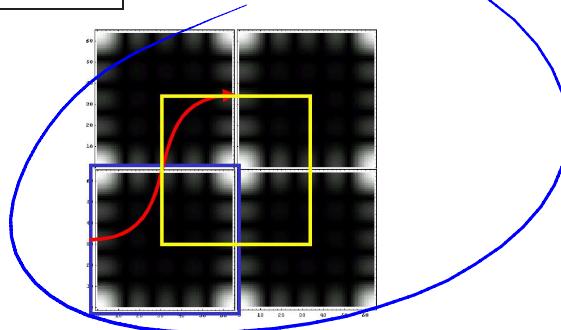
$$\begin{aligned}
 F(u, \nu) &= \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) \exp \left( -2\pi j \left( \frac{ux + \nu y}{N} \right) \right) \\
 &= \frac{1}{N} \sum_{x=0}^{N-1} \left( \frac{1}{N} \sum_{y=0}^{N-1} f(x, y) \exp \left( -2\pi j \frac{\nu y}{N} \right) \right) \exp \left( -2\pi j \frac{ux}{N} \right) \\
 &= \underbrace{\frac{1}{N} \sum_{x=0}^{N-1} \left[ \text{TF 1D de la } x \text{ ième colonne} \right] \exp \left( -2\pi j \frac{ux}{N} \right)}_{\text{TF 1D de la } y \text{ ième ligne}}
 \end{aligned}$$



# TF DISCRÈTE

## TF DISCRÈTE 2D (6)

### Recalage Cyclique

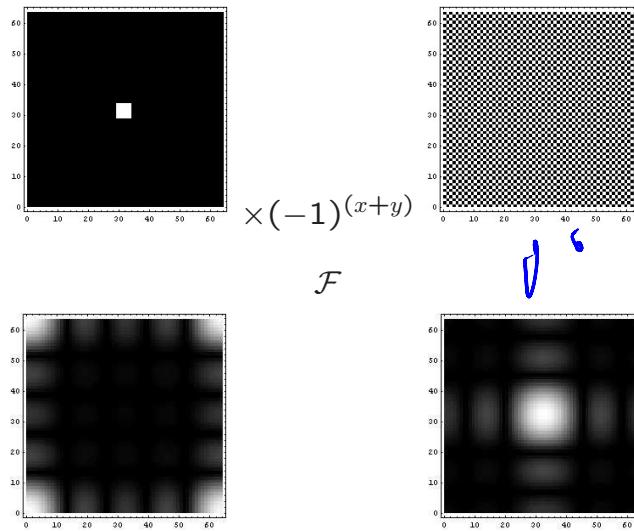


### Translation -Rappel-

$$f(x - x_0, y - y_0) \quad \blacktriangleleft \mathcal{F} \blacktriangleright \quad F(u, \nu) \exp \left( \frac{-2\pi j(ux_0 + \nu y_0)}{N} \right)$$

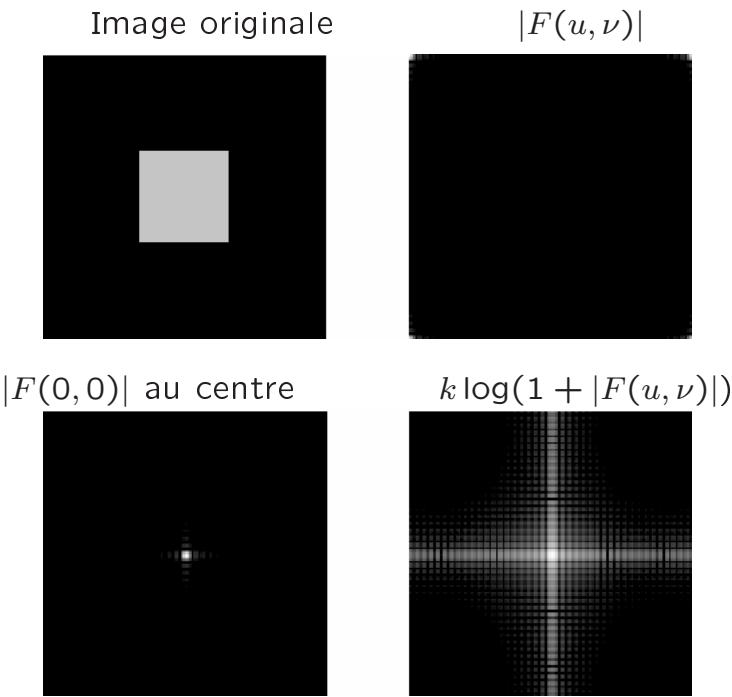
$$f(x, y) \exp \left( \frac{2\pi j(ux_0 + \nu y_0)}{N} \right) \quad \blacktriangleleft \mathcal{F} \blacktriangleright \quad F(u - u_0, \nu - \nu_0)$$

$$u_0 = \nu_0 = \frac{N}{2} \blacktriangleright \exp \left( \frac{2\pi j(ux_0 + \nu y_0)}{N} \right) = \exp (\pi j(x + y)) = (-1)^{(x+y)}$$



## TF DISCRÈTE

### TF DISCRÈTE 2D -VISUALISATION- (7)



#### Nota

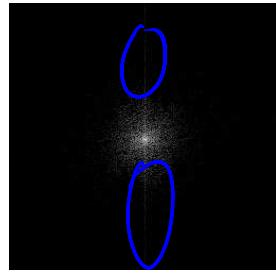
- On affiche généralement  $|F(u, \nu)|$  ( $F(u, \nu)$  complexe)
- Les fréquences élevées sont bcp plus faible que les fréquences plus basses. On affiche donc plutot  $k \log(1 + |F(u, \nu)|)$  ( $k$  est une constante de normalisation pour recaler les niveau de gris dans  $[0, 255]$ )
- On met l'origine au centre de l'image en effectuant un décalage cyclique

$$2N$$

$$\frac{2}{N}$$

# TF DISCRÈTE

11



TFD  
↓  
TF

$$\frac{32}{64} \quad 0.5$$

$$\begin{array}{r} 236 \times 4 \\ \hline 256^2 \end{array}$$

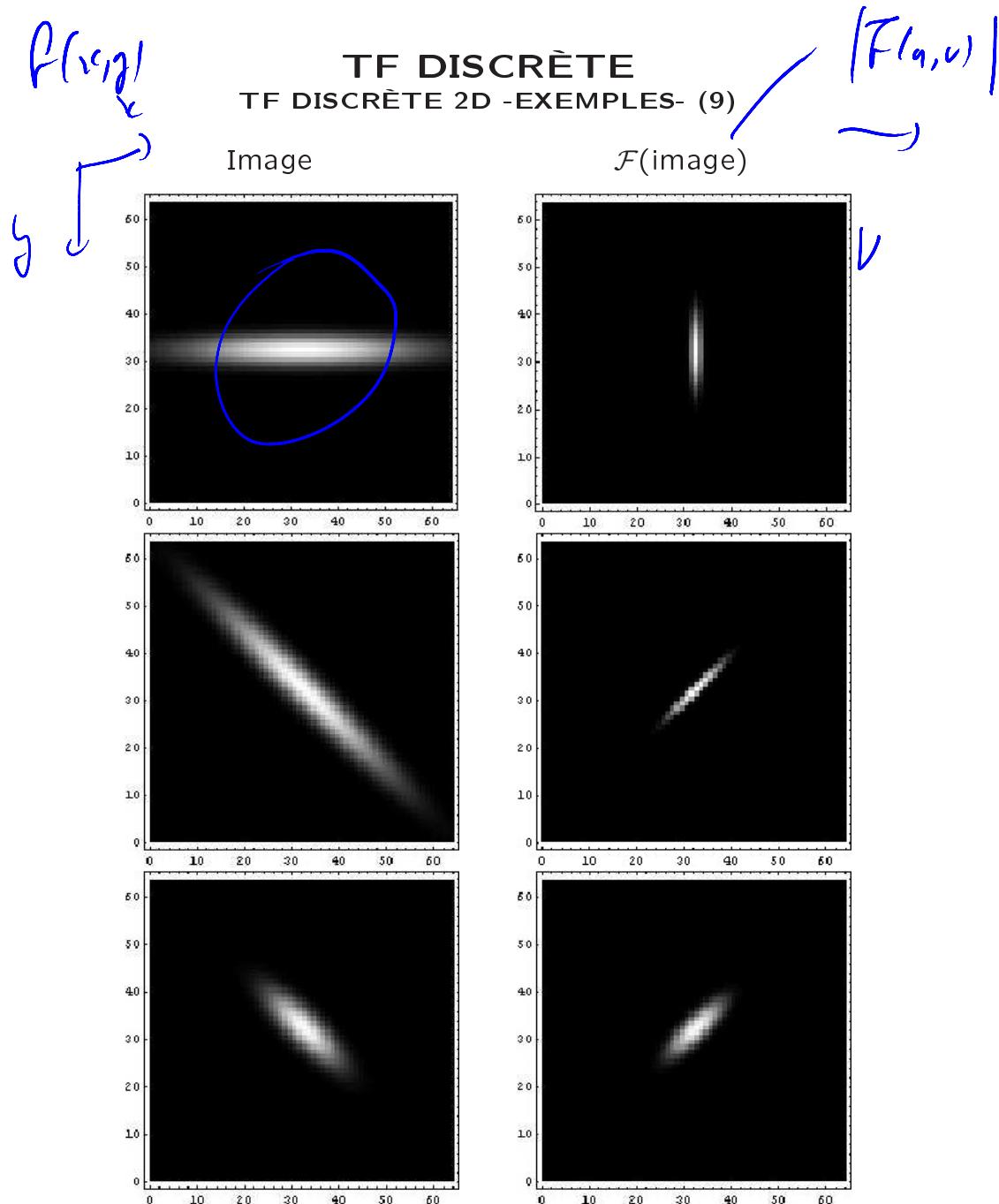
9  
251

$$\frac{2}{128} \left( \frac{1}{64} \right)$$

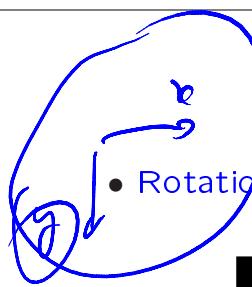
0 -

## TF DISCRÈTE

### TF DISCRÈTE 2D -EXEMPLES- (9)



## TF DISCRÈTE TF DISCRÈTE 2D -EXEMPLES- (10)



• Rotation  
Image

$f(x,y)$

$\mathcal{F}$

$(F(u,v))$   
 $\mathcal{F}(\text{image})$

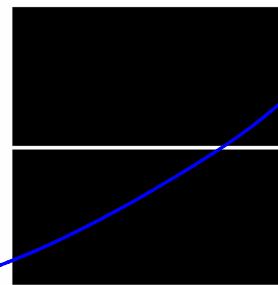
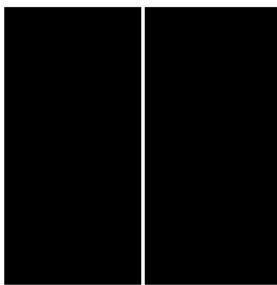


$$\begin{aligned} u &\geq 0 \\ y &\geq u \\ v &\geq v \end{aligned}$$

$$f(x,y) = \text{Cst}(x) \cdot \delta(y)$$

$\uparrow$

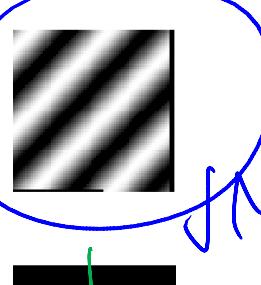
$$f(x,y) = \delta(u) \cdot \text{Cst}(v)$$



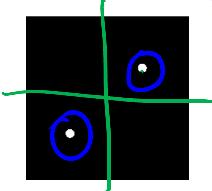
$$1 \geq \delta(v)$$

$$f(x,y) = \text{Cst}(x) \cdot \text{Cst}(y)$$

$$f(x,y) = f(u) \cdot g(v)$$



$$\delta(u) \geq 1(v) \cdot \text{Cst}(v)$$



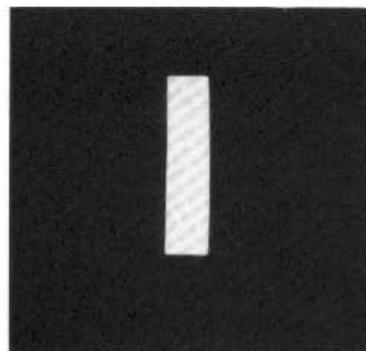
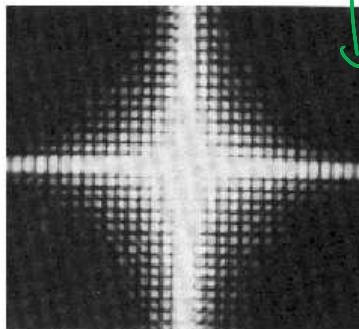
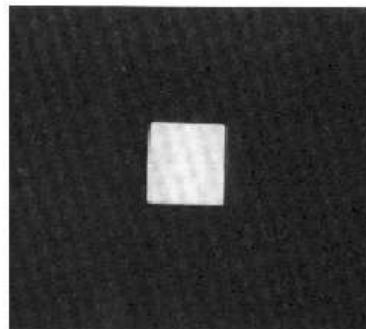
$f(x,y)$

$x$

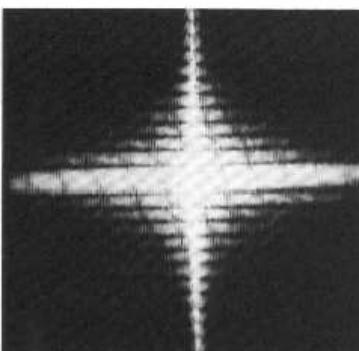
$y$

**TF DISCRÈTE**  
TF DISCRÈTE 2D -EXEMPLES- (11)

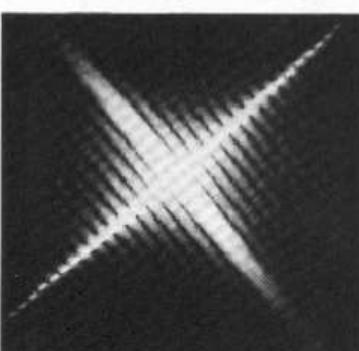
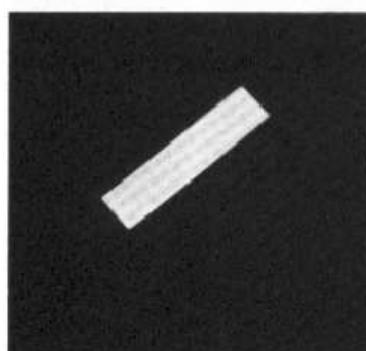
Image

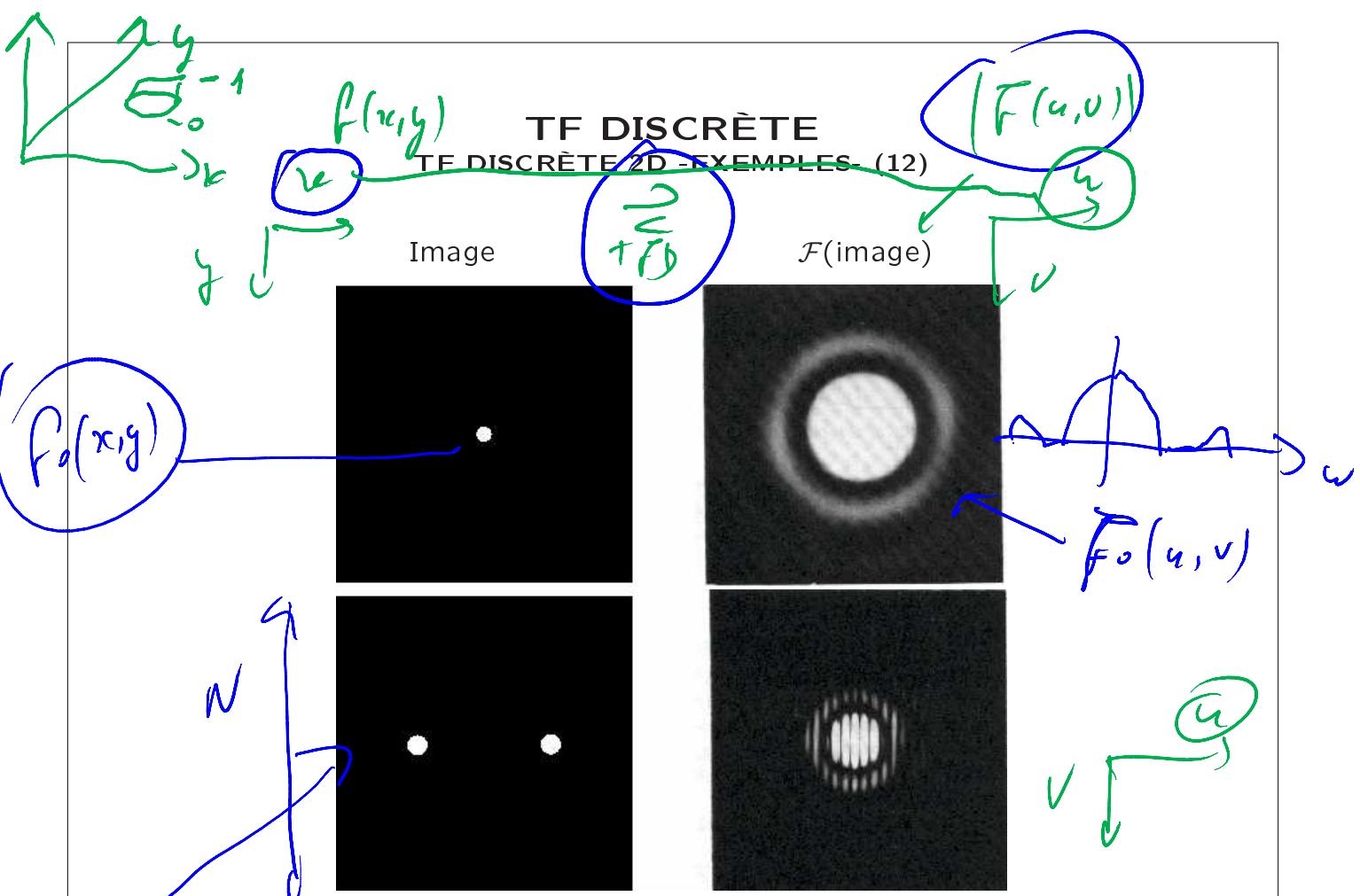


(a)



(b)



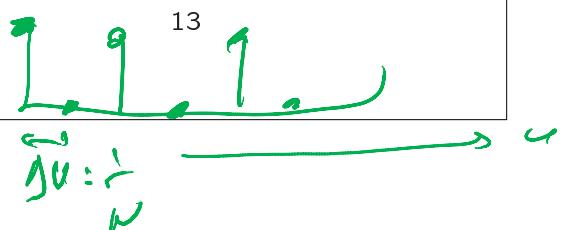


Opératé  $(F_0(x, y)) * \omega_T(x)$   $\stackrel{\mathcal{F}_{\text{DISCRÈTE}}}{\Rightarrow}$  contracté  $(F_0(u, v))$ ,  $\left(\frac{\omega_T}{N}\right)(u)$

Opératrices de discréte.

$$\Delta v = \frac{1}{N \cdot \Delta x} = \frac{1}{N T_e} = \frac{1}{N}$$

$\omega_T(u)$



Autre discréte

$\underbrace{F_0(x) \cdot \omega_T(x)}$

Autre discréte

$\underbrace{F_0(u) \cdot \omega_T(u)}$

# TF DISCRÈTE

## CONVOLUTION DISCRÈTE 1D & 2D (1)

Théorème de convolution 2D

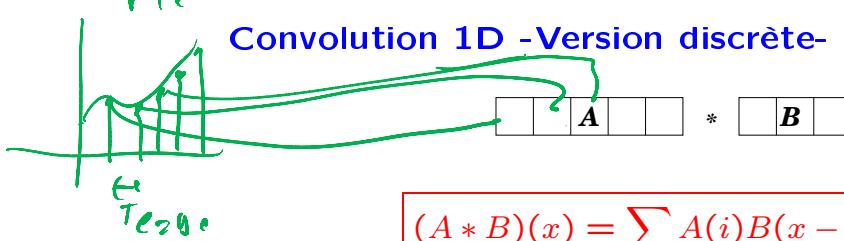
$$f(x, y) * g(x, y) \xrightarrow{\mathcal{F}} F(u, \nu) \cdot G(u, \nu)$$

$$f(x, y) \cdot g(x, y) \xrightarrow{\mathcal{F}} F(u, \nu) * G(u, \nu)$$

Convolution 2D -Version continue-

$$(f * g)(x, y) = \int_0^\infty \int_0^\infty f(\alpha, \beta) g(x - \alpha, y - \beta) d\alpha d\beta$$

Convolution 1D -Version discrète-



$$(A * B)(x) = \sum_i A(i)B(x - i)$$

-Exemple-

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \quad x=0$$

$$B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad x=0$$

$$\begin{array}{c} ? \\ \downarrow x=1 \\ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \\ \downarrow x \quad x \quad x \\ \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \\ 3 \quad 4 \quad 3 \end{array}$$

$$(A * B)(1)$$

$$\begin{array}{c} ? \\ \downarrow x=2 \\ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \\ \downarrow x \quad x \quad x \\ \begin{bmatrix} 2 & 1 \end{bmatrix} \\ 6 \quad 6 \quad 4 \end{array}$$

$$(A * B)(2)$$

# TF DISCRÈTE

## CONVOLUTION DISCRÈTE 1D & 2D (2)

### Convolution des bords

#### A) Ajout de zéro

0	0	1	2	3	4	5	0	0
3	2	1					2	1
0	4	10	16	22	22			

(2 + 1)

#### B) Enroulement

4	5	1	2	3	4	5	1	2
3	2	1					2	1
19	10	16	22	22				

(2 + 10 + 1)

#### C) Réflexion

3	2	1	2	3	4	5	4	3
3	2	1					2	1
10	10	16	22	26				

6 + 2 + 1

#### D) Défaut

1	2	3	4	5
	10	16	22	

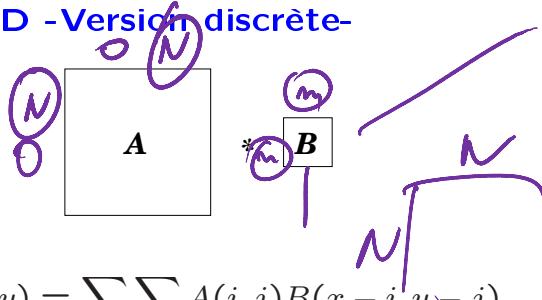
## TF DISCRÈTE

### CONVOLUTION DISCRÈTE 1D & 2D (3)

#### Convolution 2D -Version discrète-

$$O(N^2 m^2)$$

$$O(N^4)$$



$$(A * B)(x, y) = \sum_i \sum_j A(i, j)B(x - i, y - j)$$

-Exemple-

1	0	0
1	10	5
10	5	20
20	20	20
20	20	20

$$\downarrow * \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

-10	5	-15	0	0
-10	15	-10	20	20
-10	15	-10	20	20
-10	15	-10	20	20
-10	15	-10	20	20

(Ajout de zéro)

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

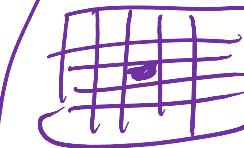
Original

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$x$  reflexion

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$y$  reflexion



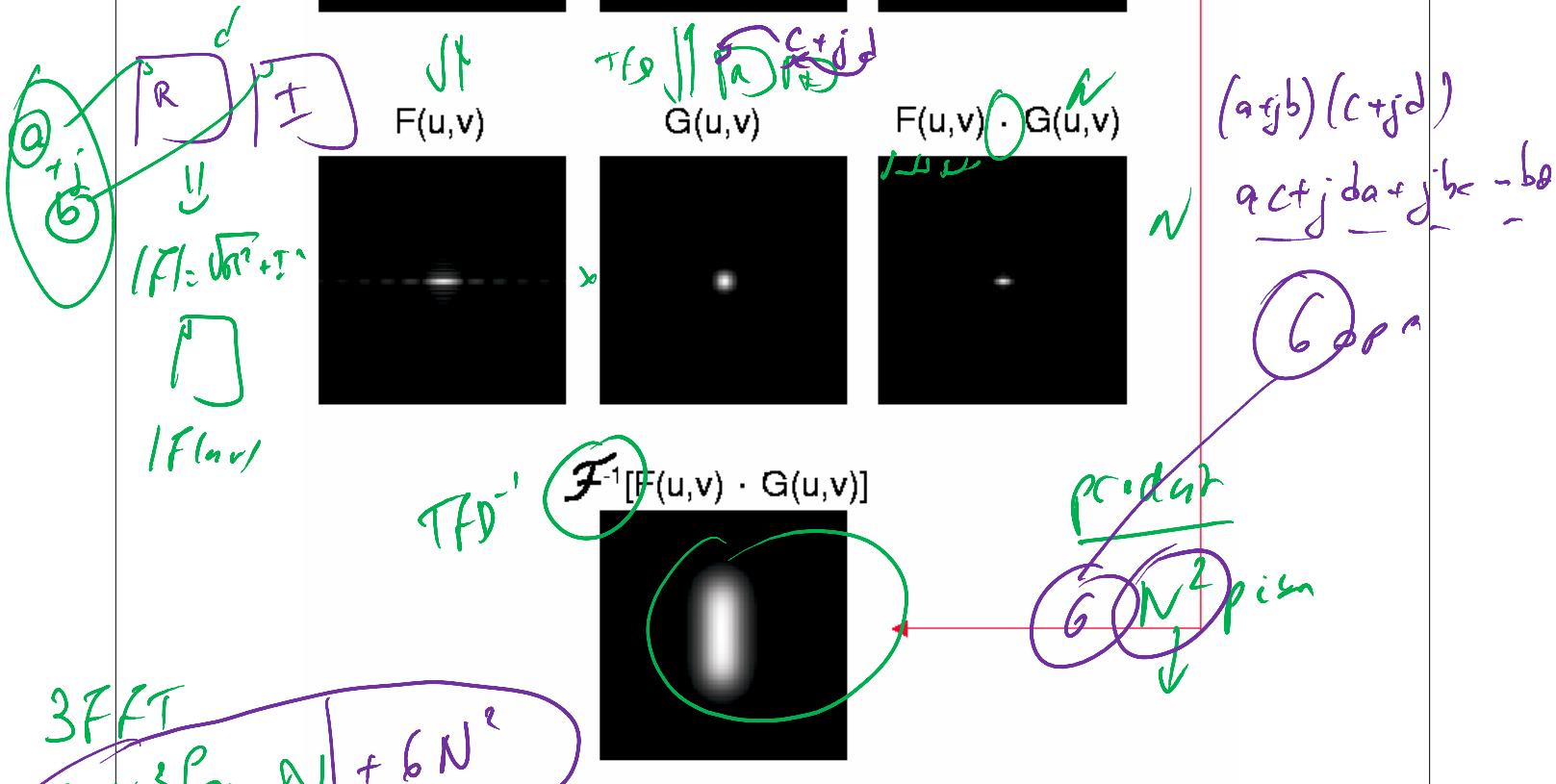
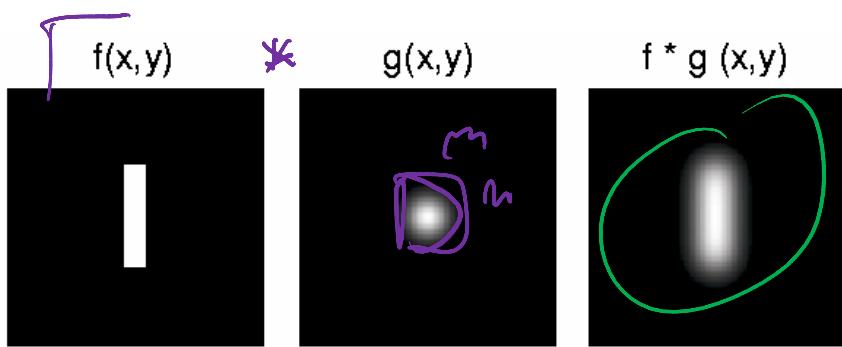
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

#### PROPRIÉTÉS

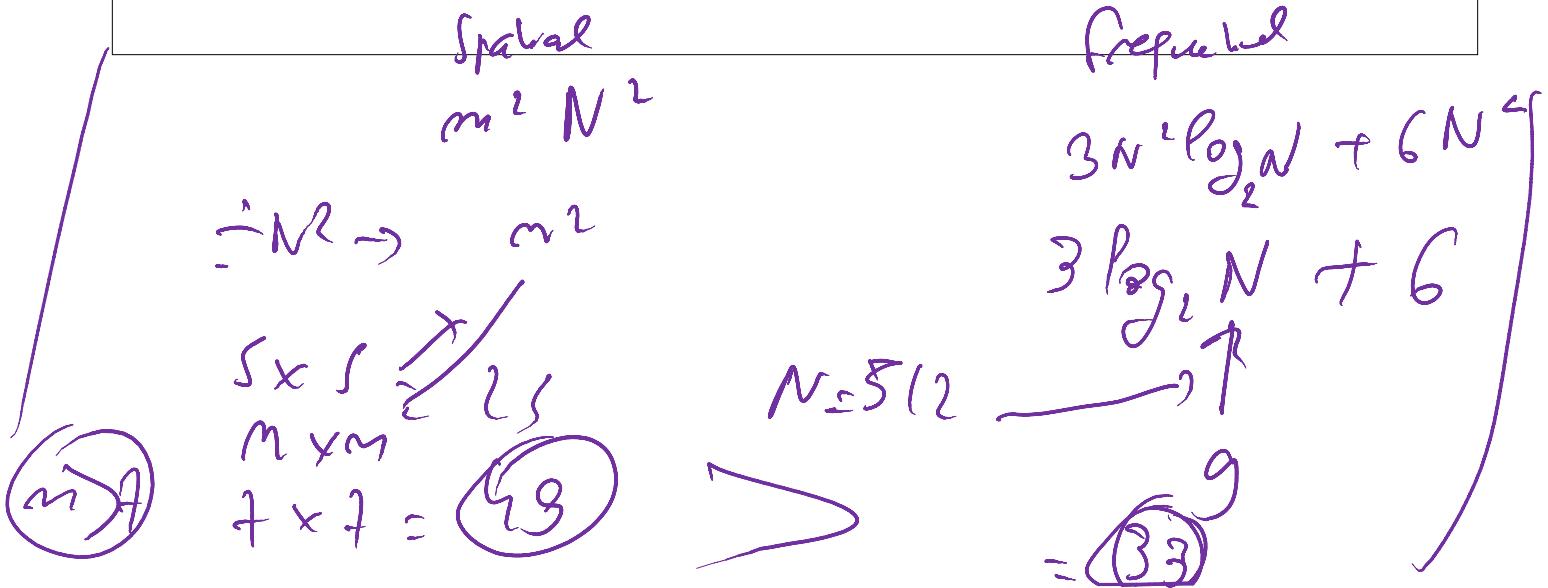
- Commutatif  $\triangleright f_1 * f_2 * f = f_2 * f_1 * f$
- Associatif  $\triangleright (f_1 * f_2) * f = f_1 * (f_2 * f)$
- Distributif  $\triangleright (f_1 + f_2) * f = f_1 * f + f_2 * f$
- $\triangleright f * (f_1 + f_2) = f * f_1 + f * f_2$

## TF DISCRÈTE

### CONVOLUTION DISCRÈTE 1D & 2D (4)

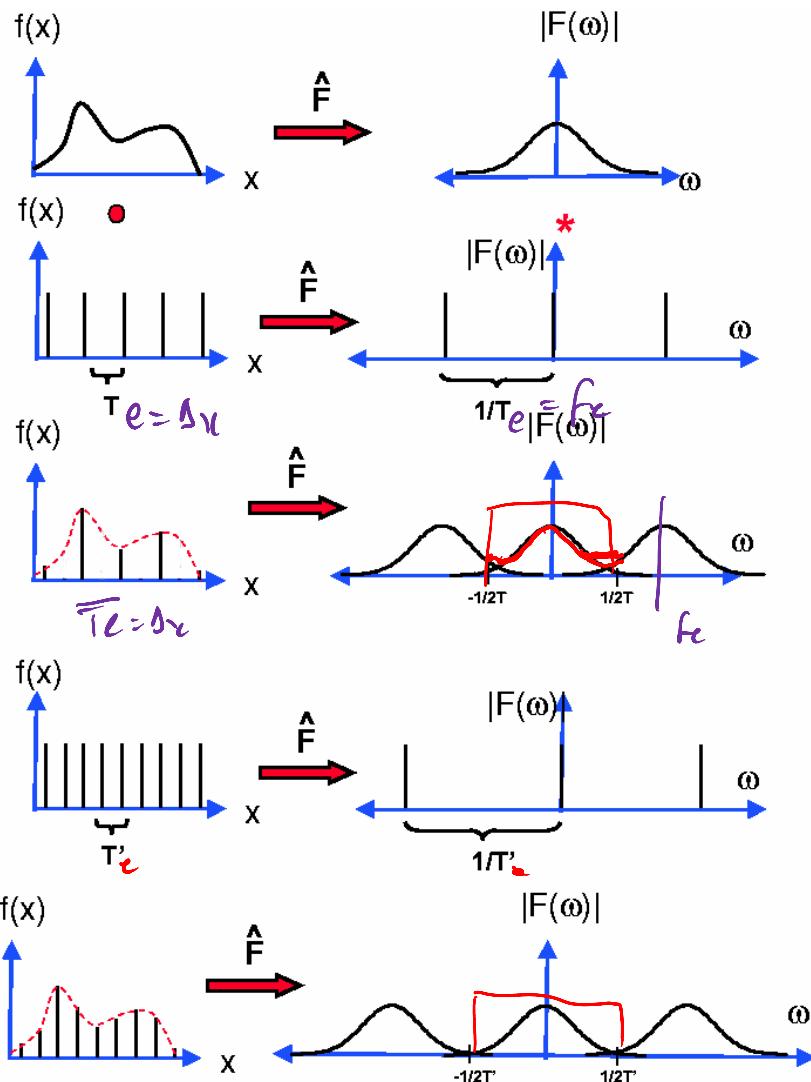


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## TF DISCRÈTE

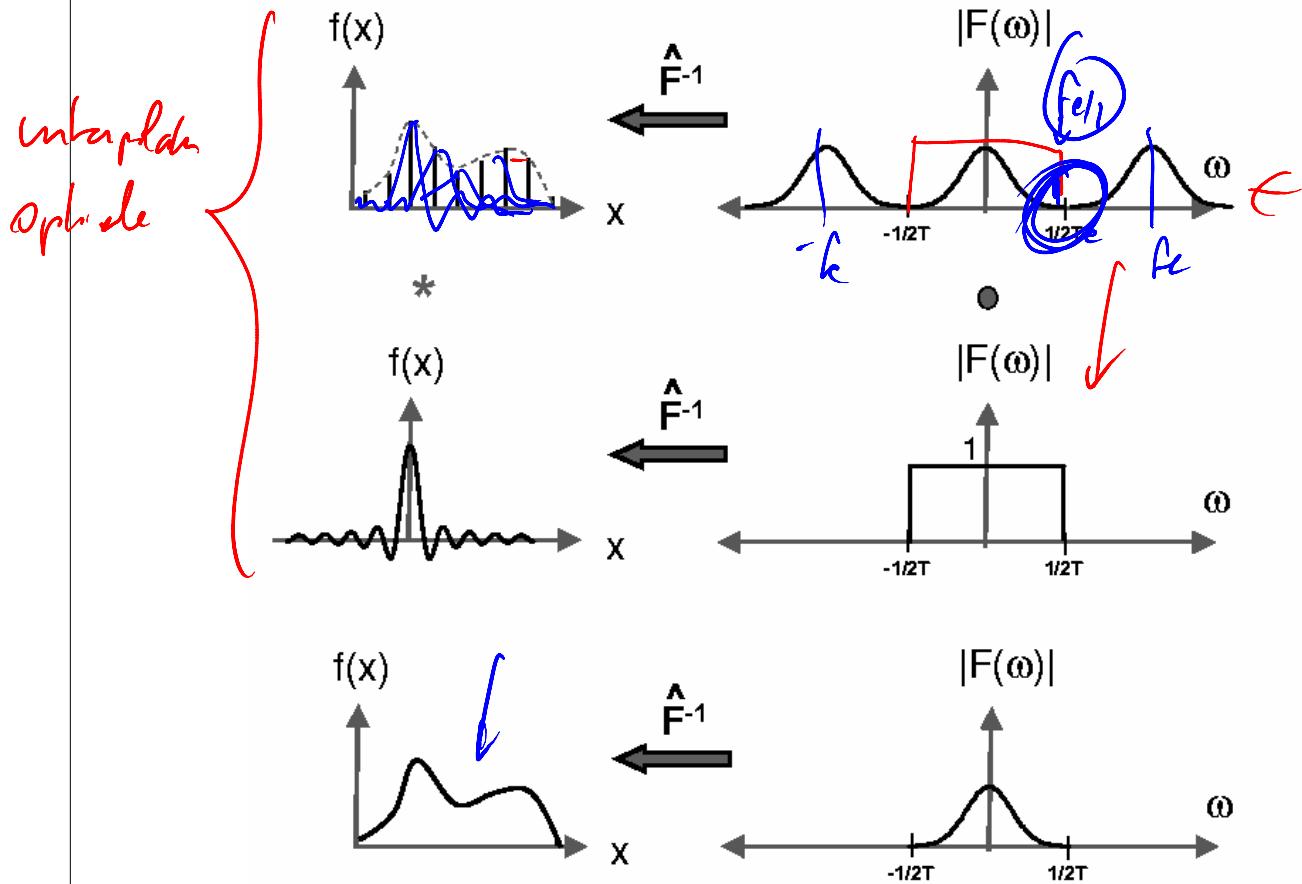
### THÉORÈME DE SCHANNON



$$T_e \leq \frac{1}{2\nu_0}$$

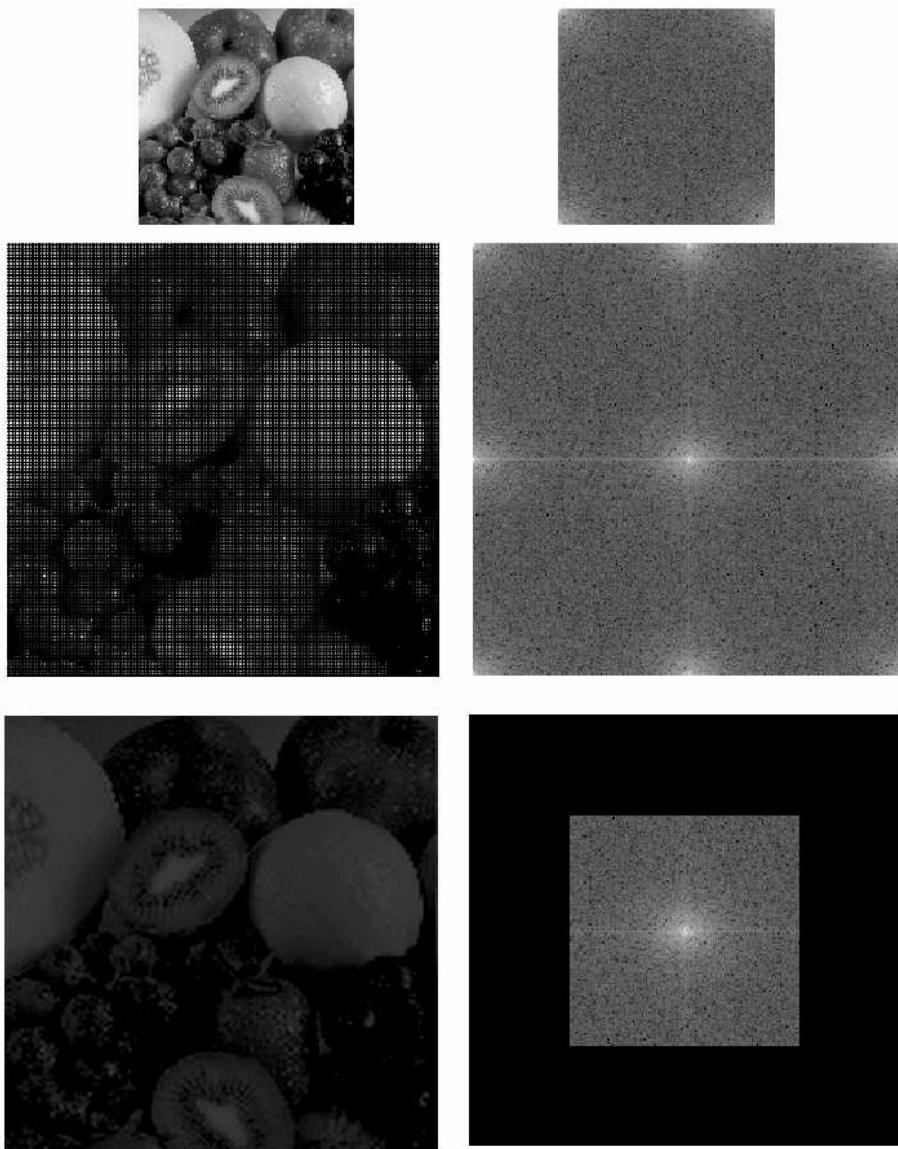
► sinon phénomène d'aliasing

## TF DISCRÈTE INTERPOLATION OPTIMALE (1)

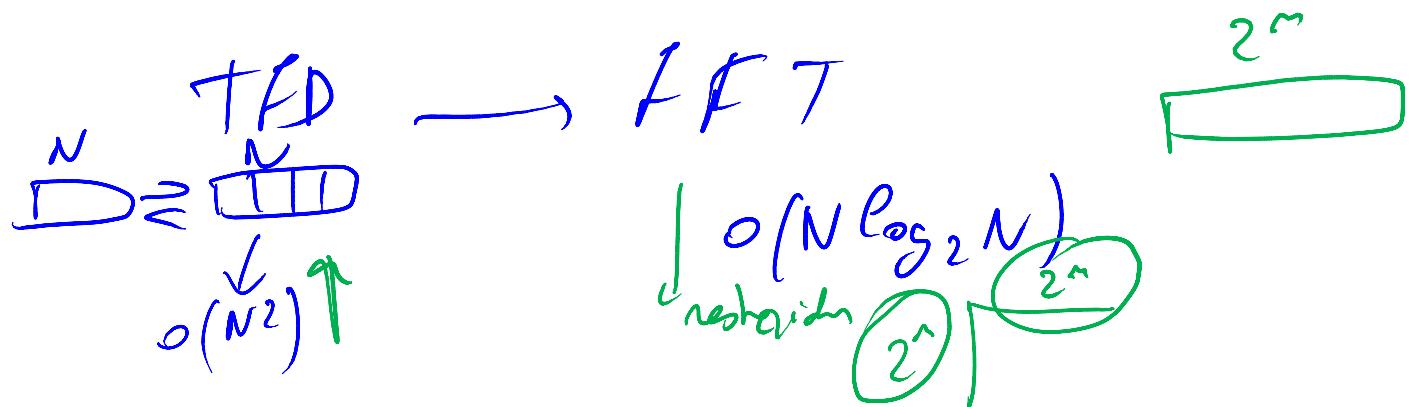


- ▶ Possibilité de reconstruire totalement le signal 2D à partir de ses échantillons

**TF DISCRÈTE**  
INTERPOLATION OPTIMALE (2)



20



Coley-Tucker  
1968

$F(u)$   $F(0)$   $F(1)$   $F(2)$   $F(3)$   $N=4$

**TF DISCRÈTE**  
FFT (1)

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi j u x}{N}\right) \quad u = 0, 1, 2, \dots, N-1$$

$O(N^2)$  opérations

$$F(u) = \underbrace{\frac{1}{N} \sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi j u 2x}{N}\right)}_{x \text{ pair}} + \underbrace{\frac{1}{N} \sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi j u (2x+1)}{N}\right)}_{x \text{ impair}}$$

$$= \frac{1}{2} \left[ \underbrace{\frac{1}{N/2} \sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi j u x}{N/2}\right)}_{\text{TF de la partie paire}} + \exp\left(\frac{-2\pi j u}{N}\right) \underbrace{\frac{1}{N/2} \sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi j u x}{N/2}\right)}_{\text{TF de la partie impaire}} \right]$$

- TF [N éléments] = 2 TF [N/2 éléments] + N/2 multiplications et N/2 additions complexes (en fait N/4 pour les multiplications)
- $Op(N) = 2Op(N/2) + N$

$$F_N(u) = \frac{1}{2} \left[ F_{N/2}^{\text{paire}}(u) + \exp\left(\frac{-2\pi j u}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

Pour  $u' = u + \frac{N}{2}$

$$\exp\left(-\frac{2\pi j u'}{N}\right) = \exp\left(-\frac{2\pi j (u + \frac{N}{2})}{N}\right) = \exp\left(-\frac{2\pi j u}{N}\right) \exp(-\pi j) = -\exp\left(-\frac{2\pi j u}{N}\right)$$

## TF DISCRÈTE

FFT (2)

Ainsi

$$F_N(u) = \frac{1}{2} [F_{N/2}^{\text{paire}}(u) + \exp\left(-\frac{2\pi j u}{N}\right) F_{N/2}^{\text{impaire}}(u)]$$

$$F_N(u + \frac{N}{2}) = \frac{1}{2} [F_{N/2}^{\text{paire}}(u) - \exp\left(-\frac{2\pi j u}{N}\right) F_{N/2}^{\text{impaire}}(u)]$$

Pour  $u = 0, 1, 2, \dots, N/2 - 1$

1 multiplication est nécessaire pour deux termes

$u = 0$

$$\mathcal{F}[\mathcal{F}(f_0)] =$$

$$\mathcal{TFD}[\mathcal{F}(f_0)] = ?$$

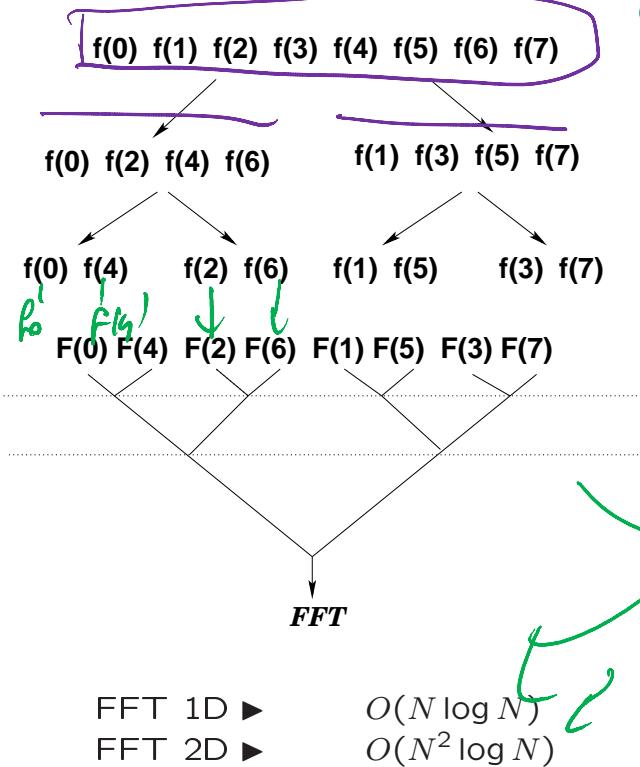
$$\boxed{f(u)} \geq \boxed{f(u)}$$

$$\mathcal{F}(u) \geq \mathcal{F}(u)$$

$$\mathcal{TFD}[\boxed{a}] = \boxed{a}$$

$$\mathcal{FFD}[\boxed{a+b}] = \boxed{a+b}$$

22



FFT 1D ►

FFT 2D ►

$O(N \log N)$

$O(N^2 \log N)$

$$\mathcal{FFT}(\boxed{11111111}) = \boxed{11000000}$$

## TF DISCRÈTE

### FFT (3)

#### Vérifions que $O(N \log N)$

(Par induction, sachant que  $\text{Op}(N) = 2 \text{Op}(N/2) + N$ )

- Vrai pour  $N = 1$ ?  $\rightarrow \text{Op}(1) = 1 \log 1 = 0 \rightarrow$  oui
- Vrai pour  $N = 2$ ?  $\rightarrow \text{Op}(2) = 2 \log 2 = 2 \rightarrow$  oui
- Vrai pour  $N \log N$ ?

- $\text{Op}(N) = N \log N$ ?
- $\text{Op}(N) = 2N/2 \log(2N/2)$
- $\text{Op}(N) = 2N/2(\log(N/2) + \log 2)$
- $\text{Op}(N) = 2N/2 \log N/2 + 2N/2 \log 2$
- $\text{Op}(N) = 2 \text{Op}(N/2) + N$

C.Q.F.D.

$$2 \cdot \frac{N}{2} \log \frac{N}{2}$$

$$\begin{aligned} \text{FFT}(\mathbf{f}_a) &= \mathbf{f}_a \\ \text{FFT}(\mathbf{f}_a \mathbf{f}_b) &= \mathbf{f}_a \mathbf{f}_b \end{aligned}$$

#### Rehaussement d'image

