Lecture 13 - UGM

Tuesday, October 18, 2022 2:59 PM

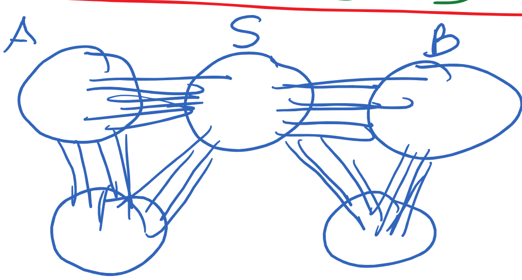
today: UGMs (cont.)
DGM vs. UGM

cond. independence for UGM

def: we say that p satisfies **global Markov property** (with respect to a undirected graph G)

iff $\forall A, B$ & $S \subseteq V$ (disjoint) s.t. S separates A from B in G

then $X_A \perp\!\!\!\perp X_B \mid X_S$



prop: $\boxed{p \in \mathcal{L}(G) \Rightarrow p \text{ satisfies the global Markov property for } G}$

proof: WLOG, we assume that $A \cup B \cup S = V$ & A separated from B by S

[why? if not, $\tilde{A} \triangleq A \cup \{a \in V : a \text{ and } A \text{ are not separated by } S\}$

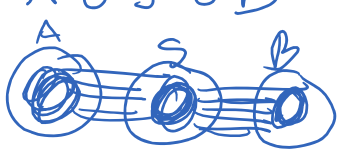
$\tilde{B} \triangleq V \setminus (S \cup \tilde{A})$ $\Rightarrow \tilde{A} \cup \tilde{B} \cup S = V$

then if have $X_{\tilde{A}} \perp\!\!\!\perp X_{\tilde{B}} \mid X_S$ $\quad$ $A \subseteq \tilde{A}$, $B \subseteq \tilde{B}$

$\overset{\text{by decomposition}}{\Rightarrow}$ $X_A \perp\!\!\!\perp X_B \mid X_S$

[exercise: show that if $b \in \tilde{B}$ $\Rightarrow b$ is separated from $\tilde{A}$ by S]

$V = A \cup S \cup B$



let $C \in \mathcal{C}$

cannot have both $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$

thus $p(x) = \frac{1}{Z} \prod_{\substack{C \in \mathcal{C} \\ C \subseteq A \cup S}} \psi_C(x_C) \prod_{\substack{C' \in \mathcal{C} \\ C' \not\subseteq A \cup S}} \psi_{C'}(x_{C'}) = f(x_{A \cup S})\, g(x_{B \cup S})$

$\underbrace{C' \not\subseteq A \cup S}$ $\Rightarrow C' \subseteq B \cup S$

$p(x_A \mid x_S) \propto p(x_A, x_S) = \sum p(x_V) = \sum f(x_{A \cup S})\, g(x_{B \cup S})$

$$\Rightarrow C' \subseteq B \cup S$$

$$p(x_A|x_S) \propto p(\underbrace{x_A, x_S}_{x_{A\cup S}}) = \sum_{x_B} p(x_V) = \sum_{x_B} \underbrace{f(x_{A\cup S})}\, g(x_{B\cup S})$$

$$= f(x_{A\cup S}) \underbrace{\sum_{x_B} g(x_{B\cup S})}_{\text{constant w.r.t. } x_A}$$

$$\Rightarrow p(x_A|x_S) = \frac{f(x_{A\cup S})}{\sum_{x_A'} f(x_A', x_S)}$$

similarly, $p(x_B|x_S) = \dfrac{g(x_{B\cup S})}{\sum_{x_B'} g(x_B', x_S)}$

$$p(x_A|x_S)\,p(x_B|x_S) = \frac{f(x_{A\cup S})\,g(x_{B\cup S})}{\sum_{x_A'} f(x_A', x_S) \sum_{x_B'} g(x_B', x_S)} = \frac{p(x_V)}{p(x_S)} = p(x_A, x_B | x_S)$$

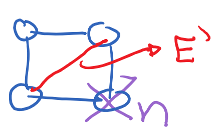
i.e. $X_A \perp\!\!\!\perp X_B \mid X_S$ //

thm.: (Hammersley-Clifford) if $p(x_V) > 0 \;\; \forall x_V$
then $p \in \mathcal{F}(G) \Leftrightarrow p$ satisfies the global Markov property for G

proof: see ch. 16 of Mike's book; use "Möbius inversion formula" (as exclusion-inclusion principle in prob.)

more properties of UGM:

* closure with respect to marginalization



let $V' = V \setminus \{n\}$

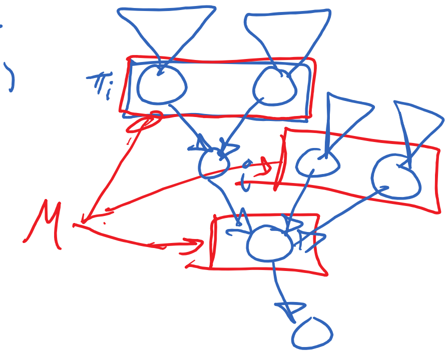
E' = edges in $G \setminus \{n\}$ + connect all neighbors of n in G together (new clique)

$\{$ marginal on $x_{1:n-1}$ for some $p \in \mathcal{F}(G) \} = \mathcal{F}(G')$

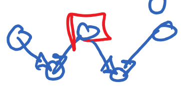
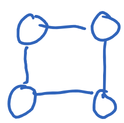
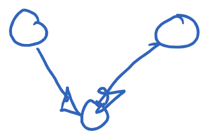
DGM vs. UGM

def: Markov blanket for i (for graph G) is the smallest set of nodes M s.t. $X_i \perp\!\!\!\perp X_{\text{"rest"}} \mid X_M$

- for UGM: $M = \{j : \{i,j\} \in E\}$ = set of neighbors of i
- for DGM: $M = \underbrace{\pi_i \cup \text{children}(i)}_{\text{like UGM}} \cup \bigcup_{j \in \text{children}(i)} \pi_j$



recap:

	DGM	UGM
factorization:	$p(x) = \prod_i p(x_i \mid x_{\pi_i})$	$\frac{1}{Z} \prod_C \psi_C(x_C)$
cond. indep.:	d-separation	separation
marginalization:	not closed in general; e.g.  but is fine for leaves	closed (connect all neighbors of removed node)
cannot exactly capture some families		

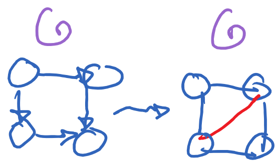
Moralization:

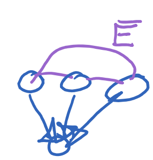
let G be a DAG; when can we get an equivalent UGM?

def: for G a DAG, we call $\bar{G}$ the moralized graph of G
where $\bar{G}$ is an undirected graph with same V
and $\bar{E} = \{\{i,j\} : (i,j) \in E\}$ } undirected version of E
$\cup \{\{k,l\} : k \neq l \in \pi_i \text{ for some } i\}$ } "moralization" → "marrying the parents" (?)

{connect all the parents of i with i in big clique
→ only needed if $|\pi_i| > 1$ (i.e. v-structure)

e.g. G → $\bar{G}$





prop: for a DAG with no v-structure [forest] then $\mathcal{L}(G) = \mathcal{L}(\bar{G})$

(proof: see hwk 3?)

β(9): for a DAG with no v-structure [forest]
then $\mathcal{I}(G) = \mathcal{I}(\bar{G})$
DGM UGM

(proof: see hwk 3?)

but in general, can only $\mathcal{I}(G) \subseteq \mathcal{I}(\bar{G})$
DAG

[note that $\bar{G}$ is the minimal undirected graph G' s.t. $\mathcal{I}(G) \subseteq \mathcal{I}(G')$]

$$p(x) = \prod_i \underbrace{p(x_i | x_{\pi_i})}_{\psi_C(x_C)}$$ where $C = \{i\} \cup \pi_i$

$\bar{G}$

16h03

general themes in this class

A) representation → DGM
→ UGM

parameterization → exponential family

} model high dim. distribution

B) inference $p(x_Q | x_E)$
"query" "evidence"

→ today: elimination algorithm
→ next: sum-product / belief propagation (for trees)

later: approximate inference e.g. MCMC, variational

C) statistical estimation/learning → MLE
maximum entropy
method of moments

Inference

want to compute:

a) marginal: $p(x_F)$ for some $F \subseteq V$

b) conditional: $p(x_F | x_E)$
"query" "evidence"

c) for UGM: partition function $Z = \sum_{x_V} \left(\prod_C \psi_C(x_C) \right)$
exponential sum

missing mark

x_V exponential sum

why? • missing data: $p(x_{\text{unobsv.}} \mid x_{\text{obsv.}})$

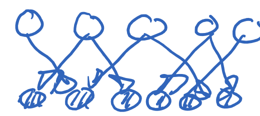
↳ example

missing pixels

"infilling task"

• prediction $p(x_{\text{future}} \mid x_{\text{past}})$

• "latent cause" $p(x_{\text{cause}} \mid x_{\text{obs}})$



diseases (causes)

symptoms

* also related inference:

$$\arg\max_{x_F} p(x_F \mid x_E)$$

could be big

(Viterbi alg.) (speech recognition)

* inference is also needed during estimation (parameter fitting MLE)
[eg during EM, E-step $p(z \mid x)$]

⊛ we present inference alg. for UGM [for simplicity and more generality]
make DGM $\rightsquigarrow \subseteq$ UGM via moralization

i.e. $p \in \text{DGM}$; $p(x) = \prod_i p(x_i \mid x_{\pi_i})$ moralize: $C_i \triangleq \{i\} \cup \pi_i$

$$= \frac{1}{Z} \prod_i \psi_{C_i}(x_{C_i})$$

$\psi_{C_i}(x_{C_i}) \triangleq p(x_i \mid x_{\pi_i})$

$Z = 1$

$$p(x_1) = \sum_{x_{2:n}} p(x) = \sum_{x_{2:n}} \prod_i \frac{1}{Z} \psi_{C_i}(x_{C_i})$$