



DIRO
IFT 6150

TRAITEMENT D'IMAGES TRANSFORMÉE DE FOURIER DISCRÈTE

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TF DISCRÈTE

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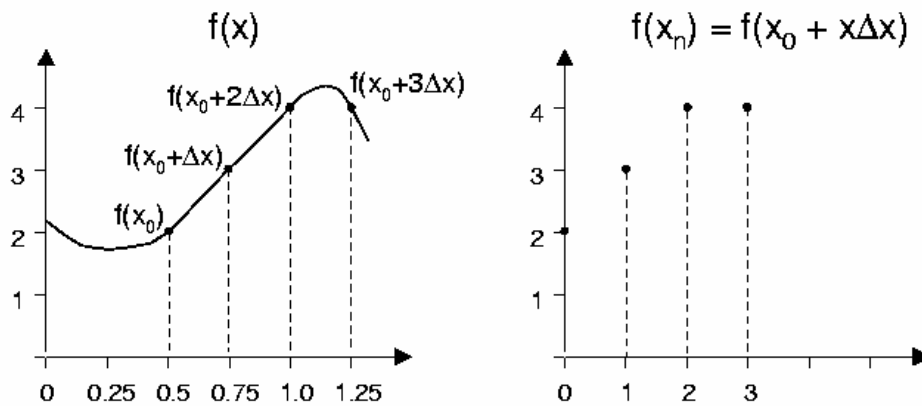
TF DISCRÈTE

TF DISCRÈTE 1D (1)

CAS CONTINU (RAPPEL)

$$\mathcal{F}[f(x)] = F(u) = \int_{-\infty}^{+\infty} f(x) \exp(-2\pi j u x) dx \quad u \in \mathbb{R}.$$

CAS DISCRET



$$f(x) = f(x_0 + x\Delta x) \quad x = 0, 1, 2, \dots, N-1$$

TFD et TFD inverse

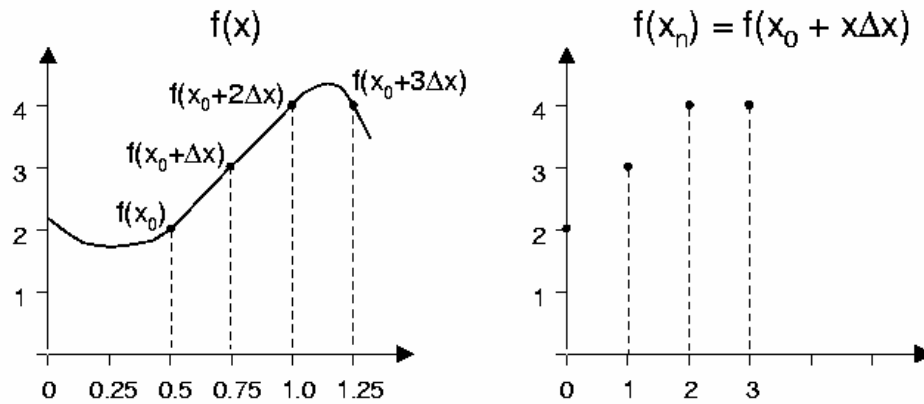
$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi j u x}{N}\right) \quad u = 0, 1, 2, \dots, N-1$$

$$f(x) = \sum_{u=0}^{N-1} F(u) \exp\left(\frac{2\pi j u x}{N}\right) \quad x = 0, 1, 2, \dots, N-1$$

$$\Delta u = \frac{1}{N\Delta x}$$

TF DISCRÈTE

TF DISCRÈTE 1D -EXEMPLE- (2)



$$\begin{aligned}
 F(0) &= \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-2\pi j 0 x}{4}\right) = \frac{1}{4} \sum_{x=0}^3 f(x) \cdot 1 \\
 &= \frac{1}{4} (f(0) + f(1) + f(2) + f(3)) = \frac{1}{4} (2 + 3 + 4 + 4) = 3.25
 \end{aligned}$$

$$\begin{aligned}
 F(1) &= \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-2\pi j x}{4}\right) \\
 &= \frac{1}{4} (2 \exp(0) + 3 \exp(-j\pi/2) + 4 \exp(-j\pi) + 4 \exp(-3j\pi/2)) \\
 &= \frac{1}{4} (-2 + j)
 \end{aligned}$$

$$F(2) = \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-4\pi j x}{4}\right) = \dots = -1/4$$

$$F(3) = \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-6\pi j 0 x}{4}\right) = \dots = 1/4(-2 - j)$$

Spectre d'amplitude

$$|F(0)| = 3.25 \quad |F(1)| = \sqrt{5}/4 \quad |F(2)| = 1/4 \quad |F(3)| = \sqrt{5}/4$$

TF DISCRÈTE

TF DISCRÈTE 2D (1)

TFD 2D et TFD 2D inverse

$$F(u, \nu) = \frac{1}{MN} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x, y) \exp \left(-2\pi j \left(\frac{ux}{N} + \frac{\nu y}{M} \right) \right)$$
$$u = 0, 1, 2, \dots, N-1$$
$$\nu = 0, 1, 2, \dots, M-1$$

$$f(x, y) = \sum_{u=0}^{N-1} \sum_{\nu=0}^{M-1} F(u, \nu) \exp \left(2\pi j \left(\frac{ux}{N} + \frac{\nu y}{M} \right) \right)$$
$$x = 0, 1, 2, \dots, N-1$$
$$y = 0, 1, 2, \dots, M-1$$

Le plus souvent, l'image à traiter est carrée $\blacktriangleleft \blacktriangleright M = N$

PROPRIÉTÉS

- Les mêmes que ceux énoncé pour la TF continue
- Cyclique (périodique)

$$F(u, \nu) = F(u + N, \nu) = F(u, \nu + N) = F(u + N, \nu + N)$$

$$f(x, y) = f(x + N, y + N)$$

car

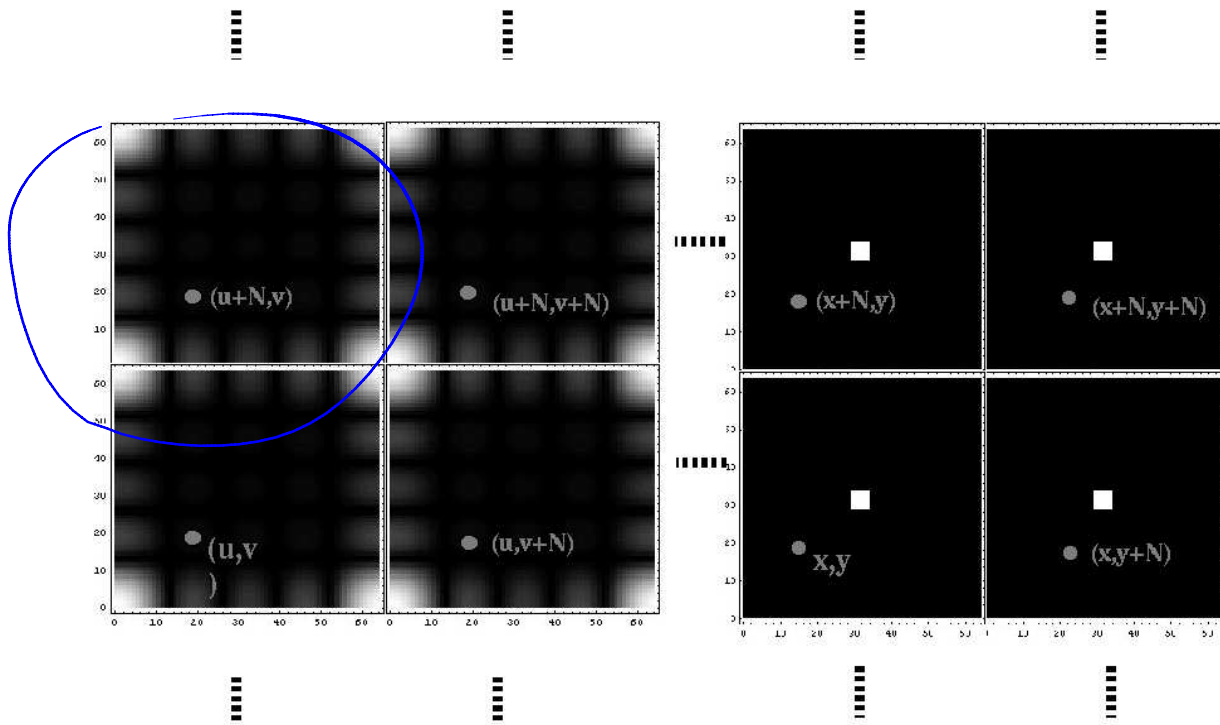
Signal périodique $-\mathcal{F} \rightarrow$ Spectre de raies

Signal échantillonné $-\mathcal{F} \rightarrow$ Spectre périodique

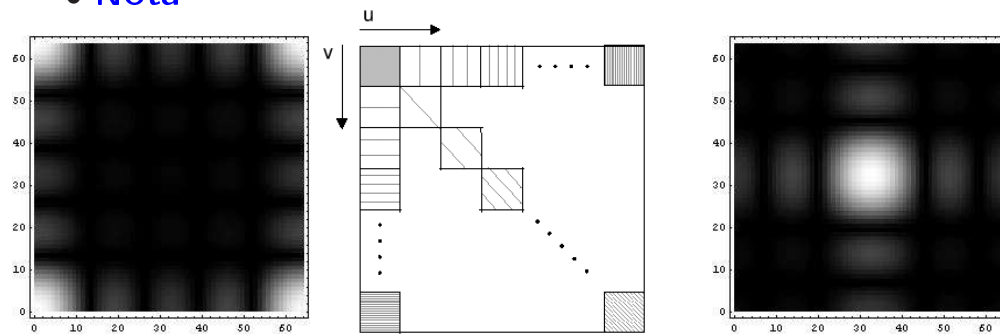
TF DISCRÈTE

TF DISCRÈTE 2D (4)

- Périodicité



- Nota



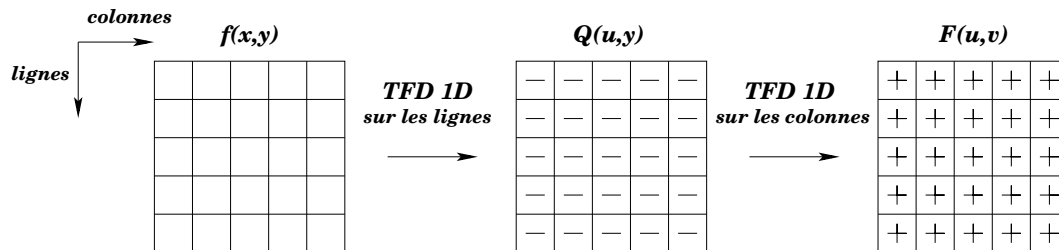
TF DISCRÈTE

TF DISCRÈTE 2D (5)

- Séparabilité

Pour une image carrée $\blacktriangleleft \blacktriangleright M = N$

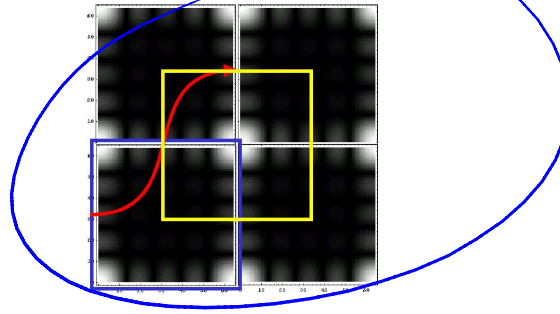
$$\begin{aligned}
 F(u, \nu) &= \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) \exp \left(-2\pi j \left(\frac{ux + \nu y}{N} \right) \right) \\
 &= \frac{1}{N} \sum_{x=0}^{N-1} \left(\frac{1}{N} \sum_{y=0}^{N-1} f(x, y) \exp \left(-2\pi j \frac{\nu y}{N} \right) \right) \exp \left(-2\pi j \frac{ux}{N} \right) \\
 &= \frac{1}{N} \sum_{x=0}^{N-1} \underbrace{\left[\text{TF 1D de la } x \text{ ième colonne} \right]}_{\text{TF 1D de la } y \text{ ième ligne}} \exp \left(-2\pi j \frac{ux}{N} \right)
 \end{aligned}$$



TF DISCRÈTE

TF DISCRÈTE 2D (6)

Recalage Cyclique

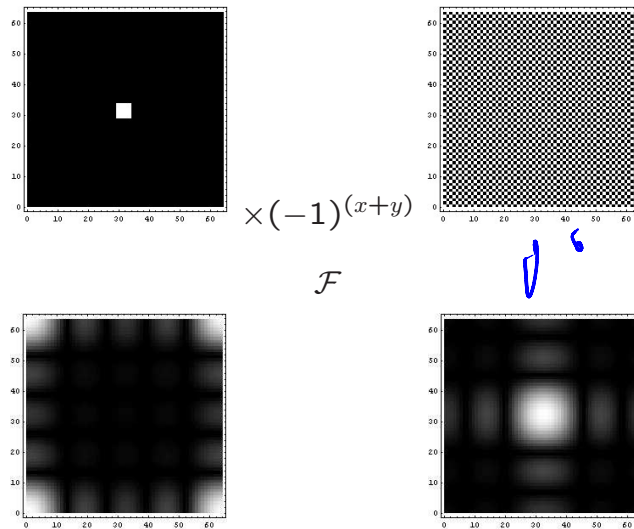


Translation -Rappel-

$$f(x - x_0, y - y_0) \quad \leftarrow \mathcal{F} \quad F(u, \nu) \exp \left(\frac{-2\pi j (ux_0 + \nu y_0)}{N} \right)$$

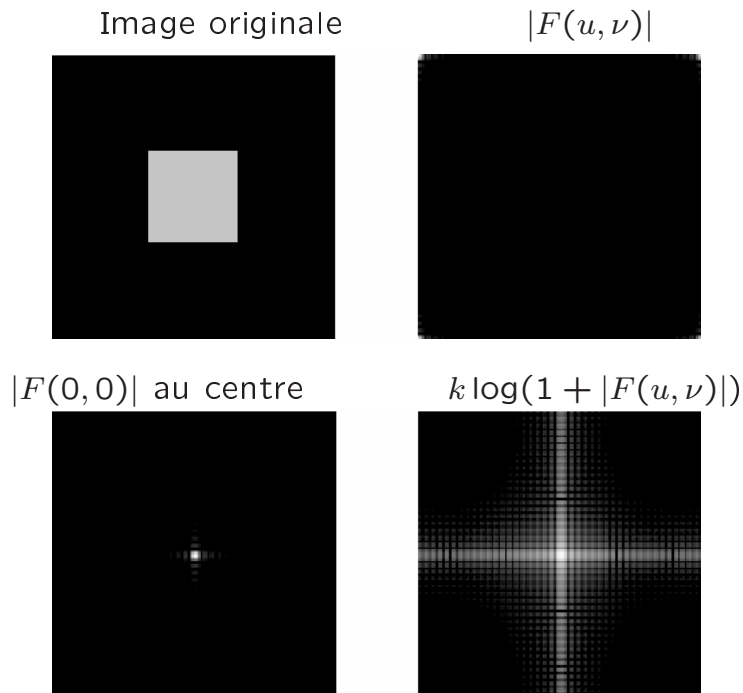
$$f(x, y) \exp \left(\frac{2\pi j (u_0 x + \nu_0 y)}{N} \right) \quad \leftarrow \mathcal{F} \quad F(u - u_0, \nu - \nu_0)$$

$$u_0 = \nu_0 = \frac{N}{2} \rightarrow \exp \left(\frac{2\pi j (u_0 x + \nu_0 y)}{N} \right) = \exp (\pi j (x + y)) = (-1)^{(x+y)}$$



TF DISCRÈTE

TF DISCRÈTE 2D -VISUALISATION- (7)



Nota

- On affiche généralement $|F(u, \nu)|$ ($F(u, \nu)$ complexe)
- Les fréquences élevées sont bcp plus faible que les fréquences plus basses. On affiche donc plutôt $k \log(1 + |F(u, \nu)|)$ (k est une constante de normalisation pour recaler les niveau de gris dans $[0, 255]$)
- On met l'origine au centre de l'image en effectuant un décalage cyclique

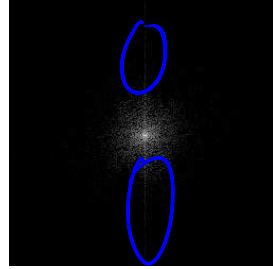
$$\frac{2N}{N^2}$$

$$\frac{2}{N}$$

TF DISCRÈTE

TF DISCRÈTE 2D -INTERPRÉTATION- (8)

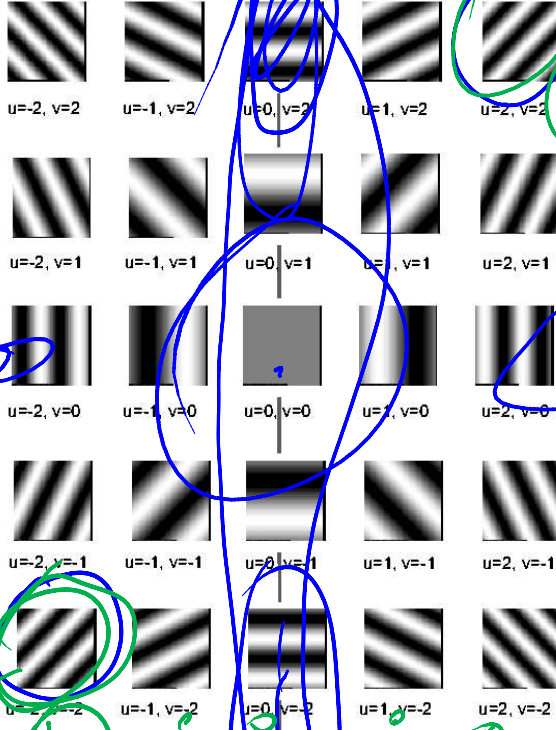
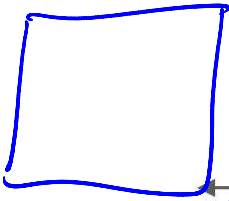
TFD
↓
TF



$$\frac{32}{64}$$

256x256

$$\frac{236 \times 4}{256^2}$$



$$\frac{4}{256}$$

$$\frac{2}{128}$$

$$\frac{1}{64}$$

TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (9)

$F(u,v)$

u

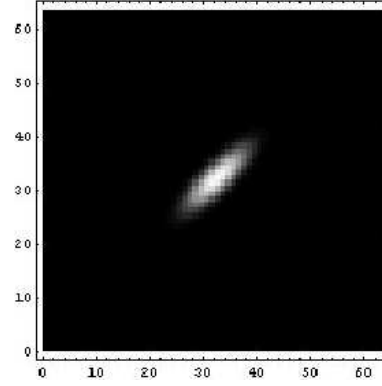
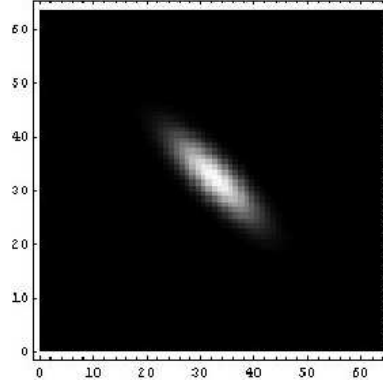
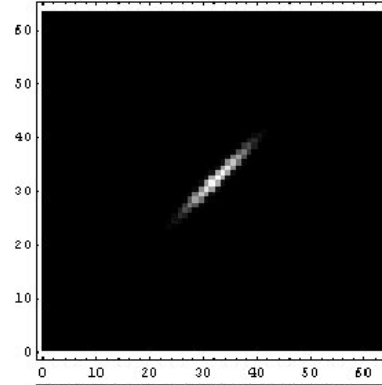
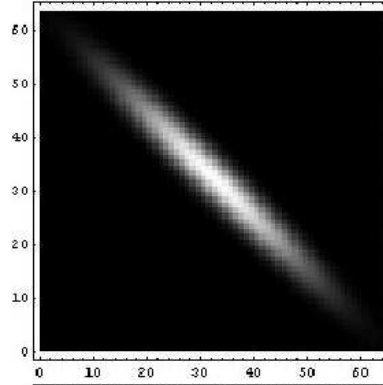
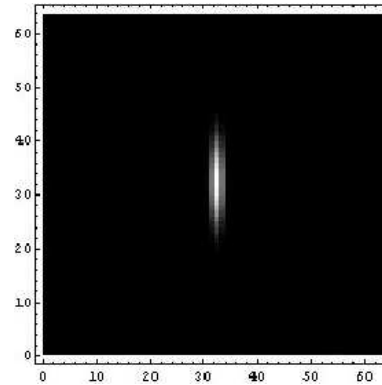
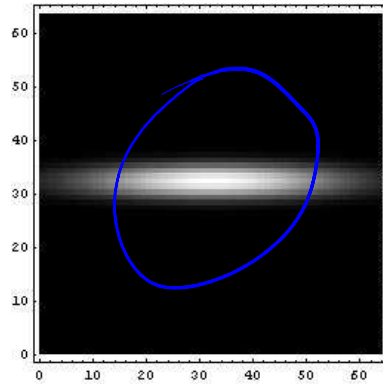
v

$|F(u,v)|$

v

Image

$\mathcal{F}(\text{image})$



TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (10)

• Rotation

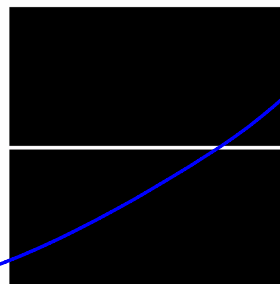
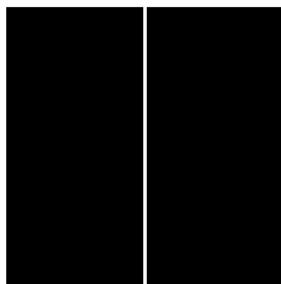
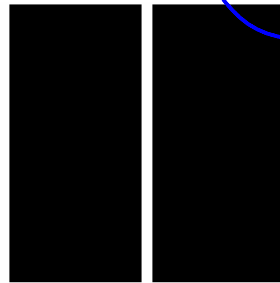
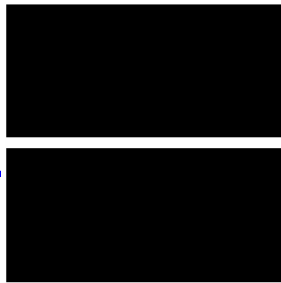
Image

$f(x,y)$

\mathcal{F}

$(F(u,v))$

$\mathcal{F}(\text{image})$



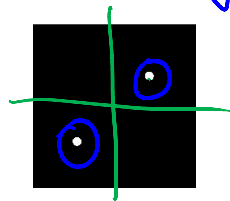
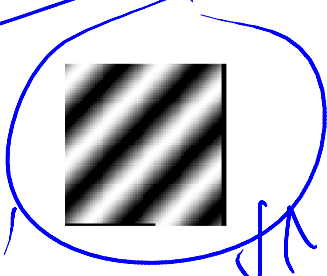
$$f(x,y) = \delta(x) \cdot \text{Cst}(y)$$

$$F(u,v) = \delta(u) \times \text{Cst}(v)$$

• Périodicité

$$F(u,v) = \delta(u) \cdot \text{Cst}(v)$$

$$F(u) \times g(v) = F(u) \cdot g(v)$$



$$\delta(x) \approx \delta(y)$$

$$\delta(y) \approx \delta(v)$$

$$1 \approx \delta(v)$$

$$\delta(y) \approx \delta(v)$$

$$\text{Cst}(v)$$

TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (11)

A

$f(x,y)$

x

y

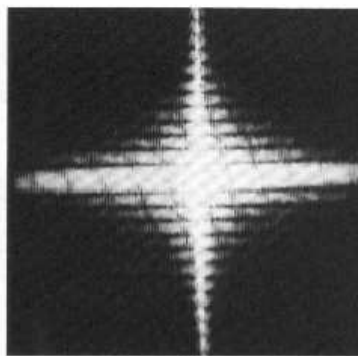
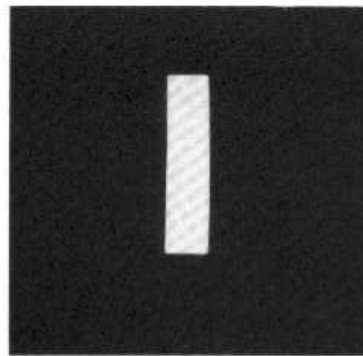
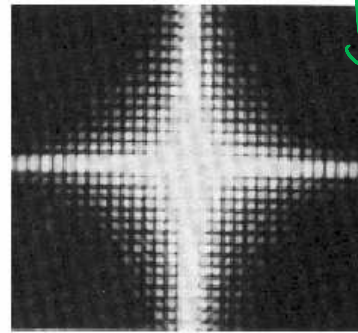
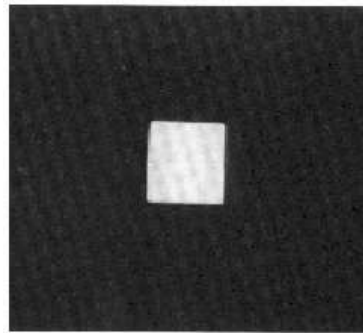
$|F(u,v)|$

u

v

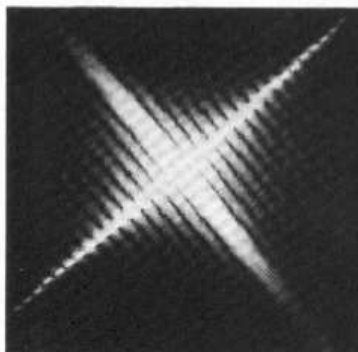
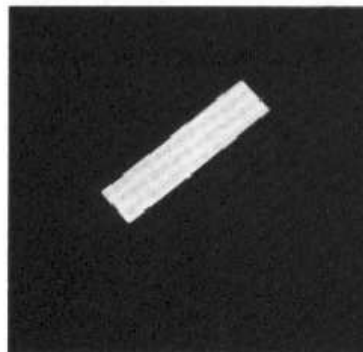
Image

$\mathcal{F}(\text{image})$



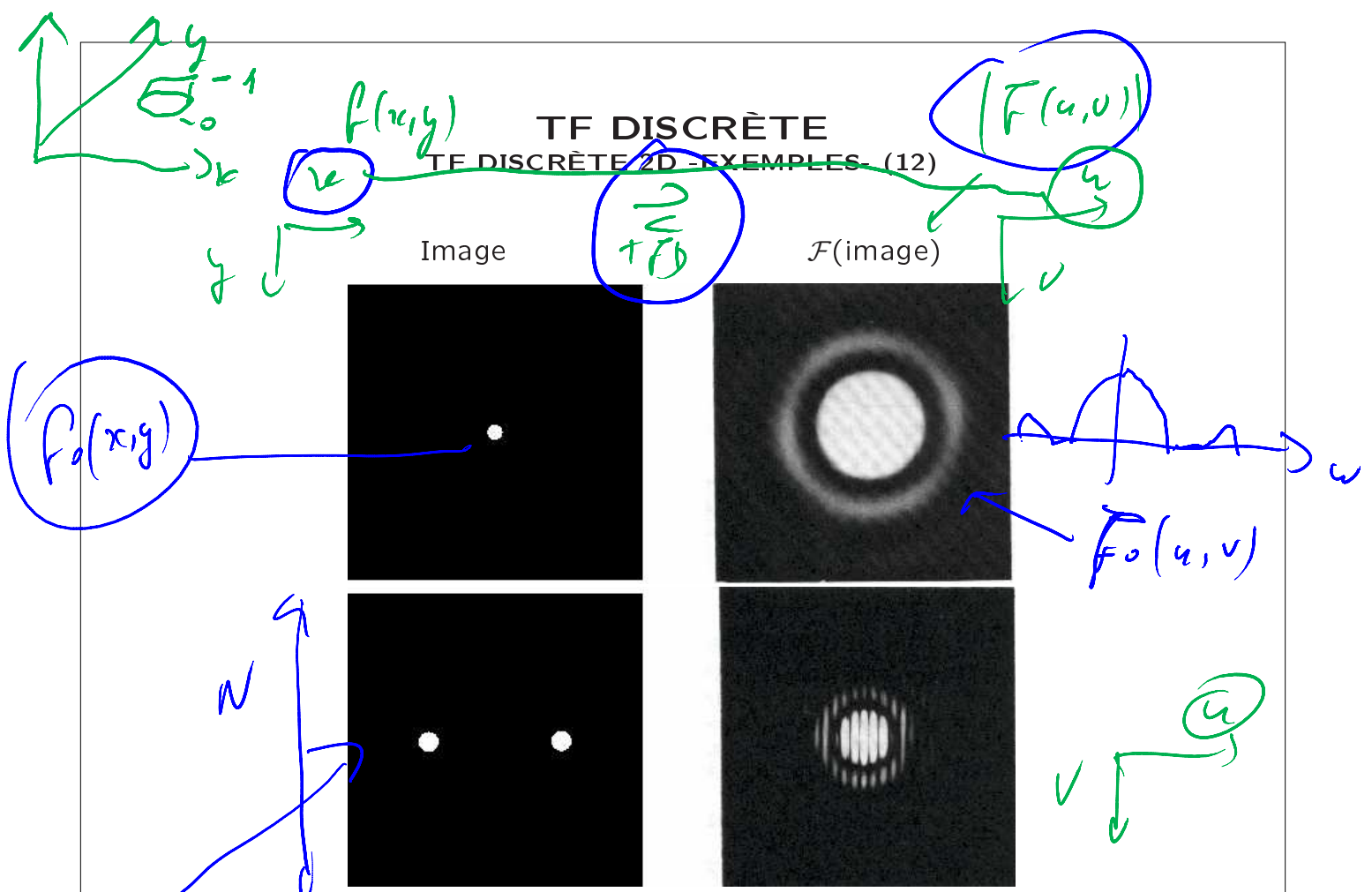
(a)

(b)



TF DISCRÈTE

TF DISCRÈTE 2D - EXEMPLES (12)

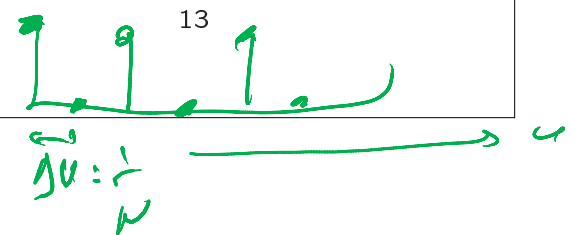


$$\text{échantillonné } (F_0(x,y)) * \underbrace{\omega_{\frac{N}{2}}(x)}_{\frac{1}{2}T} \stackrel{\text{TFD}}{=} \text{contracté } (F_0(u,v)), \underbrace{\omega_{\frac{N}{2}}(u)}_{\frac{1}{N}}$$

Opérateur de discrétisation

$$\Delta v = \frac{1}{N \cdot \Delta x} = \frac{1}{N T_e} = \frac{1}{N}$$

$$\omega_{\frac{N}{2}}(u)$$



Arbre de discrétisation

$$F(u,v) = \omega_{T_e}(x)$$

Loi de périodicité

$$F_0(u,v) * \omega_T(x)$$

TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (1)

Theoreme de convolution 2D

$$\begin{aligned} f(x, y) * g(x, y) &\xrightarrow{-\mathcal{F}} F(u, v) \cdot G(u, v) \\ f(x, y) \cdot g(x, y) &\xrightarrow{-\mathcal{F}} F(u, v) * G(u, v) \end{aligned}$$

Convolution 2D -Version continue-

$$(f * g)(x, y) = \int_0^\infty \int_0^\infty f(\alpha, \beta) g(x - \alpha, y - \beta) d\alpha d\beta$$

Convolution 1D -Version discrète-

$$(A * B)(x) = \sum_i A(i) B(x - i)$$

-Exemple-

A
1 2 3 4 5

x=0

x
1 2 3 4 5

3 2 1

3 4 3

?	10			
---	----	--	--	--

(A*B)(1)

B
1 2 3

x=0

x
1 2 3 4 5

3 2 1

6 6 4

10	16	22	
----	----	----	--

(A*B)(2)

Convolution des bords

Handwritten diagram illustrating the calculation of the next row in a Fibonacci sequence. The top row contains the numbers 0, 0, 1, 2, 3, 4, 5, 0, 0. The bottom row contains the numbers 0, 4, 10, 16, 22, 22. Arrows and circles indicate the addition of adjacent numbers from the top row to produce the bottom row.

4 5 1 2 3 4 5 1 2

3 2 1 3 2 1

19 10 16 22 23

3 2 **1** 2 3 4 5 4 3
 3 2 1
10 10 16 22 26

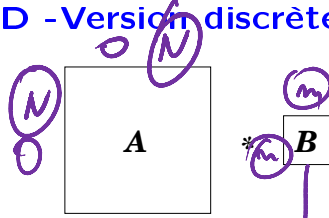
1	2	3	4	5
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	10	16	22	
--	-----------	-----------	-----------	--

TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (3)

Convolution 2D -Version discrète-



$$(A * B)(x, y) = \sum_i \sum_j A(i, j) B(x - i, y - j)$$

-Exemple-

10	5	20	20	20
10	5	20	20	20
10	5	20	20	20
10	5	20	20	20
10	5	20	20	20

*

-1	1
0	1

-10	5	-15	0	0
-10	15	-10	20	20
-10	15	-10	20	20
-10	15	-10	20	20
-10	15	-10	20	20

(Ajout de zéro)

-1	1
0	1

1	-1
1	0

1	0
1	-1

Original

x reflexion

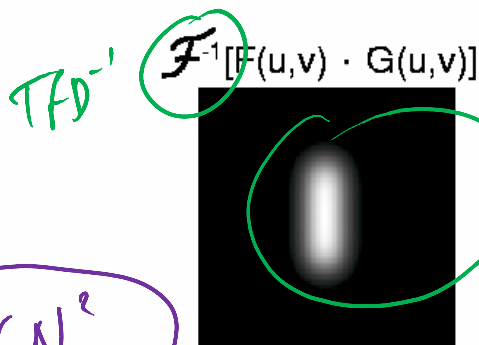
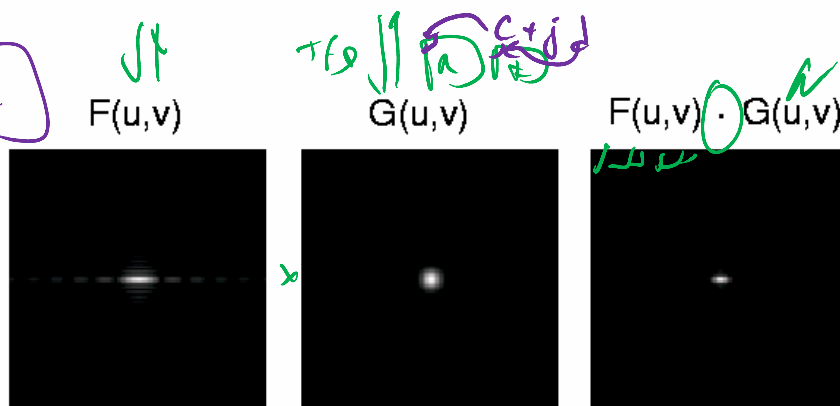
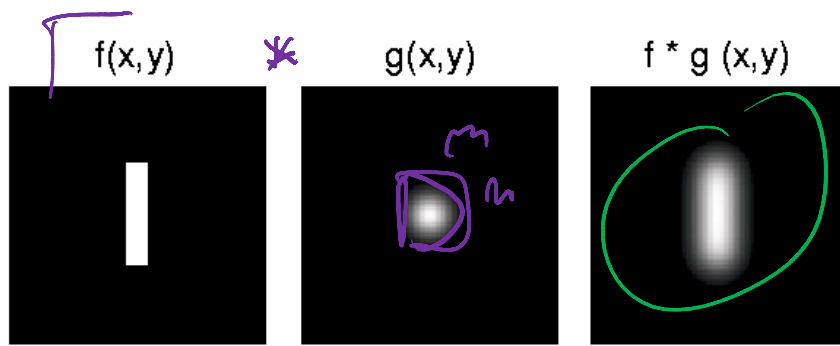
y reflexion

PROPRIÉTÉS

- Commutatif ► $f_1 * f_2 * f = f_2 * f_1 * f$
- Associatif ► $(f_1 * f_2) * f = f_1 * (f_2 * f)$
- Distributif ► $(f_1 + f_2) * f = f_1 * f + f_2 * f$
 ► $f * (f_1 + f_2) = f * f_1 + f * f_2$

TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (4)



$$(a+jb)(c+jd) = ac + jda + jbc - bd$$

product

$6N^2$ pixels

3FFT

$$3N^2 \log_2 N + 6N^2$$

17

Spatial

$m^2 N^2$

Frequency

$$3N^2 \log_2 N + 6N^2$$

$$3 \log_2 N + 6$$

$$\frac{1}{N^2} \rightarrow m^2$$

5x5

m x m

7x7 = 49

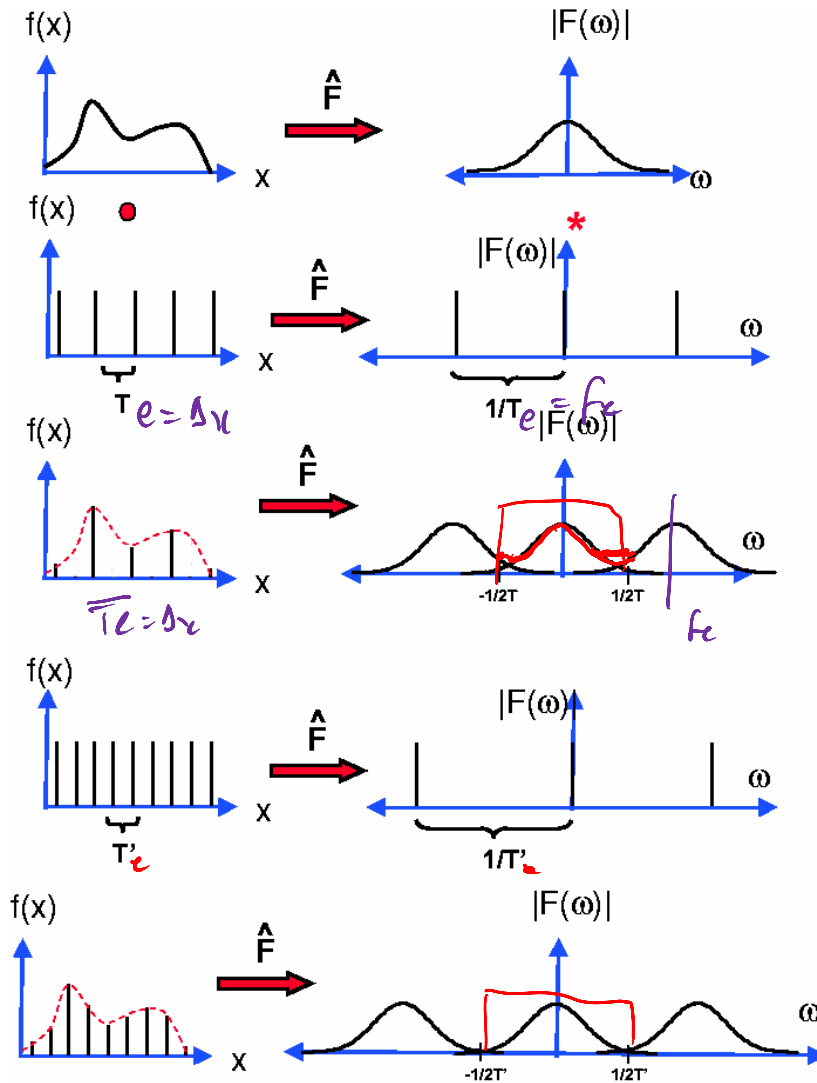
N=512

9

(33)

TF DISCRÈTE

THÉORÈME DE SCHANNON

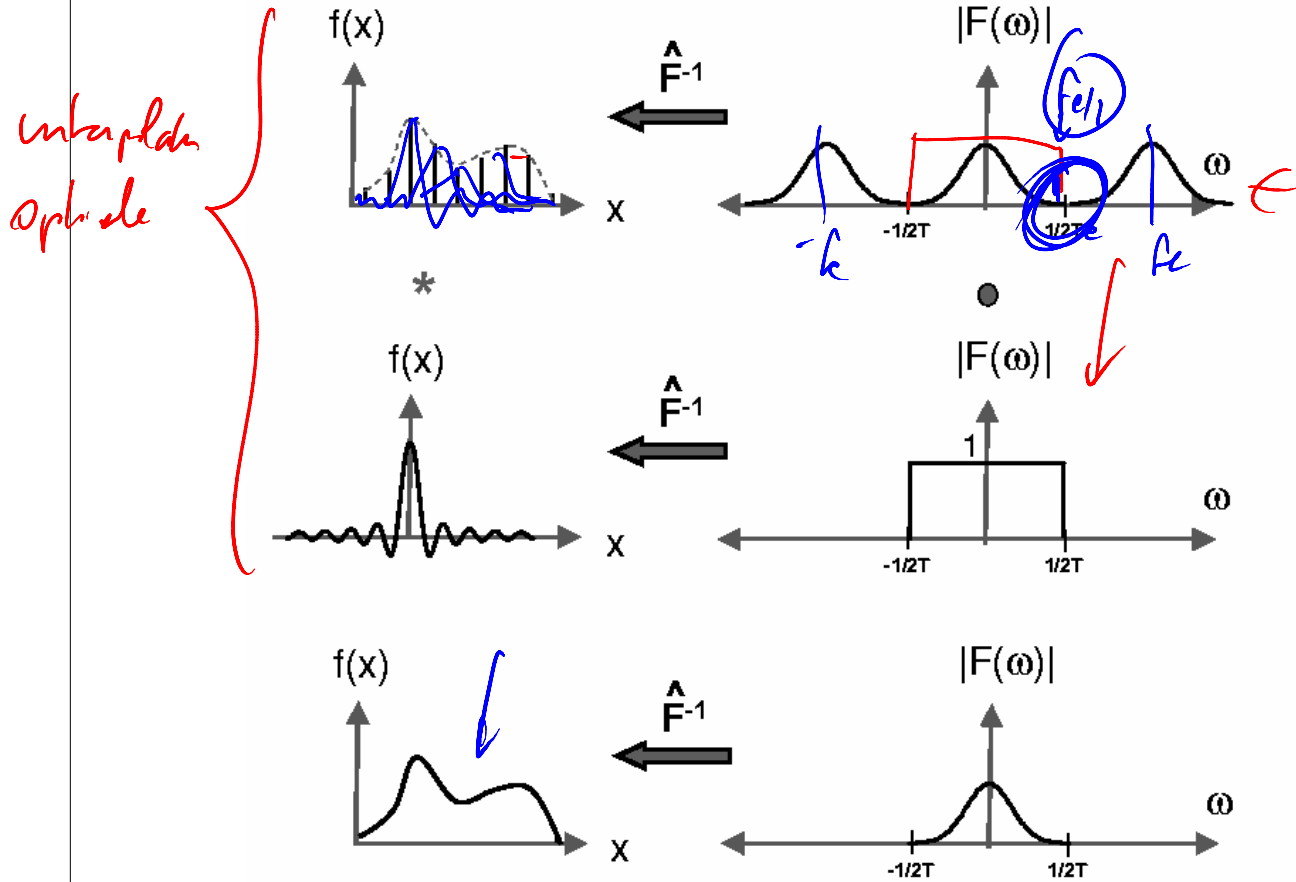


$$T_e \leq \frac{1}{2\nu_0}$$

► sinon phénomène d'aliasing

TF DISCRÈTE

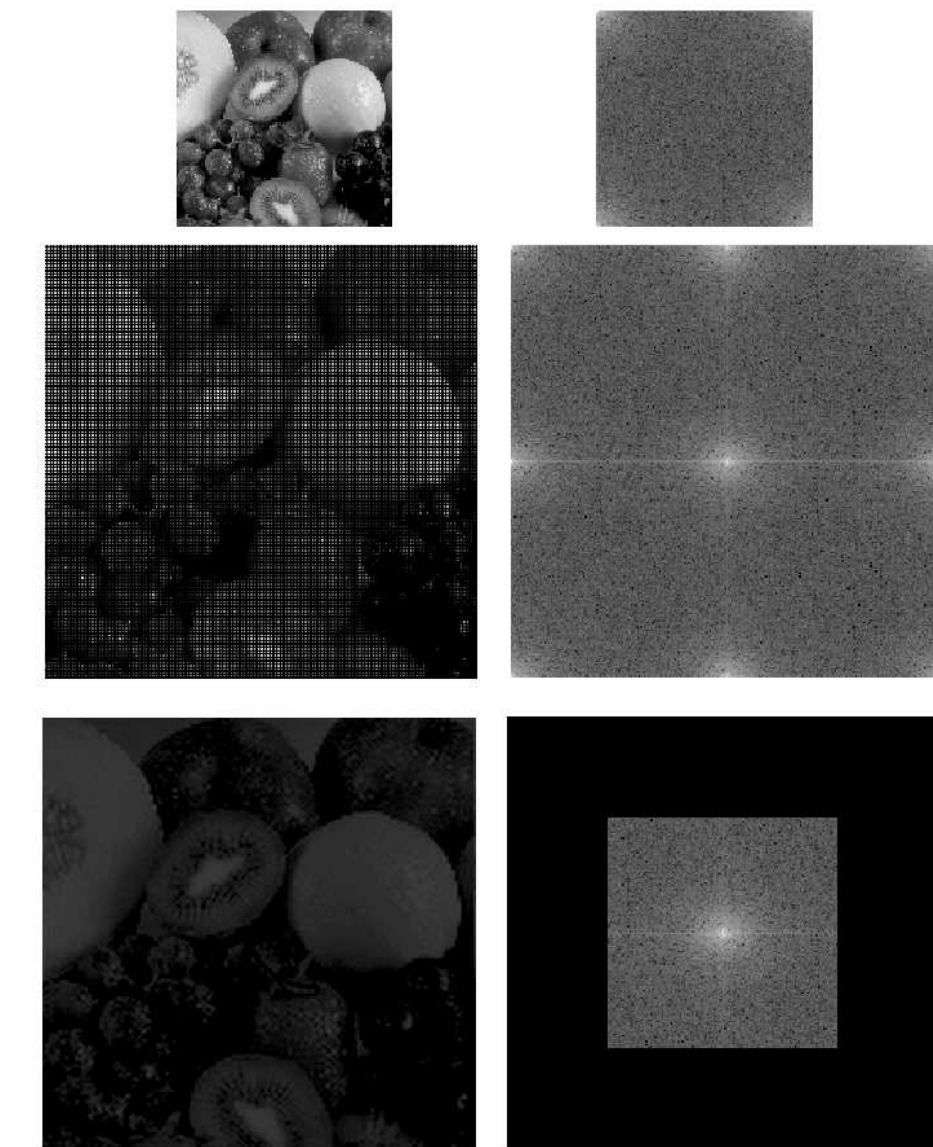
INTERPOLATION OPTIMALE (1)



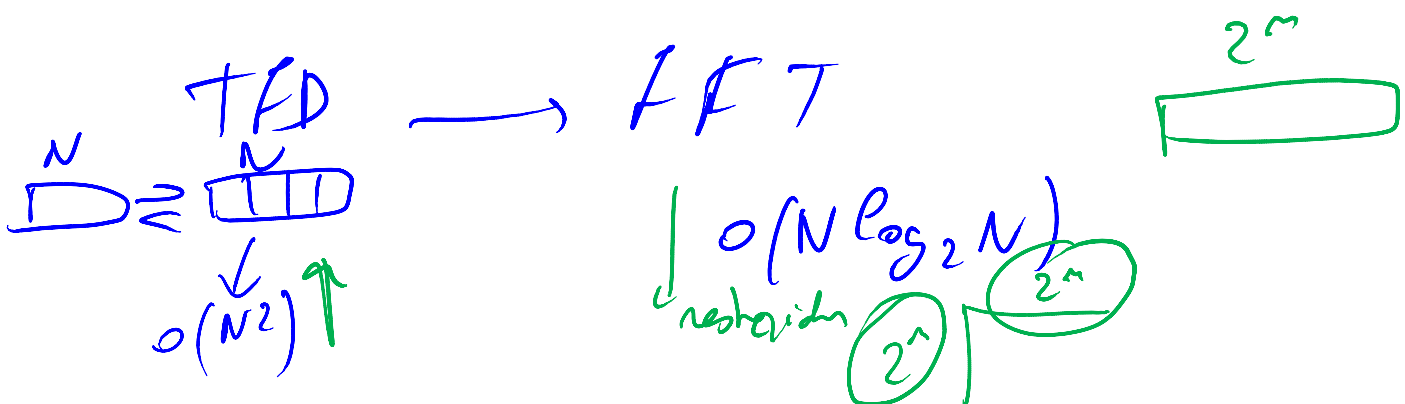
► Possibilité de reconstruire totalement le signal 2D à partir de ses échantillons

TF DISCRÈTE

INTERPOLATION OPTIMALE (2)



20



Cooley-Tukey 1968

$F(0) \quad F(1) \quad F(2) \quad F(3)$ $N=4$

TF DISCRÈTE

FFT (1)

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi j u x}{N}\right) \quad u = 0, 1, 2, \dots, N-1$$

$O(N^2)$ opérations

ICBIN

$$F(u) = \underbrace{\frac{1}{N} \sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi j u 2x}{N}\right)}_{x \text{ pair}} + \underbrace{\frac{1}{N} \sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi j u (2x+1)}{N}\right)}_{x \text{ impair}}$$

$\exp(a+b) = \exp(a) \exp(b)$

$$= \frac{1}{2} \left[\underbrace{\frac{1}{N/2} \sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi j u x}{N/2}\right)}_{\text{TF de la partie paire}} \underbrace{\exp\left(\frac{-2\pi j u}{N}\right)}_{\text{facteur}} + \underbrace{\frac{1}{N/2} \sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi j u x}{N/2}\right)}_{\text{TF de la partie impaire}} \right]$$

- TF [N éléments] = 2 TF [$N/2$ éléments] + $N/2$ multiplications et $N/2$ additions complexes (en fait $N/4$ pour les multiplications)

- $Op(N) = 2Op(N/2) + N$

$$F_N(u) = \frac{1}{2} \left[F_{N/2}^{\text{paire}}(u) + \exp\left(-\frac{2\pi j u}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

Pour $u' = u + \frac{N}{2}$

$$\exp\left(-\frac{2\pi j u'}{N}\right) = \exp\left(-\frac{2\pi j (u + \frac{N}{2})}{N}\right) = \exp\left(-\frac{2\pi j u}{N}\right) \exp(-\pi j) = -\exp\left(-\frac{2\pi j u}{N}\right)$$

TF DISCRÈTE FFT (2)

Ainsi

$$F_N(u) = \frac{1}{2} \left[F_{N/2}^{\text{paire}}(u) + \exp\left(-\frac{2\pi ju}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

$$F_N(u + \frac{N}{2}) = \frac{1}{2} \left[F_{N/2}^{\text{paire}}(u) - \exp\left(-\frac{2\pi ju}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

Pour $u = 0, 1, 2, \dots, N/2 - 1$

1 multiplication est nécessaire pour deux termes

$f(0) f(1) f(2) f(3) f(4) f(5) f(6) f(7)$

$f(0) f(2) f(4) f(6) \quad f(1) f(3) f(5) f(7)$

$f(0) f(4) \quad f(2) f(6) \quad f(1) f(5) \quad f(3) f(7)$
 $F(0) F(4) \quad F(2) F(6) \quad F(1) F(5) \quad F(3) F(7)$

FFT

FFT 1D ►

FFT 2D ►

$O(N \log N)$

$O(N^2 \log N)$

$$\text{FFT} \left(\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 4 & 0 & 0 & 0 \end{bmatrix}$$

TF DISCRÈTE

FFT (3)

Vérifions que $O(N \log N)$

(Par induction, sachant que $Op(N) = 2Op(N/2) + N$)

- Vrai pour $N = 1$? $\triangleright Op(1) = 1 \log 1 = 0 \triangleright$ oui
- Vrai pour $N = 2$? $\triangleright Op(2) = 2 \log 2 = 2 \triangleright$ oui
- Vrai pour $N \log N$?

- $Op(N) = N \log N$?
- $Op(N) = 2N/2 \log(2N/2)$
- $Op(N) = 2N/2(\log(N/2) + \log 2)$
- $Op(N) = 2N/2 \log N/2 + 2N/2 \log 2$
- $Op(N) = 2Op(N/2) + N$

C.Q.F.D.

$$FFT(1a) = 1a$$

$$FFT(1a \ 1b) = \begin{bmatrix} \frac{a+b}{2} & \frac{a-b}{2} \end{bmatrix}$$

$$2 \cdot \frac{N \log N}{2} = N \log N$$

Rehaussement d'image

