

Simplex algorithm
for
problems with bounded variables

Simplex method for problems with bounded variables

- Consider the linear programming problem with bounded variables

$$\begin{aligned} \min \quad & c^T g \\ \text{s.t.} \quad & Ag = h \\ & l_j \leq g_j \leq q_j \quad j = 1, 2, \dots, n \end{aligned}$$

where $g, c, l, q \in R^n$, $h \in R^m$, and A is a $m \times n$ matrix

- Complete the following change of variables to reduce the lower bound to 0

$$x_j = g_j - l_j \quad (\text{i.e., } g_j = x_j + l_j)$$

Simplex method for problems with bounded variables

the problem becomes

$$\begin{aligned} \min \quad & c^T(x+l) \\ \text{s.t.} \quad & A(x+l) = h \\ & l_j \leq x_j + l_j \leq q_j \quad j = 1, 2, \dots, n \end{aligned}$$

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where $c, x, l, q \in R^n$, $h \in R^m$, and A is a $m \times n$ matrix

$$\begin{aligned} \min \quad & c^T x + c^T l \\ \text{s.t.} \quad & Ax = h - Al \\ & l_j - l_j \leq x_j + l_j - l_j \leq q_j - l_j \quad j = 1, 2, \dots, n \end{aligned}$$

replacing : $u_j = q_j - l_j$ and $b = h - Al$

$$\begin{aligned} \min \quad & c^T x + c^T l \\ \text{s.t.} \quad & Ax = b \\ & 0 \leq x_j \leq u_j \quad j = 1, 2, \dots, n \end{aligned}$$

Simplex method for problems with bounded variables

- In this problem

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & 0 \leq x_j \leq u_j \quad j = 1, 2, \dots, n \end{aligned}$$

since $c^T l$ is a constant, we can eliminate it from the minimisation without modifying the optimal solution.

Then in the rest of the presentation we consider the problem without this constant.

$$\begin{array}{ll}
\min & c^T x \\
\text{s.t.} & Ax = b \\
& 0 \leq x_j \leq u_j \quad j = 1, 2, \dots, n
\end{array}$$

- Consider the explicit formulation of the problem

$$\min z = \sum_{j=1}^n c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^n a_{ij} x_j = b_i \quad i = 1, 2, \dots, m$$

$$0 \leq x_j \leq u_j \quad j = 1, 2, \dots, n$$

$$x_j, y_j \geq 0 \quad j = 1, 2, \dots, n$$

- One way of solving the problem is to introduce slack variables y_j , and then use the simplex algorithm.

$$\min z = \sum_{j=1}^n c_j x_j$$

Tableau with $m + n$ rows

$$\text{s.t.} \quad \sum_{j=1}^n a_{ij} x_j = b_i \quad i = 1, 2, \dots, m$$

$$x_j + y_j = u_j \quad j = 1, 2, \dots, n$$

$$x_j, y_j \geq 0 \quad j = 1, 2, \dots, n$$

$$\min z = \sum_{j=1}^n c_j x_j$$

Tableau with m rows

$$\text{s.t.} \quad \sum_{j=1}^n a_{ij} x_j = b_i \quad i = 1, 2, \dots, m$$

$$0 \leq x_j \leq u_j \quad j = 1, 2, \dots, n$$

account implicitly

$$\min \quad z = \sum_{j=1}^n c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^n a_{ij} x_j = b_i \quad i = 1, 2, \dots, m$$

$$x_j + y_j = u_j \quad j = 1, 2, \dots, n$$

$$x_j, y_j \geq 0 \quad j = 1, 2, \dots, n$$

Non degeneracy:
all the basic variables
are positive at
each iteration

- Consider a basic feasible solution of this problem
- Because of the constraints $x_j + y_j = u_j$, at least one of the variables x_j or y_j is basic, $j = 1, 2, \dots, n$.
- Then for all $j = 1, 2, \dots, n$, one of the three situations holds:
 - a) $x_j = u_j$ is basic and $y_j = 0$ is non basic
 - b) $x_j = 0$ is non basic and $y_j = u_j$ is basic
 - c) $0 < x_j < u_j$ is basic and $0 < y_j < u_j$ is basic

$$\min \quad z = \sum_{j=1}^n c_j x_j$$

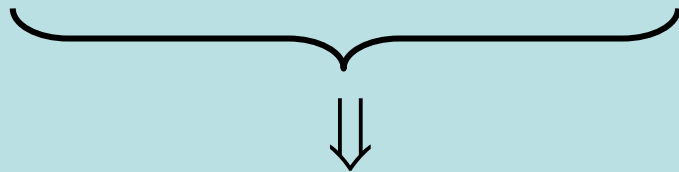
$$\text{Sujet à} \quad \sum_{j=1}^n a_{ij} x_j = b_i \quad i = 1, 2, \dots, m$$

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- a) x_j basic; y_j non basic
- b) x_j non basic; y_j basic
- c) x_j basic; y_j basic

$m + n$ basic variables required
There are n variables y_j



There are at least m variables x_j
that are basic

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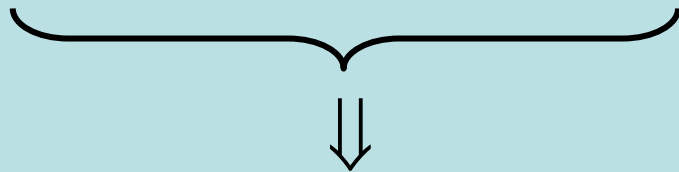
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Exactly m variables x_j satisfying
 $0 < x_j < u_j$.

For contradiction, if $m^0 \neq m$ variables x_j satisfy the relation, then the m^0 corresponding variables y_j would be basic.

Furthermore, for the $n - m^0$ other indices j , either $x_j = u_j$ (case a) or $y_j = u_j$ (case b) would be verified.

Then the number of basic variables would be equal to

$$2m^0 + (n - m^0) = m^0 + n \neq m + n$$

$$\begin{aligned}
\min \quad & z = \sum_{j=1}^n c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j = b_i \quad i = 1, 2, \dots, m \\
& x_j + y_j = u_j \quad j = 1, 2, \dots, n \\
& x_j, y_j \geq 0 \quad j = 1, 2, \dots, n
\end{aligned}$$

$$\begin{aligned}
\min \quad & z = c^T x \\
\text{s.t.} \quad & Ax + 0y = b \\
& Ix + Iy = u \\
& x, y \geq 0
\end{aligned}$$

$$\begin{bmatrix} A & 0 \\ I & I \end{bmatrix}$$

To simplify notation, assume the following basic variables:

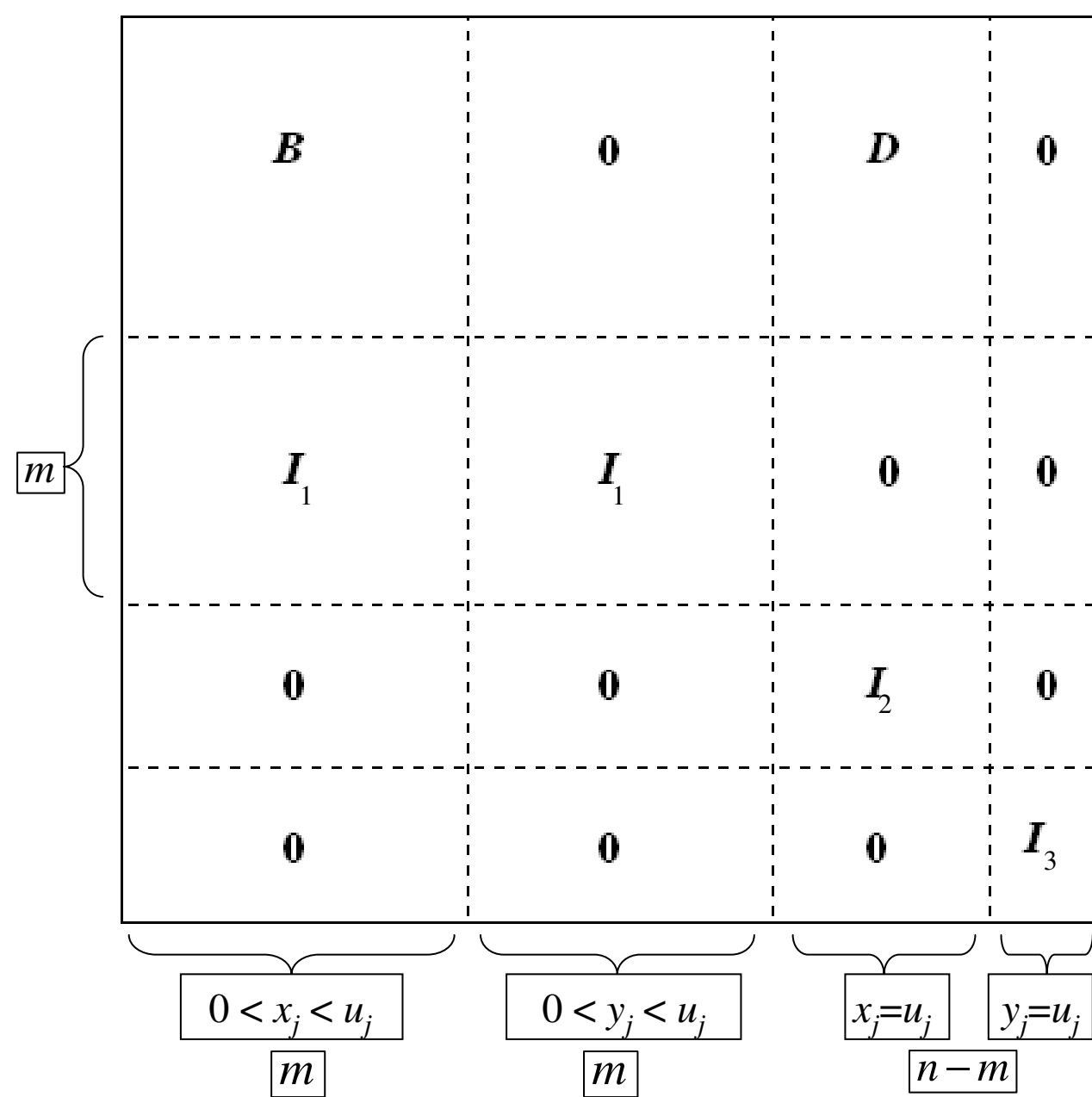
$$0 < x_i < u_i \quad i = 1, \dots, m$$

$$0 < y_i < u_i \quad i = 1, \dots, m$$

$$x_i = u_i \quad i = m + 1, \dots, m + l$$

$$y_i = u_i \quad i = m + l + 1, \dots, m + n$$

The basis has the following form



$$\min z = \sum_{j=1}^n c_j x_j$$

$$\text{s.t. } \sum_{j=1}^n a_{ij} x_j = b_i \quad i=1,2,\dots,m$$

$$x_j + y_j = u_j \quad j=1,2,\dots,n$$

$$x_j, y_j \geq 0 \quad j=1,2,\dots,n$$

m

$$\min z = c^T x$$

$$\text{s.t. } Ax + 0y = b$$

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$$x, y \geq 0$$

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n

To simplify notation, assume the following basic variables:

$$0 < x_i < u_i \quad i = 1, \dots, m$$

$$0 < y_i < u_i \quad i = 1, \dots, m$$

$$x_i = u_i \quad i = m + 1, \dots, m + l$$

$$y_i = u_i \quad i = m + l + 1, \dots, m + n$$

$$[x] = \begin{bmatrix} B & 0 & D & 0 \\ I_1 & I_1 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix}$$

$$\begin{aligned}
 [x] &= \begin{bmatrix} B & 0 & D & 0 \\ I_1 & I_1 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix} \\
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 \end{aligned}$$

where the matrix I is $n \times n$

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 \end{aligned}$$

$$\begin{aligned}
 \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= ad - bc \\
 &= d(a - bd^{-1}c)
 \end{aligned}$$

where the matrix I is $n \times n$

$$\begin{aligned} \begin{bmatrix} \mathbf{x} \end{bmatrix} &= \begin{bmatrix} B & 0 & D & 0 \\ I_1 & I_1 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix} \\ &= \begin{bmatrix} B & 0 & D & 0 \\ I_1 & & & \\ 0 & I & & \\ 0 & & & \end{bmatrix} \end{aligned}$$

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where the matrix I is $n \times n$

$$\begin{aligned} \det(\Xi) &= \det(I) \left\{ \det(B) - \det \left([0 \ D \ 0] I^{-1} \begin{bmatrix} I_1 \\ 0 \\ 0 \end{bmatrix} \right) \right\} \\ &= \det(I) \left\{ \det(B) - \det(\tilde{0}) \right\} \end{aligned}$$

where the matrix $\tilde{0}$ is $m \times m$ since
 $[0 \ D \ 0]$ is a $m \times n$ matrix and

$$\begin{bmatrix} I_1 \\ 0 \\ 0 \end{bmatrix} \text{ is a } n \times m \text{ matrix}$$

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Then $\det(\Xi) = \det(I) \det(B)$.

$$\begin{aligned} [\mathfrak{B}] &= \begin{bmatrix} B & 0 & D & 0 \\ I_1 & I_1 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix} \\ &= \begin{bmatrix} B & 0 & D & 0 \\ I_1 & & & \\ 0 & I & & \\ 0 & & & \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= ad - bc \\ &= d(a - bd^{-1}c) \end{aligned}$$

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Then $\det(\mathfrak{B}) = \det(I) \det(B)$.

Since \mathfrak{B} is a basis, then $\det(\mathfrak{B}) \neq 0$.

Consequently $\det(B) = \det(\mathfrak{B}) \neq 0$ and then B is non singular.

$$\begin{aligned} [\mathfrak{B}] &= \begin{bmatrix} B & 0 & D & 0 \\ I_1 & I_1 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix} \\ &= \begin{bmatrix} B & 0 & D & 0 \\ I_1 & & & \\ 0 & I & & \\ 0 & & & \end{bmatrix} \end{aligned}$$

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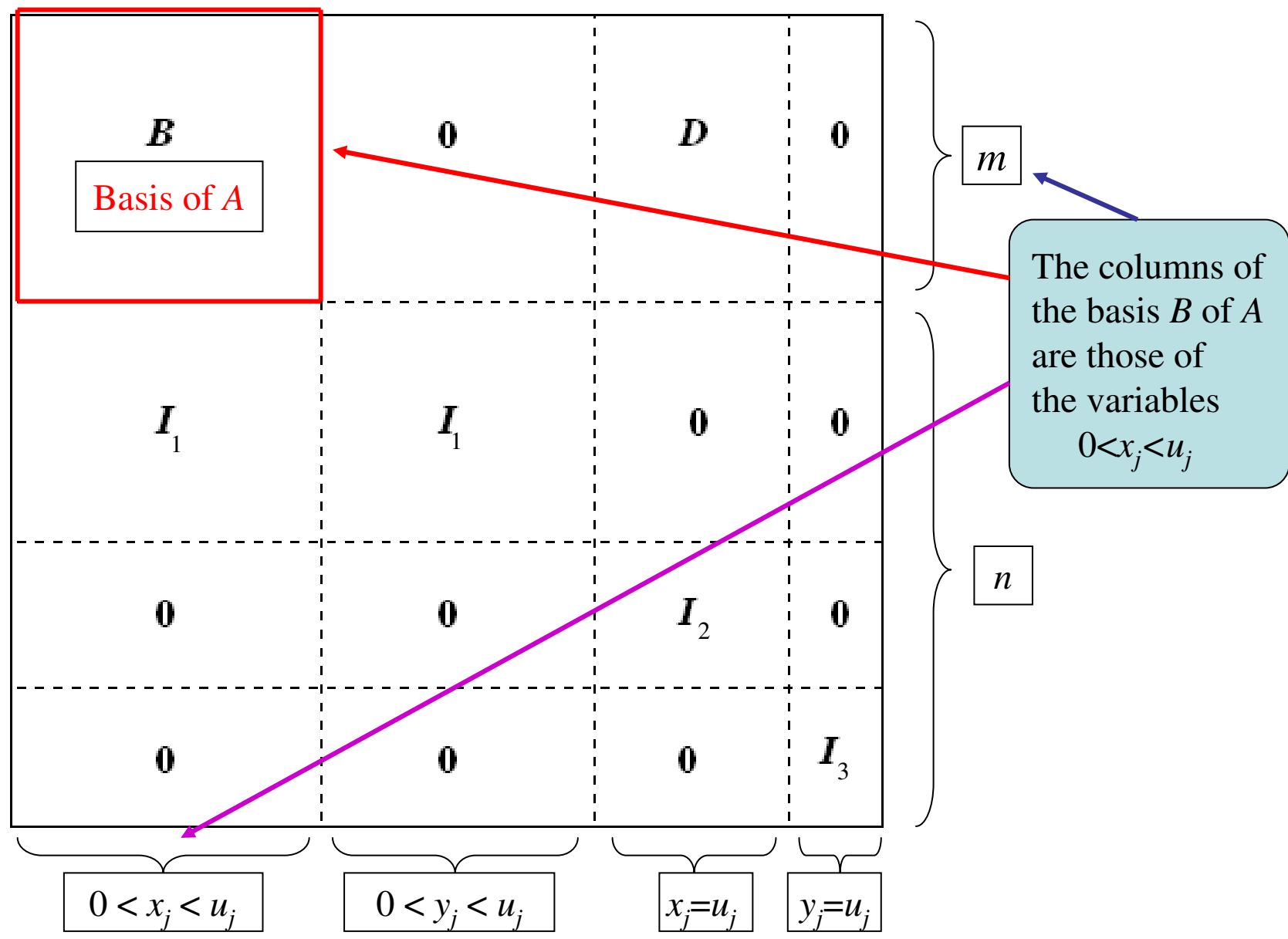
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Consequently $\det(B) = \det(\mathfrak{B}) \neq 0$ and then B is non singular.

Then B is a basis of A .

The basis has the following form



- Then, we can specify a variant of the simplex method to solve this problem specifically:

$$\begin{aligned} \min \quad & z = \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j = b_i \quad i = 1, 2, \dots, m \\ & 0 \leq x_j \leq u_j \quad j = 1, 2, \dots, n \end{aligned}$$

by dealing implicitly with the upper bound u_j . At each iteration, we consider a solution (basic) associated with a basis B de A having

$$\begin{array}{lll} m \text{ basic variables} & 0 < x_j < u_j & j \in IB \\ n - m \text{ non basic variables} & x_j = 0 \text{ ou } u_j & j \in JB \end{array}$$

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\min \quad & z = \sum_{j=1}^n c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j = b_i \quad i = 1, 2, \dots, m \\
& x_j + y_j = u_j \quad j = 1, 2, \dots, n \\
& x_j, y_j \geq 0 \quad j = 1, 2, \dots, n
\end{aligned}$$

- At each iteration, we consider a solution (basic) associated with a basis B de A having

$$m \text{ basic variables} \quad 0 < x_j < u_j \quad j \in IB$$

$$n - m \text{ non basic variables} \quad x_j = 0 \text{ or } u_j \quad j \in JB$$

- Denote the indices of the basic variables $IB = \{j_1, j_2, \dots, j_m\}$ where j_i is the index of the basic variable in the i^{th} row, then

$$x_j = 0 \text{ ou } u_j \quad j \in JB$$

$$x_{j_i} = \bar{b}_i - \sum_{j \in JB} \bar{a}_{ij} x_j \quad i = 1, 2, \dots, m$$

Dependent variables	x_1 x_2 \dots x_r \dots x_m x_{m+1} \dots x_s \dots x_n $-z$	r.h.s.
x_1	1	\bar{b}_1
x_2		\bar{b}_2
\vdots		\vdots
x_r		\bar{b}_r
\vdots		\vdots
x_m		\bar{b}_m
$-z$		\bar{z}

We find similar values as in problems where there are no upper bounds, except for non basic variables

$$x_j = 0 \text{ or } u_j \quad j \in JB$$

We find similar values as in problems where there are no upper bounds, except for non basic variables

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We have to modify the entering criterion and the leaving criterion accordingly to generate a variant of the simplex algorithm for this problem

Step 1: Selecting the entering variable

The criterion to select the entering variable must be modified to account for the non basic variables x_j being equal to their upper bounds u_j since these variables can be reduced.

Hence, for an index $j \in JB$

if $x_j = 0$ and $\bar{c}_j < 0$, it is interesting to increase x_j
if $x_j = u_j$ and $\bar{c}_j > 0$, it is interesting to decrease x_j

Determine $\bar{c}_{s_1} = \min_{j \in JB} \{ \bar{c}_j : x_j = 0 \}$ and $\bar{c}_{s_2} = \max_{j \in JB} \{ \bar{c}_j : x_j = u_j \}$

Let $\bar{c}_s = \min \{ \bar{c}_{s_1}, -\bar{c}_{s_2} \}$ $\left(\max \{ |\bar{c}_{s_1}|, |\bar{c}_{s_2}| \} \right)$

If $\bar{c}_s \geq 0$, **then the solution is optimal**, and the algorithm stops.

If $\bar{c}_s < 0$ and $\bar{c}_s = \bar{c}_{s_1}$, **then the non basic variable x_{s_1} increases**, and go to Step 2.1.

If $\bar{c}_s < 0$ et $\bar{c}_s = -\bar{c}_{s_2}$, **then the non basic variable x_{s_2} decreases**, and go to Step 2.2.

Step 2.1: Selecting the leaving variable

- The increase θ of the entering variable x_s is stop by the first of the following three situations happening:

i) x_s reach its upper bound u_s
 ii) a basic variable x_{j_r} decreases to 0 (in this case $\bar{a}_{rs} > 0$)

iii) a basic variable x_{j_r} increases to reach its upper bound u_{j_r} (in this case $\bar{a}_{rs} < 0$)

$$\begin{array}{l} 0 \leq x_{j_1} = g_1 - \bar{a}_{1s}(0+\theta) \leq u_{j_1} \\ \vdots \\ 0 \leq x_{j_r} = g_r - \bar{a}_{rs}(0+\theta) \leq u_{j_r} \\ \vdots \\ 0 \leq x_{j_m} = g_m - \bar{a}_{ms}(0+\theta) \leq u_{j_m} \\ x_s = 0 + \theta \leq u_s \end{array}$$

Let
$$g_i = \bar{b}_i - \sum_{j \in JB} \bar{a}_{ij} x_j$$

Value of the basic variables x_{j_i}

$$\theta = \min \left\{ u_s, \min_{1 \leq i \leq m} \left\{ \frac{g_i}{\bar{a}_{is}} : \bar{a}_{is} > 0 \right\}, \min_{1 \leq i \leq m} \left\{ \frac{u_{j_i} - g_i}{-\bar{a}_{is}} : \bar{a}_{is} < 0 \right\} \right\}$$

For all i such that $\bar{a}_{is} > 0$, then x_{j_i} decreases when x_s increases of the value θ . It follows $x_{j_i} = g_i - \bar{a}_{is}\theta \geq 0 \Leftrightarrow \bar{a}_{is}\theta \leq g_i$

$$\Leftrightarrow \theta \leq \frac{g_i}{\bar{a}_{is}}$$

Then
$$\theta \leq \min_{1 \leq i \leq m} \left\{ \frac{g_i}{\bar{a}_{is}} : \bar{a}_{is} > 0 \right\}$$

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For all i such that $\bar{a}_{is} < 0$, then x_{j_i} increases when x_s increases of the value θ . It follows $x_{j_i} = g_i - \bar{a}_{is}\theta \leq u_{j_i} \Leftrightarrow -\bar{a}_{is}\theta \leq u_{j_i} - g_i$

$$\Leftrightarrow \theta \leq \frac{u_{j_i} - g_i}{-\bar{a}_{is}}$$

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If $\theta = \infty$, then the problem is not bounded from below, and the algorithm stops.

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Value of the basic variables x_{j_i}

$$\theta = \min \left\{ u_s, \min_{1 \leq i \leq m} \left\{ \frac{g_i}{\bar{a}_{is}} : \bar{a}_{is} > 0 \right\}, \min_{1 \leq i \leq m} \left\{ \frac{u_{j_i} - g_i}{-\bar{a}_{is}} : \bar{a}_{is} < 0 \right\} \right\}$$

If $x_s = u_s$, then the set of basic variables is not modified, and the same basis is used at the next iteration.

the variable x_s remains non basic but its value is modified from 0 to u_s .

Go to step 1.

Step 2.1: Selecting the leaving variable

- The increase θ of the entering variable x_s is stop by the first of the following three situations happening:

i) x_s reach its upper bound u_s

ii) a basic variable x_{j_r} decreases to 0 (in this case $\bar{a}_{rs} > 0$)

iii) a basic variable x_{j_r} increases to reach its upper bound u_{j_r} (in this case

$\bar{a}_{rs} < 0$)

$$\begin{aligned} 0 \leq x_{j_1} &= g_1 - \bar{a}_{1s}(0 + \theta) \leq u_{j_1} \\ \vdots \\ 0 \leq x_{j_r} &= g_r - \bar{a}_{rs}(0 + \theta) \leq u_{j_r} \\ \vdots \\ 0 \leq x_{j_m} &= g_m - \bar{a}_{ms}(0 + \theta) \leq u_{j_m} \\ x_s &= 0 + \theta \leq u_s \end{aligned}$$

$$\text{Let } g_i = \bar{b}_i - \sum_{j \in JB} \bar{a}_{ij} x_j$$

Value of the basic variables x_{j_i}

$$\theta = \min \left\{ u_s, \min_{1 \leq i \leq m} \left\{ \frac{g_i}{\bar{a}_{is}} : \bar{a}_{is} > 0 \right\}, \min_{1 \leq i \leq m} \left\{ \frac{u_{j_i} - g_i}{-\bar{a}_{is}} : \bar{a}_{is} < 0 \right\} \right\}$$

$$\text{If } \theta = \frac{g_r}{\bar{a}_{rs}} = \min_{1 \leq i \leq m} \left\{ \frac{g_i}{\bar{a}_{is}} : \bar{a}_{is} > 0 \right\},$$

then the value on the entretring variable x_s increases to θ .

The entering variable x_s becomes a basic variable replacing the leaving variable x_{j_r} becoming a non basic variable equal to 0
Pivot on $a_{j_r,s}$, and go to step 1.

Step 2.1: Selecting the leaving variable

- The increase θ of the entering variable x_s is stop by the first of the following three situations happening:

i) x_s reach its upper bound u_s

ii) a basic variable x_{j_r} decreases to 0 (in this case $\bar{a}_{rs} > 0$)

iii) a basic variable x_{j_r} increases to reach its upper bound u_{j_r} (in this case $\bar{a}_{rs} < 0$)

$$\begin{aligned} 0 &\leq x_{j_1} = g_1 - \bar{a}_{1s}(0 + \theta) \leq u_{j_1} \\ &\vdots \\ 0 &\leq x_{j_r} = g_r - \bar{a}_{rs}(0 + \theta) \leq u_{j_r} \\ &\vdots \\ 0 &\leq x_{j_m} = g_m - \bar{a}_{ms}(0 + \theta) \leq u_{j_m} \\ x_s &= 0 + \theta \leq u_s \end{aligned}$$

$$\text{Let } g_i = \bar{b}_i - \sum_{j \in JB} \bar{a}_{ij} x_j$$

Value of the basic variables x_{j_i}

$$\theta = \min \left\{ u_s, \min_{1 \leq i \leq m} \left\{ \frac{g_i}{\bar{a}_{is}} : \bar{a}_{is} > 0 \right\}, \min_{1 \leq i \leq m} \left\{ \frac{u_{j_i} - g_i}{-\bar{a}_{is}} : \bar{a}_{is} < 0 \right\} \right\}$$

$$\text{If } \theta = \frac{u_{j_r} - g_r}{-\bar{a}_{rs}} = \min_{1 \leq i \leq m} \left\{ \frac{u_{j_i} - g_i}{-\bar{a}_{is}} : \bar{a}_{is} < 0 \right\},$$

then the value on the entretring variable x_s increases to θ .

The entering variable x_s becomes a basic variable replacing the leaving variable x_{j_r} becoming a non basic variable equal to u_{j_r}

Pivot on $a_{j_r,s}$, and go to step 1.

Step 1: Selecting the entering variable

The criterion to select the entering variable must be modified to account for the non basic variables x_j being equal to their upper bounds u_j since these variables can be reduced.

Hence, for an index $j \in JB$

if $x_j = 0$ and $\bar{c}_j < 0$, it is interesting to increase x_j
if $x_j = u_j$ and $\bar{c}_j > 0$, it is interesting to decrease x_j

Determine $\bar{c}_{s_1} = \min_{j \in JB} \{ \bar{c}_j : x_j = 0 \}$ and $\bar{c}_{s_2} = \max_{j \in JB} \{ \bar{c}_j : x_j = u_j \}$

Let $\bar{c}_s = \min \{ \bar{c}_{s_1}, -\bar{c}_{s_2} \}$ $\left(\max \{ |\bar{c}_{s_1}|, |\bar{c}_{s_2}| \} \right)$

If $\bar{c}_s \geq 0$, **then the solution is optimal**, and the algorithm stops.

If $\bar{c}_s < 0$ and $\bar{c}_s = \bar{c}_{s_1}$, **then the non basic variable x_{s_1} increases**, and go to Step 2.1.

If $\bar{c}_s < 0$ et $\bar{c}_s < \bar{c}_{s_1}$, **then the non basic variable x_{s_2} decreases**, and go to Step 2.2.

Step 2.2: Selecting the leaving variable

- The decrease θ of the entering variable x_s is stop by the first of the following three situations happening:

i) x_s reduces to 0

ii) a basic variable x_{j_r} decreases to 0 (in this case $\bar{a}_{rs} < 0$)

iii) a basic variable x_{j_r} increases to reach its upper bound u_{j_r} (in this case $\bar{a}_{rs} > 0$)

$$\begin{array}{l} 0 \leq x_{j_1} = g_1 - \bar{a}_{1s}(-\theta) \leq u_{j_1} \\ \vdots \\ 0 \leq x_{j_r} = g_r - \bar{a}_{rs}(-\theta) \leq u_{j_r} \\ \vdots \\ 0 \leq x_{j_m} = g_m - \bar{a}_{ms}(-\theta) \leq u_{j_m} \\ x_s = u_s - \theta \geq 0 \end{array}$$

$$\text{Let } g_i = \bar{b}_i - \sum_{j \in JB} \bar{a}_{ij} x_j$$

Value of the basic variables x_{j_i}

$$\theta = \min \left\{ u_s, \min_{1 \leq i \leq m} \left\{ \frac{g_i}{-a_{is}} : \bar{a}_{is} < 0 \right\}, \min_{1 \leq i \leq m} \left\{ \frac{u_{j_i} - g_i}{a_{is}} : \bar{a}_{is} > 0 \right\} \right\}$$

For all i such that $\bar{a}_{is} < 0$, then x_{j_i} decreases when x_s decreases of the value θ . It follows $x_{j_i} = g_i - \bar{a}_{is}(-\theta) \geq 0 \Leftrightarrow -\bar{a}_{is}\theta \leq g_i$

$$\Leftrightarrow \theta \leq \frac{g_i}{-\bar{a}_{is}}$$

$$\text{Then } \theta \leq \min_{1 \leq i \leq m} \left\{ \frac{g_i}{-\bar{a}_{is}} : \bar{a}_{is} < 0 \right\}$$

Step 2.2: Selecting the leaving variable

- The decrease θ of the entering variable x_s is stop by the first of the following three situations happening:

i) x_s reduces to 0

ii) a basic variable x_{j_r} decreases to 0 (in this case $\bar{a}_{rs} < 0$)

iii) a basic variable x_{j_r} increases to reach its upper bound u_{j_r} (in this case $\bar{a}_{rs} > 0$)

$$\begin{array}{l} 0 \leq x_{j_1} = g_1 - \bar{a}_{1s}(-\theta) \leq u_{j_1} \\ \vdots \\ 0 \leq x_{j_r} = g_r - \bar{a}_{rs}(-\theta) \leq u_{j_r} \\ \vdots \\ 0 \leq x_{j_m} = g_m - \bar{a}_{ms}(-\theta) \leq u_{j_m} \\ x_s = u_s - \theta \geq 0 \end{array}$$

$$\text{Let } g_i = \bar{b}_i - \sum_{j \in JB} \bar{a}_{ij} x_j$$

Value of the basic variables x_{j_i}

$$\theta = \min \left\{ u_s, \min_{1 \leq i \leq m} \left\{ \frac{g_i}{-a_{is}} : a_{is} < 0 \right\}, \min_{1 \leq i \leq m} \left\{ \frac{u_{j_i} - g_i}{a_{is}} : a_{is} > 0 \right\} \right\}$$

For all i such that $\bar{a}_{is} > 0$, then x_{j_i} increases when x_s decreases of the value θ . It follows $x_{j_i} = g_i - \bar{a}_{is}(-\theta) \leq u_{j_i} \Leftrightarrow \bar{a}_{is} \theta \leq u_{j_i} - g_i$

$$\Leftrightarrow \theta \leq \frac{u_{j_i} - g_i}{\bar{a}_{is}}$$

$$\text{Then } \theta \leq \min_{1 \leq i \leq m} \left\{ \frac{u_{j_i} - g_i}{\bar{a}_{is}} : \bar{a}_{is} > 0 \right\}$$

Step 2.2: Selecting the leaving variable

- The decrease θ of the entering variable x_s is stop by the first of the following three situations happening:

i) x_s reduces to 0

ii) a basic variable x_{j_r} decreases to 0 (in this case $\bar{a}_{rs} < 0$)

iii) a basic variable x_{j_r} increases to reach its upper bound u_{j_r} (in this case $\bar{a}_{rs} > 0$)

$$\begin{aligned} 0 &\leq x_{j_1} = g_1 - \bar{a}_{1s}(-\theta) \leq u_{j_1} \\ &\vdots \\ 0 &\leq x_{j_r} = g_r - \bar{a}_{rs}(-\theta) \leq u_{j_r} \\ &\vdots \\ 0 &\leq x_{j_m} = g_m - \bar{a}_{ms}(-\theta) \leq u_{j_m} \\ x_s &= u_s - \theta \geq 0 \end{aligned}$$

$$\text{Let } g_i = \bar{b}_i - \sum_{j \in JB} \bar{a}_{ij} x_j$$

Value of the basic variables x_{j_i}

$$\theta = \min \left\{ u_s, \min_{1 \leq i \leq m} \left\{ \frac{g_i}{-a_{is}} : \bar{a}_{is} < 0 \right\}, \min_{1 \leq i \leq m} \left\{ \frac{u_{j_i} - g_i}{a_{is}} : \bar{a}_{is} > 0 \right\} \right\}$$

If $\theta = u_s$, then the set of basic variables is not modified, and the same basis is used at the next iteration.

the variable x_s remains non basic but its value is modified from u_s to 0.

Go to step 1.

Step 2.2: Selecting the leaving variable

- The decrease θ of the entering variable x_s is stop by the first of the following three situations happening:

i) x_s reduces to 0

ii) a basic variable x_{j_r} decreases to 0 (in this case $\bar{a}_{rs} < 0$)

iii) a basic variable x_{j_r} increases to reach its upper bound u_{j_r} (in this case $\bar{a}_{rs} > 0$)

$$\begin{array}{l} 0 \leq x_{j_1} = g_1 - \bar{a}_{1s}(-\theta) \leq u_{j_1} \\ \vdots \\ 0 \leq x_{j_r} = g_r - \bar{a}_{rs}(-\theta) \leq u_{j_r} \\ \vdots \\ 0 \leq x_{j_m} = g_m - \bar{a}_{ms}(-\theta) \leq u_{j_m} \\ x_s = u_s - \theta \geq 0 \end{array}$$

$$\text{Let } g_i = \bar{b}_i - \sum_{j \in JB} \bar{a}_{ij} x_j$$

Value of the basic variables x_{j_i}

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$$\text{If } \theta = \frac{g_r}{-a_{rs}} = \min_{1 \leq i \leq m} \left\{ \frac{g_i}{-a_{is}} : \bar{a}_{is} < 0 \right\},$$

then the value on the entering variable x_s is reduced by θ (i.e., $x_s \leftarrow u_s - \theta$).

The entering variable x_s becomes a basic variable replacing the leaving variable x_{j_r} becoming a non basic variable equal to 0
Pivot on $a_{j_r,s}$, and go to step 1.

Step 2.2: Selecting the leaving variable

- The decrease θ of the entering variable x_s is stop by the first of the following three situations happening:

i) x_s reduces to 0

ii) a basic variable x_{j_r} decreases to 0 (in this case $\bar{a}_{rs} < 0$)

iii) a basic variable x_{j_r} increases to reach its upper bound u_{j_r} (in this case $\bar{a}_{rs} > 0$)

$$\begin{array}{l} 0 \leq x_{j_1} = g_1 - \bar{a}_{1s}(-\theta) \leq u_{j_1} \\ \vdots \\ 0 \leq x_{j_r} = g_r - \bar{a}_{rs}(-\theta) \leq u_{j_r} \\ \vdots \\ 0 \leq x_{j_m} = g_m - \bar{a}_{ms}(-\theta) \leq u_{j_m} \\ x_s = u_s - \theta \geq 0 \end{array}$$

$$\text{Let } g_i = \bar{b}_i - \sum_{j \in JB} \bar{a}_{ij} x_j$$

Value of the basic variables x_{j_i}

$$\theta = \min \left\{ u_s, \min_{1 \leq i \leq m} \left\{ \frac{g_i}{-a_{is}} : \bar{a}_{is} < 0 \right\}, \min_{1 \leq i \leq m} \left\{ \frac{u_{j_i} - g_i}{a_{is}} : \bar{a}_{is} > 0 \right\} \right\}$$

$$\text{If } \theta = \frac{u_{j_r} - g_r}{a_{rs}} = \min_{1 \leq i \leq m} \left\{ \frac{u_{j_i} - g_i}{a_{is}} : \bar{a}_{is} > 0 \right\},$$

then the value on the entering variable x_s is reduced by θ (i.e., $x_s \leftarrow u_s - \theta$).

The entering variable x_s becomes a basic variable replacing the leaving variable x_{j_r} becoming a non basic variable equal to u_{j_r} .

Pivot on $a_{j_r,s}$, and go to step 1.

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