

### 1. Cautionary tale about importance sampling (10 points)

Suppose that we wish to estimate the normalizing constant  $Z_p$  for an un-normalized Gaussian  $\tilde{p}(x) = \exp(-\frac{1}{2\sigma_p^2}x^2)$ . In other words, we have  $p(\cdot) \sim \mathcal{N}(\cdot|0, \sigma_p^2)$  with  $p(x) = \tilde{p}(x)/Z_p$ .

Given  $N$  i.i.d. samples  $x^{(1)}, \dots, x^{(N)}$  from a standard normal  $q(\cdot) \sim \mathcal{N}(0, 1)$ , consider the importance sampling estimate:

$$\hat{Z} = \frac{1}{N} \sum_{i=1}^N \frac{\tilde{p}(x^{(i)})}{q(x^{(i)})}.$$

- (a) Show that  $\hat{Z}$  is an unbiased estimator of  $Z_p$ .
- (b) Letting  $f(x) := \tilde{p}(x)/q(x)$ , show that  $\text{var}(\hat{Z}) = \frac{1}{N} \text{Var}(f(X))$  whenever  $\text{Var}(f(X)) < \infty$ .
- (c) For what values of  $\sigma_p^2$  is this variance finite?  
*Hint: Keep your variances non-negative!*

### 2. Gibbs sampling and mean field variational inference (10 points)

Consider the Ising model with binary variables  $X_s \in \{0, 1\}$  and a factorization of the form:

$$p(x; \eta) = \frac{1}{Z_p} \exp \left( \sum_{s \in V} \eta_s x_s + \sum_{\{s, t\} \in E} \eta_{st} x_s x_t \right).$$

- (a) Derive the Gibbs sampling updates for this model.
- (b) Consider fully factorized approximation  $q$ . Use the notation  $q(X_s = 1) = \tau_s$ . Derive an expression for  $KL(q||p) - \log(Z_p)$ .  
*Note: We can't directly compute the value of the KL divergence, as it would depend on the intractable quantity  $\log(Z_p)$ . When we subtract it, the remainder  $KL(q||p) - \log(Z_p)$  can be computed efficiently.*
- (c) Derive the naive mean field updates, based on a fully factorized approximation. More specifically, we do cyclic coordinate descent on  $KL(q||p)$ , sequentially updating the parameter  $\tau_s \in [0, 1]$ . Note that  $Z_p$  is a constant with respect to the parameters  $\tau_s$ .

### 3. Implementation of Gibbs sampling and mean field variational inference (20 points)

Follow the instructions in this Colab notebook:

<https://colab.research.google.com/drive/15MSQ6fNlmStRrRCNbsgxoAT7mn829vNt?usp=sharing>