





An Introduction to **Generative Flow Networks**

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Outline

- Motivation
- Applications
- GFlowNets
 - O Flow Networks
 - O Generative Flow Networks
 - O Training GFlowNets
- Comparison to meethods seen in class



Important References

- GFlowNet Foundations. Yoshua Bengio, et al. JMLR 2023.
- [NeurIPS 2021 GFlowNet paper] Flow Network based Generative Models for Non-Iterative Diverse Candidate Generation. Emmanuel Bengio, et al. NeurIPS 2021.
- [Structure Learning with GFlowNets] Bayesian Structure Learning with Generative Flow Networks. Tristan Deleu, et al. UAI 2022.



Setting

We want to do **inference** over an *intractable* distribution:

- Sampling: $x \sim p(x)$.
- Computing expectations: $E_{x \sim p(x)}[f(x)]$.

Sampling is intractable, but:

- Samples are discrete* and can be built compositionally.
- Can not evaluate p(x), but can evaluate a reward function "R" such that $p(x) \propto R(x)$.

^{*}For continuous GFlowNets, see A Theory of Continuous Generative Flow Networks (S. Lahlou et al., 2023)

Simplified Taxonomy for Sampling

- **Easy**: Know p(x), can efficiently sample from p(x)
 - O Uniform
 - O Mixture of Gaussians
 - O HMM (with Gaussian emission probabilities)
- Approximate: sampling is expensive or impossible, but can evaluate p(x)
 - O Rejection sampling: for using an easy distribution q(x)
 - O Importance sampling
 - O MCMC: exploit structure and low dimensionality of $p(x_i | x_{i})$
 - O Variational Inference: sample from easy $q \in \text{argmin } KL(q \parallel p)$
 - O **GFlowNets** (can evaluate $R(x) \propto p(x)$; next slide)



GFlowNets Main Idea

Given a reward function $\mathbf{R}(x)$,

- 1. Construct a DAG **G** over all possible samples
 - **a.** States: possible samples x (e.g. all possible chemical molecules)
 - **b. Transitions**: composition of parent to produce child (e.g. adding a basic molecule to produce a more complex one)
- 2. Learn a probability distribution $p \in L(G)$ such that $p(x) \propto R(x)$
 - a. In practice, GFlowNets model unnormalized "flows" such that $\mathbf{F}(x) = \mathbf{R}(x)$
 - b. Also, approximate F(x) variationaly (i.e. use a neural network to output F(x))
- 3. Sample over G.
 - a. Ancestral sampling, beam search, ...



Applications



Scientific Discovery

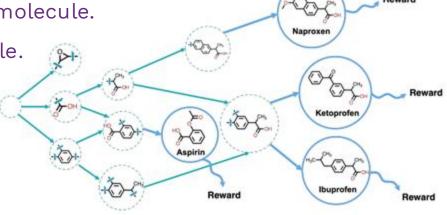
For instance, constructing molecules.

• States: sets of (partially constructed) molecules.

• Transitions: addition of a basic molecule.

• Reward: property of the molecule.

Building blocks

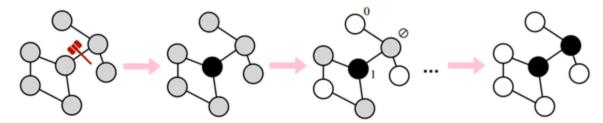


GFlowNets for AI-Driven Scientific Discovery. Moksh Jain, et al. Digital Discovery 2023.

Combinatorial Optimization Problems

For instance, finding the largest clique in a UGM.

- States: sets of fully connected nodes.
- Transitions: addition of a basic molecule.
- Reward: size of the set.

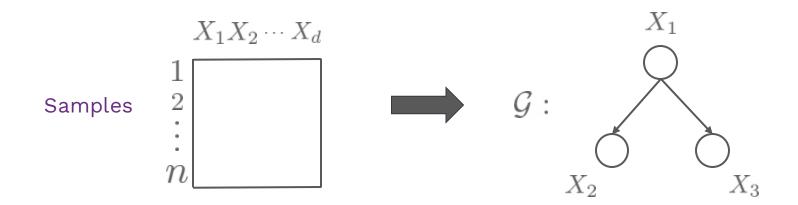


Let the Flows Tell: Solving Graph Combinatorial Optimization Problems with GFlowNets. Dinghuai Zhang, et al. NeurIPS 2023.

Structure Learning

Let X_1, X_2, \ldots, X_d be rvs. We want to construct a "minimal" graph \mathcal{G} such that:

$$p(x_1, x_2, \dots, x_d) \in \mathcal{L}(\mathcal{G})$$



Structure Learning

- States: DAGs.
- Transitions: addition of a directed edge.
- Reward: likelihood of data given the graph.

The transitions exploit the <u>compositional</u> aspect. Instead of looking at all possible DAGs without structure, the GFlowNet depth equals the **number of edges** in the DAG.



Structure Learning s_0



- ullet DAG $G=(\mathcal{S},\mathbb{A})$ with source and sink states, $s_0,s_f\in\mathcal{S}$
- ullet Trajectories $au=(s_1,\ldots,s_n)$, where $s_t{
 ightarrow}s_{t+1}\in\mathbb{A}$
- ullet Let ${\mathcal T}$ be the set of all trajectories.
- Forward transition probabilities $\hat{P}_F(s' \mid s)$

$$\hat{P}_F(s'\mid s)$$
 is a probability

$$\sum_{s' \in Child(s)} \hat{P}_F(s' \mid s) = 1$$

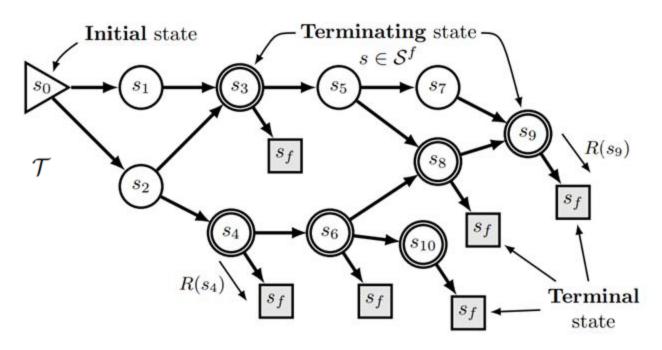
$$\sum_{s' \in Child(s)} \hat{P}_F(s' \mid s) = 1 \qquad \hat{P}_F(\tau) := \prod_{t=1}^{n-1} \hat{P}_F(s_{t+1} \mid s_t)$$

Instead of modeling probabilities, consider (unnormalized) flows:

$$F: \mathcal{T} \mapsto \mathbb{R}^+$$

$$F(s \rightarrow s') := F(\{\tau \in \mathcal{T} : s \rightarrow s' \in \tau\}) = \sum_{\tau \in \mathcal{T} : s \rightarrow s' \in \tau} F(\tau)$$

$$F(s) := F(\{\tau \in \mathcal{T} : s \in \tau\}) = \sum_{\tau \in \mathcal{T} : s \in \tau} F(\tau)$$



*Fig 2 in GFlowNet Foundations

The flow induces a distribution over the terminating states.

$$F(s_0) = \sum_{\tau \in \mathcal{T}} F(\tau) = Z,$$

$$F(s_f) = \sum F(\tau) = Z.$$

$$P(s) := \frac{F(s)}{Z}$$

$$P(s \rightarrow s' \mid s) = \frac{F(s \rightarrow s')}{F(s)}$$

The flow induces a distribution over the terminating states.

$$P_T(s) := P(s \to s_f) = \frac{F(s \to s_f)}{Z}$$

$$\sum_{s \in \mathcal{S}^f} P_T(s) = 1.$$

If we learn a flow **F** that matches a reward function on terminating states:

- A probability distribution over terminating states follows.
- The probabilities are proportional to the reward.
- And the graph **G** can be used to efficiently **sample**.

But parameterizing a flow is expensive! -> One value per trajectory in **G**.



Markovian Flow Networks

Instead, consider **Markovian** flow networks:

$$P(s \rightarrow s' \mid \tau) = P(s \rightarrow s' \mid s)$$

- Cheaper to model.
- Induces a unique forward transition probability **P(s'|s)**.
- **Proposition 23** in GFlowNet Foundations: the set of markovian flows is expressive enough to represent **all flow functions** over trajectories.

Markovian Flow Networks

- Even Markovian flows are expensive (need flow over every node).
- Approximate F with a neural network respecting flow conservation:

$$F(s) = \sum_{s' \in Child(s)} F(s \to s')$$
$$F(s') = \sum_{s \in Par(s')} F(s \to s')$$

Generative Flow Networks - GFlowNets

A **GFlowNet** is a (Markovian) flow network where:

$$\forall s \in \mathcal{S}^f \quad F(s \to s_f) = R(s)$$

- R is a given reward function.
 - O If **s** is not a valid terminating state, set a reward of 0.
- F is parameterized with (say) a NN.



The NeurIPS 2021 GFlowNet paper enforces flow matching:

$$\sum_{s \in \operatorname{Pa}(s')} F_{\theta}(s \to s') - \sum_{s'' \in \operatorname{Ch}(s')} F_{\theta}(s' \to s'') = R(s')$$

$$F(s \to s_f)$$

$$0 \text{ for invalid states}$$

Which leads to the following objective:

$$\mathcal{L}(\phi) = \mathbb{E}_{\pi} \left[\left[\log \frac{\sum_{s} F_{\phi}(s \to s')}{R(s') + \sum_{s''} F_{\phi}(s' \to s'')} \right]^{2} \right]$$

The expectation is over all trajectories -> intractable.

$$\mathcal{L}(\phi) = \mathbb{E}_{\pi} \left[\left[\log \frac{\sum_{s} F_{\phi}(s \to s')}{R(s') + \sum_{s''} F_{\phi}(s' \to s'')} \right]^{2} \right]$$
trajectories

Sample trajectories instead

- If there is structure, the GFlowNet could generalize across trajectories.
- Trade-off between sampling likely trajectories and exploration.

The expectation is over all trajectories -> intractable.

$$\mathcal{L}(\phi) = \mathbb{E}_{\pi} \left[\left[\log \frac{\sum_{s} F_{\phi}(s \to s')}{R(s') + \sum_{s''} F_{\phi}(s' \to s'')} \right]^{2} \right]$$
trajectories

Note: this is different from supervised learning and reinforcement learning

- **SL**: the distribution over trajectories is non-stationary
- RL: F is not trained to maximize a reward, just to match it

Alternative Loss Functions

Detailed balance (Bengio et al., 2021):

$$F_{\theta}(s)P_{F}(s'\mid s) = F_{\theta}(s')P_{B}(s\mid s')$$

Trajectory balance (Malkin et al., 2022):

$$Z\prod_{t=1}^{n} P_{F}(s_{t} \mid s_{t-1}) = R(s_{n})\prod_{t=1}^{n} P_{B}(s_{t-1} \mid s_{t})$$

Trajectory Balance: Improved Credit Assignment in GFlowNets. Nikolay Malkin et al., NeurIPS 2022.

GFlowNets in Context

Markov Chain Monte Carlo

MCMC

- No "setup" cost.
- Samples are not independent
- Sampling is costly
 - O Mixing time can scale poorly
 - O Mode mixing

GFlowNets

- Needs to be trained.
- Samples are independent.
- Sampling is efficient: do ancestral sampling



Generative Modeling

Generative Modeling

- Trained on *data*, to maximize its likelihood.
- Prone to overfitting.
- Improves with more data.

GFlowNets

- Trained to match a reward function.
- Prone to underfitting.
- Improves with more trajectories.



Reinforcement Learning

Reinforcement Learning

GFlowNets

See section 7.2 in GFlowNet Foundations





Thanks!

References

- GFlowNets for AI-Driven Scientific Discovery. Moksh Jain, et al. Digital Discovery 2023.
- Let the Flows Tell: Solving Graph Combinatorial Optimization Problems with GFlowNets. Dinghuai Zhang, et al. NeurIPS 2023.
- Trajectory Balance: Improved Credit Assignment in GFlowNets. Nikolay Malkin et al., NeurIPS 2022.
- GFlowNet Foundations. Yoshua Bengio, et al. JMLR 2023.
- Flow Network based Generative Models for Non-Iterative Diverse Candidate Generation. Emmanuel Bengio, et al. NeurIPS 2021.
- Bayesian Structure Learning with Generative Flow Networks. Tristan Deleu, et al. UAI 2022.





Generative Flow Networks and Bayesian Structure Learning

Structure Learning

Credit to Tristan Deleu

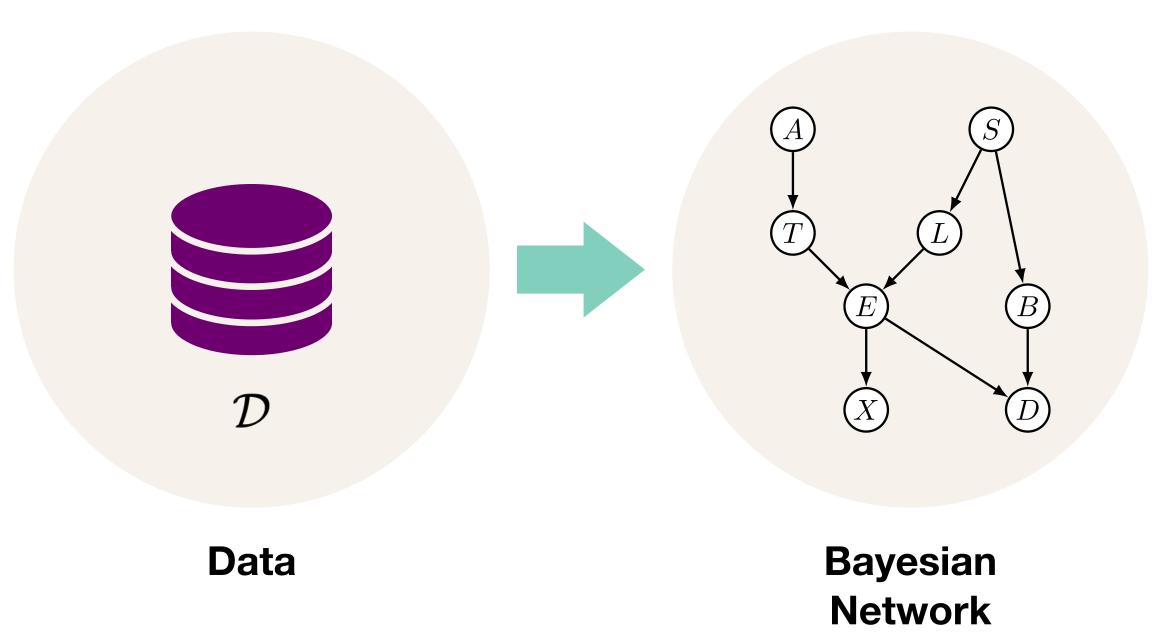


Structure Learning

- Graphical representation of the conditional independences in a distribution, represented as a Directed Acyclic Graph (DAG).
- The joint distribution is decomposed as:

$$P(X_1,\ldots,X_d) = \prod_{k=1}^d P(X_k \mid \text{Pa}_G(X_k))$$

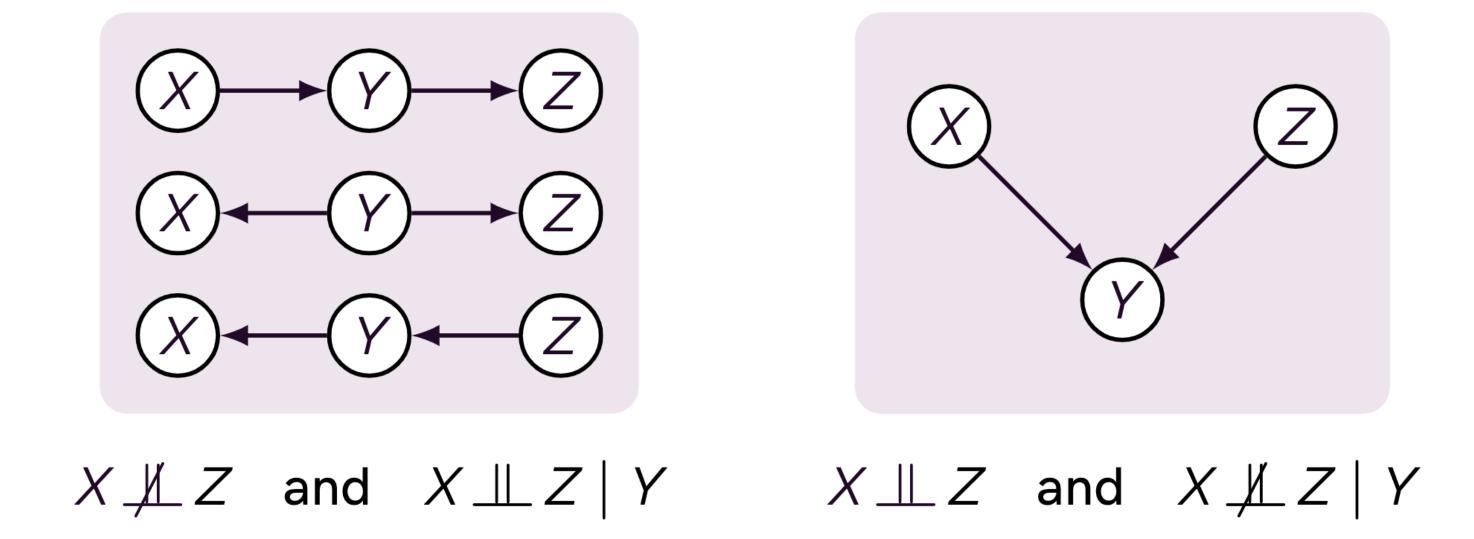
• Structure learning: Given a dataset of observations \mathcal{D} , find the graph structure G.





Markov Equivalence

- Recall: A Directed Graphical Model encodes the Conditional Independence of a distribution.
- Multiple DAGs may encode the same Conditional Independence statements.

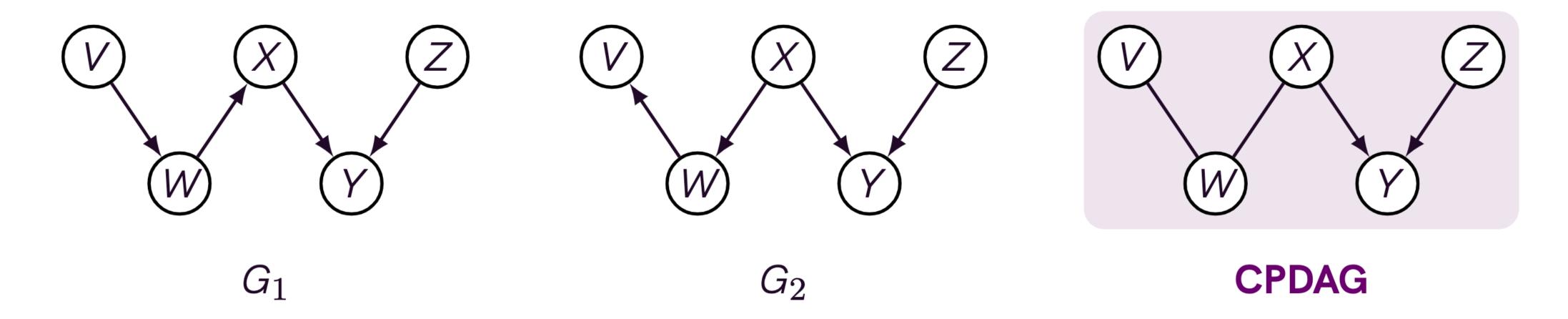


 Two DAGs encoding the same Conditional Independence statements are called Markov Equivalent.

Markov Equivalence

Theorem (Verma & Pearl, 1991)

Two DAGs G_1 and G_2 are **Markov Equivalent** if and only if they have the same skeleton and the same v-structures.



 Markov Equivalence Classes can be represented as a Completed Partially Directed Acyclic Graph (CPDAG).

Faithfulness

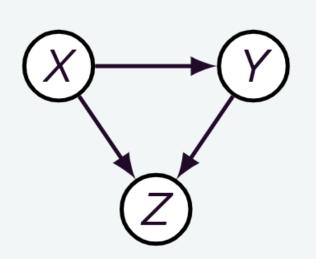
A & B are d-separated by C in $\mathcal G$



Faithfulness

$$X_A \perp \!\!\! \perp X_B \mid X_C$$

Exercise: Violation of Faithfulness



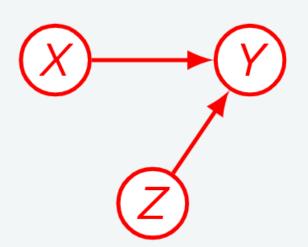
$$X := N_X$$

$$Y := X + N_Y$$

$$Z := X - Y + N_Z$$

with
$$N_X, N_Y, N_Z \stackrel{iid}{\sim} \mathcal{N}(0, \sigma^2)$$

Structure Learning



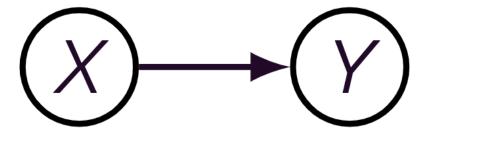
p(X, Y, Z) is a Multivariate Normal distribution, where the only conditional independence statements are: $X \perp\!\!\!\perp Z$ and $X \not\perp\!\!\!\perp Z \mid Y$.

Structure Identifiability

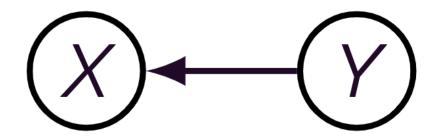
Theorem

If p is faithful wrt. \mathcal{G}^0 , then the Markov Equivalence class of \mathcal{G}^0 is **identifiable** from p.

 Only the Markov Equivalence class is identifiable from observations, not an individual graph. Two Markov Equivalent graphs may lead to different causal conclusions!



or



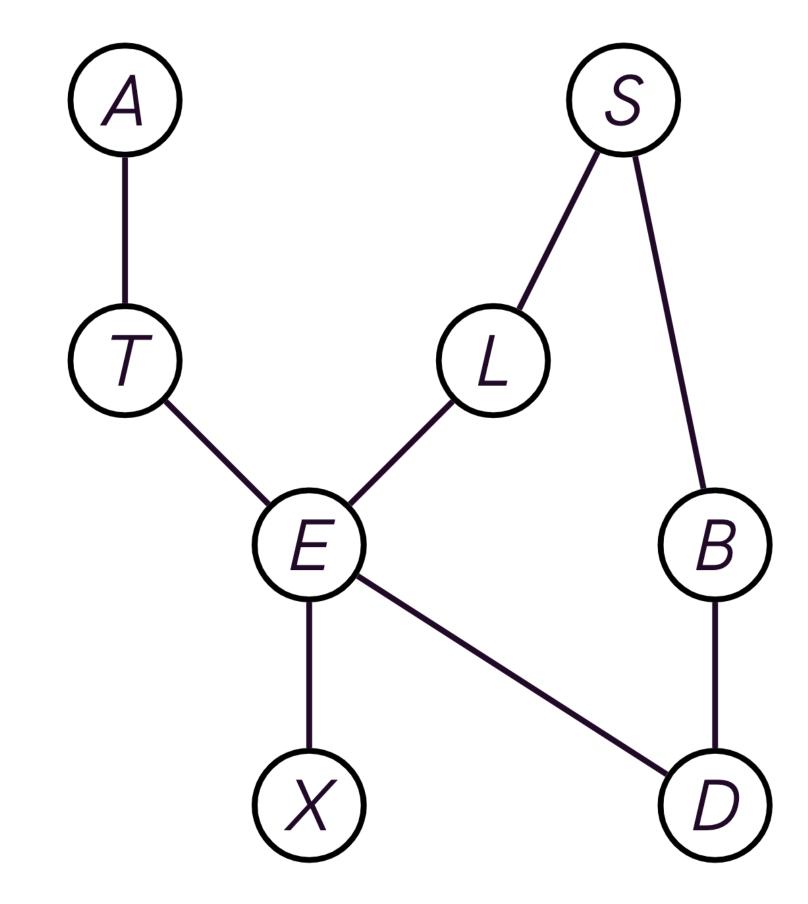
- Under different assumptions, an individual DAG may be identifiable
 - Additive Noise Model (ANM): $X_j := f_j(X_{\operatorname{Pa}_j}) + N_j$, $N_j \stackrel{iid}{\sim} \mathcal{N}(0, \sigma^2)$, where fj are nonlinear.
 - Using interventional data (i.e. data resulting from controlled experiments).

Constraint-based methods

Step 1: Identify the skeleton

For each pair of nodes X & Y, and $\mathbf{A} \subseteq \mathbf{V} \setminus \{X, Y\}$, test if $X \perp \!\!\! \perp_{\mathcal{D}} Y \mid \mathbf{A}$.

If there is no set \mathbf{A} s.t. $X \perp \!\!\! \perp_{\mathcal{D}} Y \mid \mathbf{A}$, then add an edge $X -\!\!\!\! - Y$.



Constraint-based methods

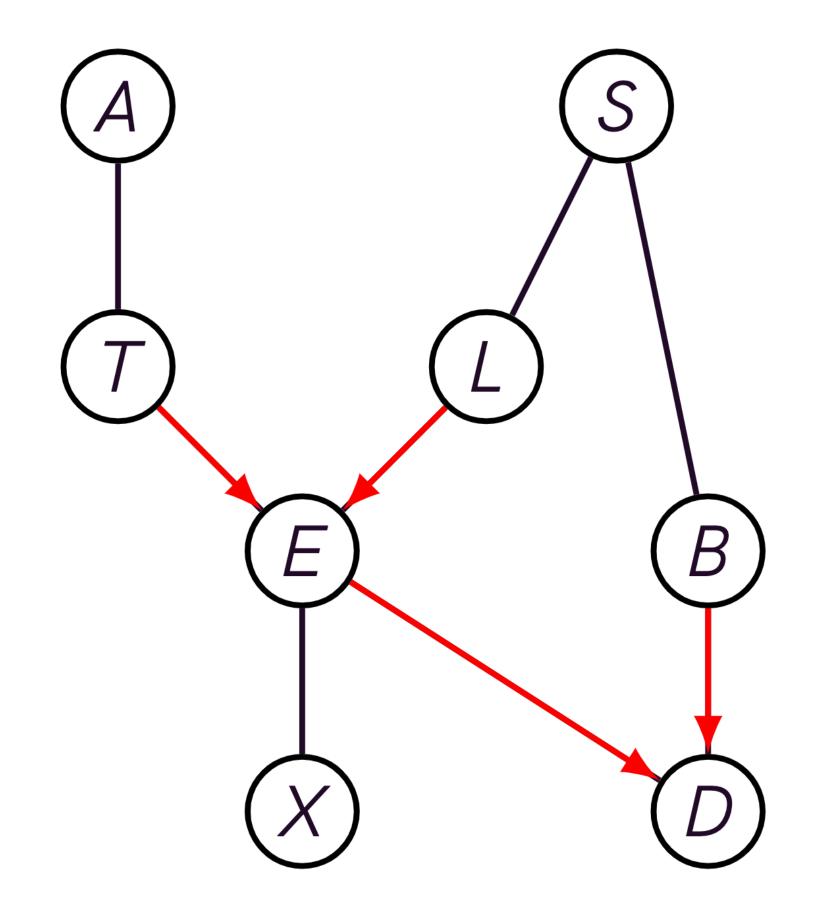
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Step 2: Identify the v-structures

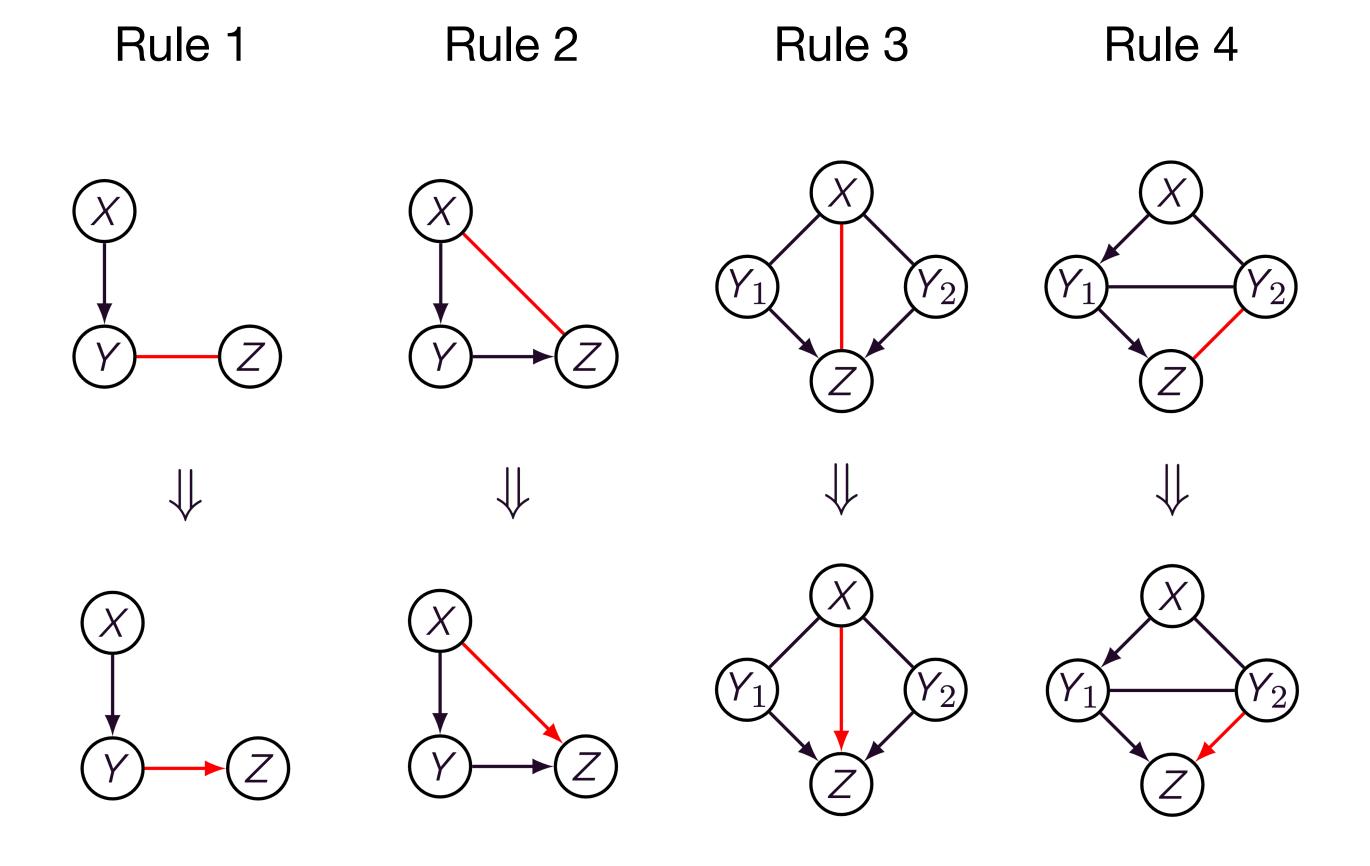
For each structure X - Z - Y with no edge between X & Y, orient $X \to Z \leftarrow Y$ iff $Z \notin \mathbf{A}$, where \mathbf{A} is such that $X \perp\!\!\!\perp_{\mathcal{D}} Y \mid \mathbf{A}$.



Constraint-based methods

Step 2': Additional orientations

Use Meek's orientation rules to orient some of the remaining edges.



Score-based methods

• Treat the problem of learning the structure of the DAG as a model selection problem

$$\max_{\mathcal{G} \in \mathrm{DAG}} \mathrm{score}(\mathcal{G} \mid \mathcal{D})$$

Choice of scores

- Likelihood score: $\operatorname{score}_{\mathcal{L}}(\mathcal{G} \mid \mathcal{D}) = \log p(\mathcal{D} \mid \hat{\theta}^{\mathrm{MLE}}_{\mathcal{G}}, \mathcal{G})$
- Bayesian score: $\operatorname{score}_B(\mathcal{G} \mid \mathcal{D}) = \log p(\mathcal{D} \mid \mathcal{G}) + \log p(\mathcal{G})$
- Bayesian Information Criterion (BIC):

$$\operatorname{score}_{\mathcal{B}/\mathcal{C}}(\mathcal{G} \mid \mathcal{D}) = \log p(\mathcal{D} \mid \hat{\theta}_{\mathcal{G}}^{\mathrm{MLE}}, \mathcal{G}) - \frac{\log N}{2} \mathrm{Dim}[\mathcal{G}]$$

Score-based methods

 $\max_{\mathcal{G} \in \mathrm{DAG}} \mathrm{score}(\mathcal{G} \mid \mathcal{D})$

- How to search over the space of DAGs?
- The number of DAGs over n nodes is **super-exponential** in n: $2^{\Theta(n^2)}$

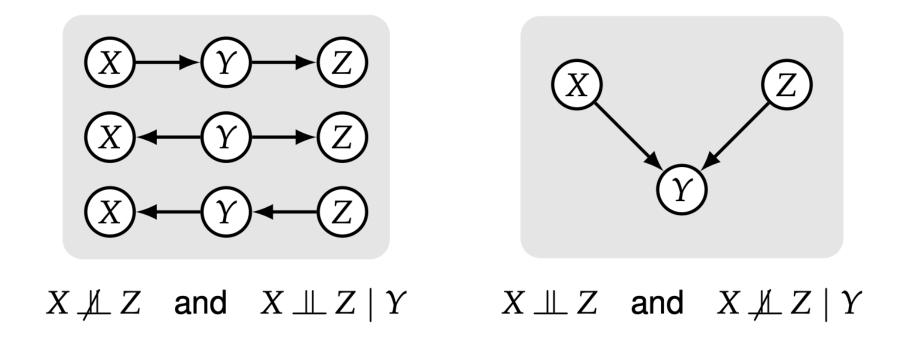
Theorem

Let $G_{\leq d} = \{ \mathcal{G} \ a \ DAG \mid every \ node \ has \ at \ most \ d \ parents \}$. Finding a DAG in $G_{\leq d}$ that maximizes a score is **NP-hard** for $d \geq 2$.

- Heuristic solutions:
 - Greedy algorithms: Hill climbing, GES
 - Genetic algorithms
 - Constrained continuous optimization: NOTEARS, Gran-DAG, DCDI, etc...

Bayesian Structure Learning

- When the dataset is small, we want to take into account the epistemic uncertainty over the graph structures of the Bayesian Network.
- Markov Equivalence: There may be multiple graphs encoding the same conditional independences.



• From the point of view of observations, Markov equivalent graphs fit the data equally well.

Bayesian Structure Learning: Instead of finding a single graph from observations, characterize the whole posterior distribution over graphs:

$$P(G \mid \mathcal{D}) = \frac{P(\mathcal{D} \mid G)P(G)}{P(\mathcal{D})}$$



Bayesian Structure Learning

Bayesian Structure Learning:

Instead of finding a single graph from observations, characterize the whole **posterior distribution** over graphs:

$$P(G \mid \mathcal{D}) = \frac{P(\mathcal{D} \mid G)P(G)}{\uparrow P(\mathcal{D})}$$

Graphs are **discrete** and **composite** objects

The number of DAGs is **super-exponential** in the number of nodes (eg. there are 10⁷² DAGs over 15 nodes)

The marginal likelihood is in general intractable

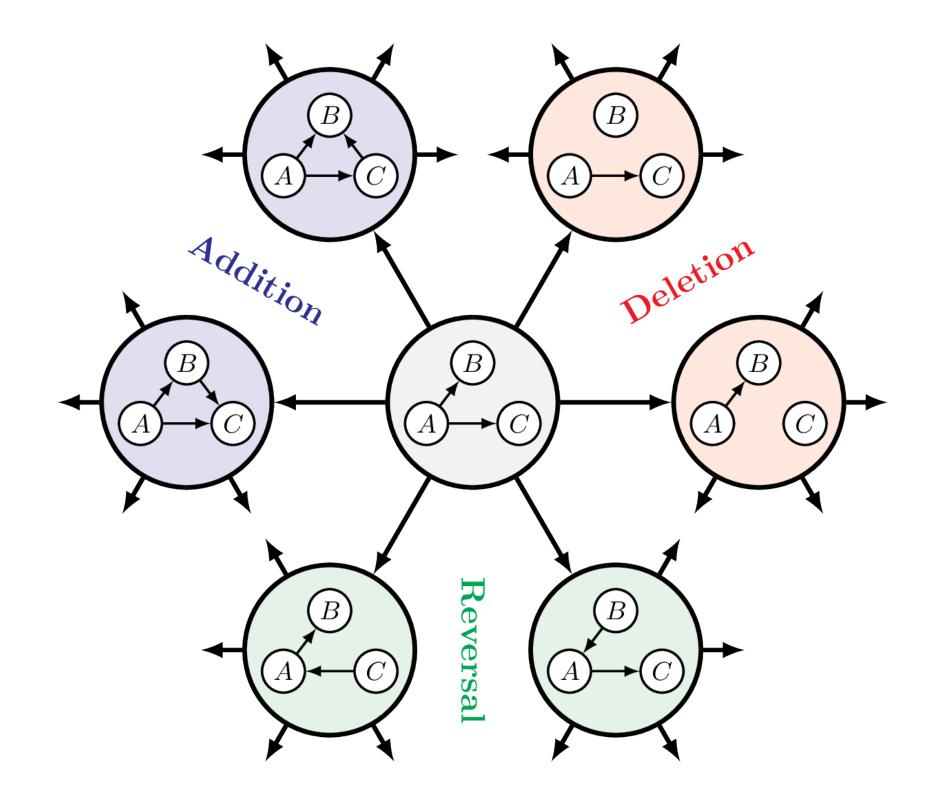
$$P(\mathcal{D} \mid G) = \int_{\Theta} P(\mathcal{D} \mid \theta, G) P(\theta \mid G) d\theta$$

We will choose models so that this can be computed efficiently in closed form.



Markov Chain Monte Carlo

- Approximate the posterior distribution using Markov Chain Monte Carlo (MCMC).
- Build a Markov chain by adding, removing, or reversing edges uniformly at random.
- Issue: Highly multimodal distribution (Markov equivalence), leading to poor mixing of the Markov chain.



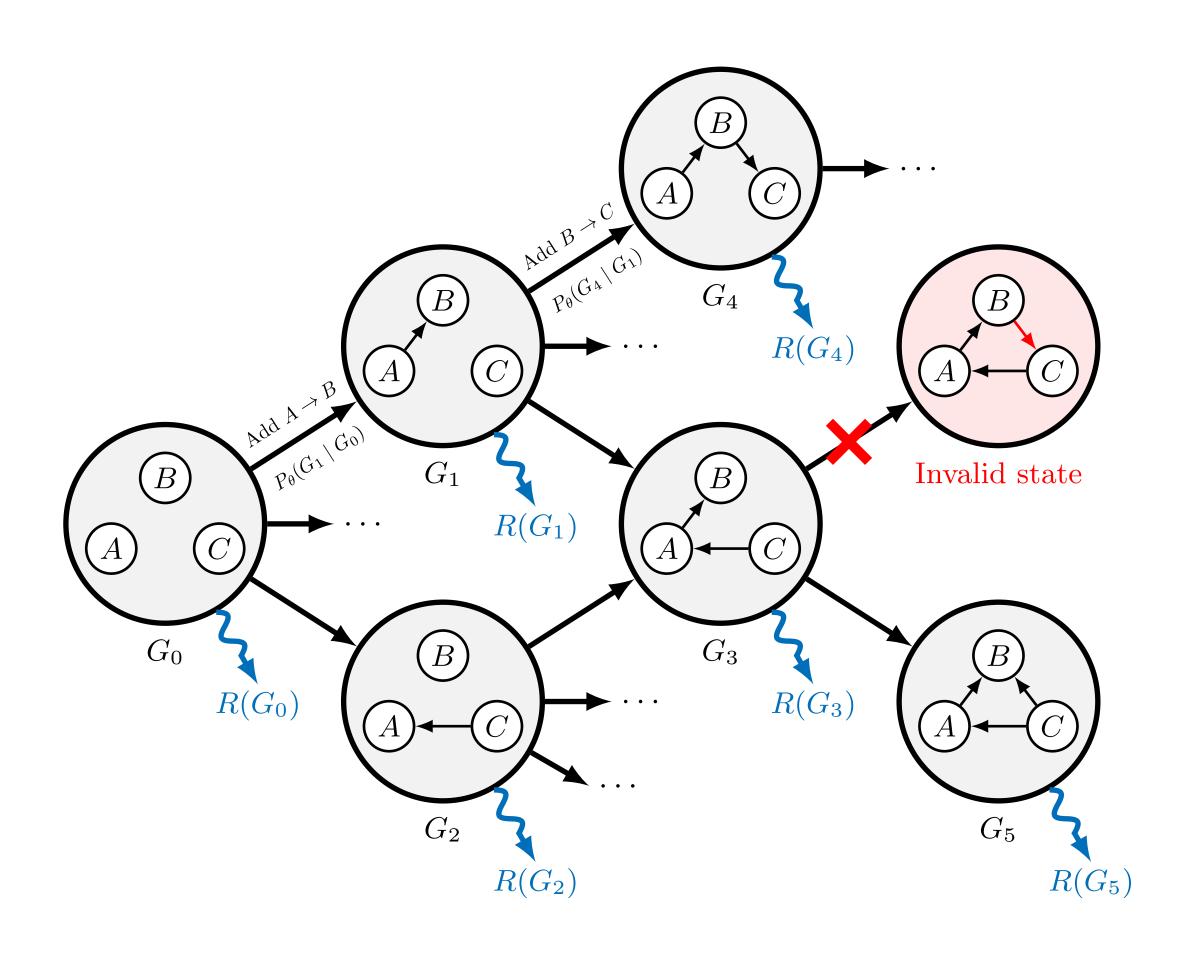


DAG-GFlowNet



GFlowNet over DAGs

- DAGs are constructed sequentially one edge at a time, starting from the empty graph.
- All the states of the GFlowNet are valid DAGs, meaning that all the states are terminating.
- A new edge to be added to a DAG:
 - must not already be present;
 - must not introduce a cycle.
- We can filter out invalid actions using a mask, that can also be updated online.





Detailed balance condition

Flow matching condition (Bengio et al., 2021)

$$\sum_{s \in \operatorname{Pa}(s')} F_{\theta}(s \to s') - \sum_{s'' \in \operatorname{Ch}(s')} F_{\theta}(s' \to s'') = R(s')$$



$$P(s_{t+1} \mid s_t) \propto F_{\theta}(s_t \rightarrow s_{t+1})$$

Detailed balance condition (Ours)

$$R(s')P_B(s \mid s')P_{\theta}(s_f \mid s) = R(s)P_{\theta}(s' \mid s)P_{\theta}(s_f \mid s')$$

Fixed backward transition probability e.g. Uniform distribution

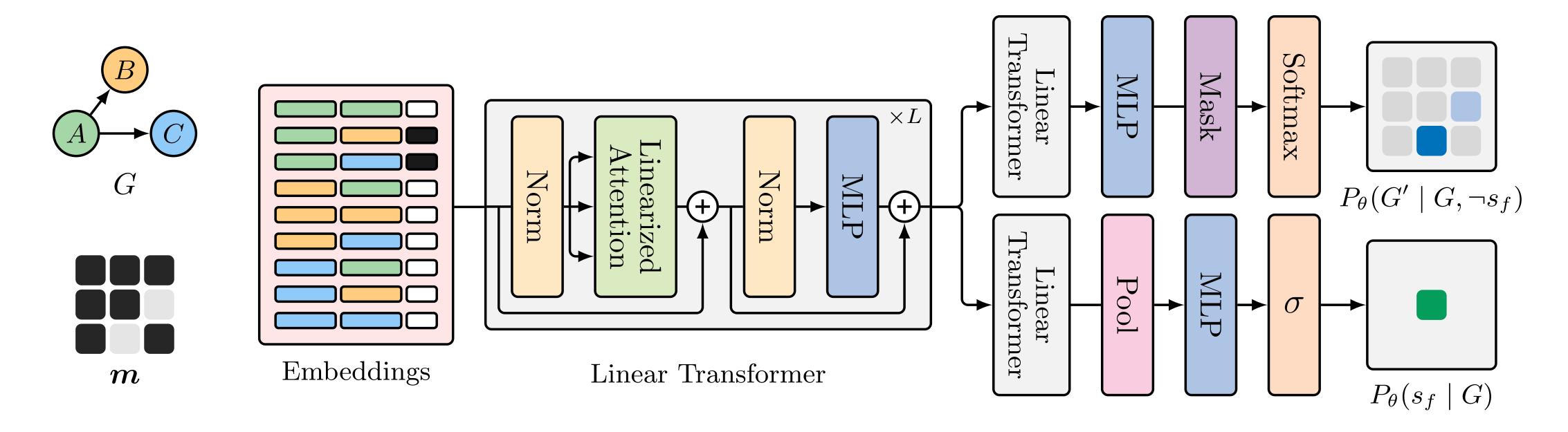
Learned forward transition probability

- √ Valid when all the states of the GFlowNet are terminating
- ✓ Induces a **distribution** $P(s) \propto R(s)$
- √ It does not depend on flows anymore (flow-matching or detailed balance conditions).
- It does not depend on the total flow Z (trajectory balance condition).



Forward Transition Probabilities

Hierarchical model for the forward transition probabilities: $P_{\theta}(G' \mid G) = (1 - P_{\theta}(s_f \mid G))P_{\theta}(G' \mid G, \neg s_f)$



- ✓ Independent of the order of edges.
- √ Set-to-set architecture.

- The **number of parameters** does not scale too much with the size of the graph.
- √ No quadratic scale with the input size.



Application to Bayesian Structure Learning

Bayesian Structure Learning

Characterize the **posterior distribution** over DAGs

$$P(G \mid \mathcal{D}) = \frac{P(\mathcal{D} \mid G)P(G)}{P(\mathcal{D})}$$



GFlowNet

A GFlowNet induces a distribution

$$P(s) \propto R(s)$$



DAG-GFlowNet

$$R(G) = P(\mathcal{D} \mid G)P(G)$$



Tools from Reinforcement Learning

The GFlowNet is trained off-policy

We use a **replay buffer** to store transitions over the course of training, and **sample transitions** from the replay buffer

$$\mathcal{L}(\theta) = \mathbb{E}_{\pi} \left[\left[\log \frac{R(G')P_B(G \mid G')P_{\theta}(s_f \mid G)}{R(G)P_{\theta}(G' \mid G)P_{\theta}(s_f \mid G')} \right]^2 \right]$$

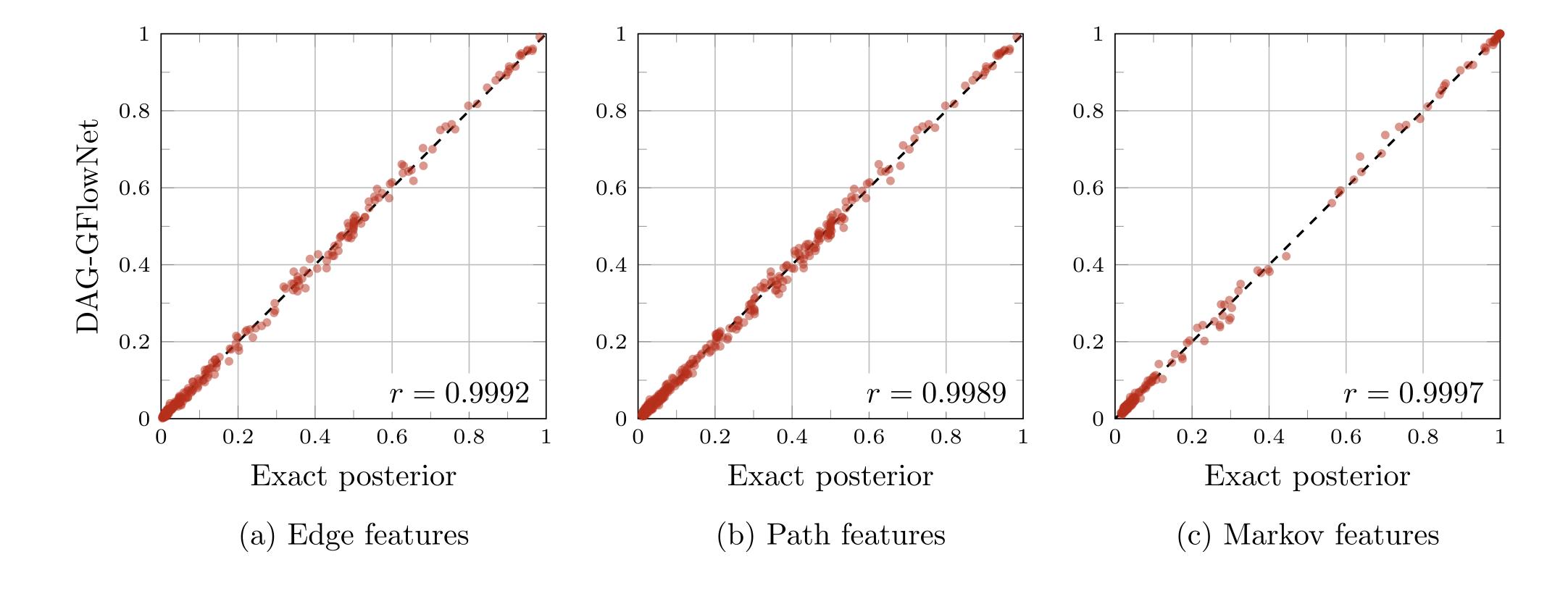


Experimental results



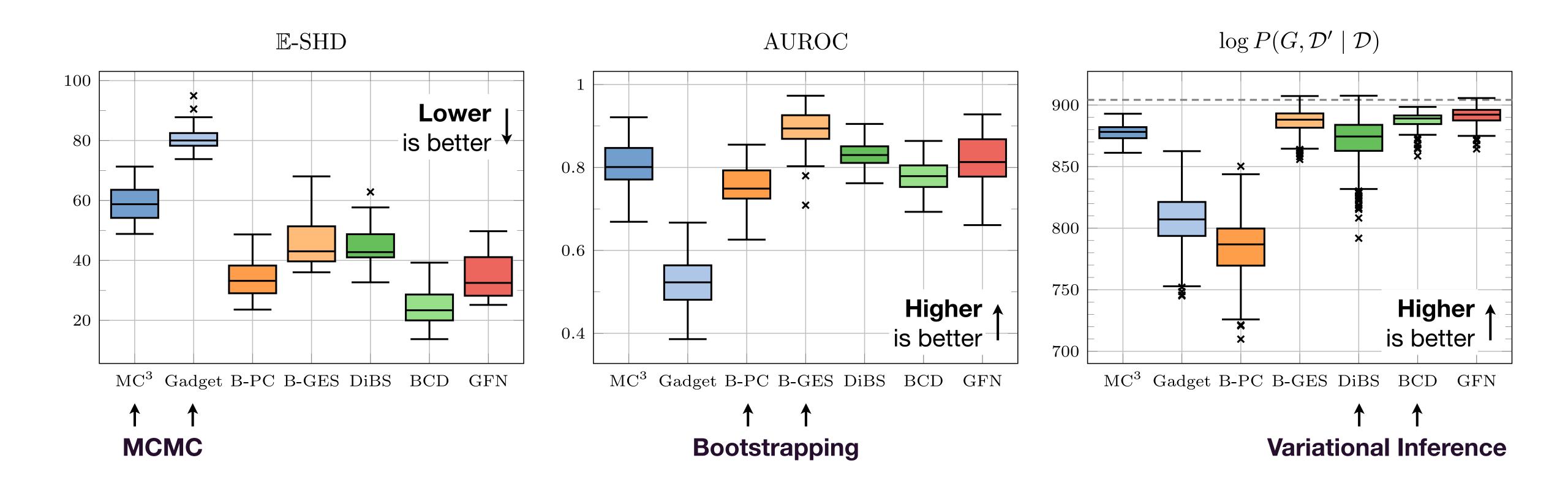
Experimental results - Accurate approximation

Comparison with the **exact posterior** distribution $P(G \mid \mathcal{D})$ on graphs with d = 5 nodes, computed by enumerating the **29,281 possible DAGs**.





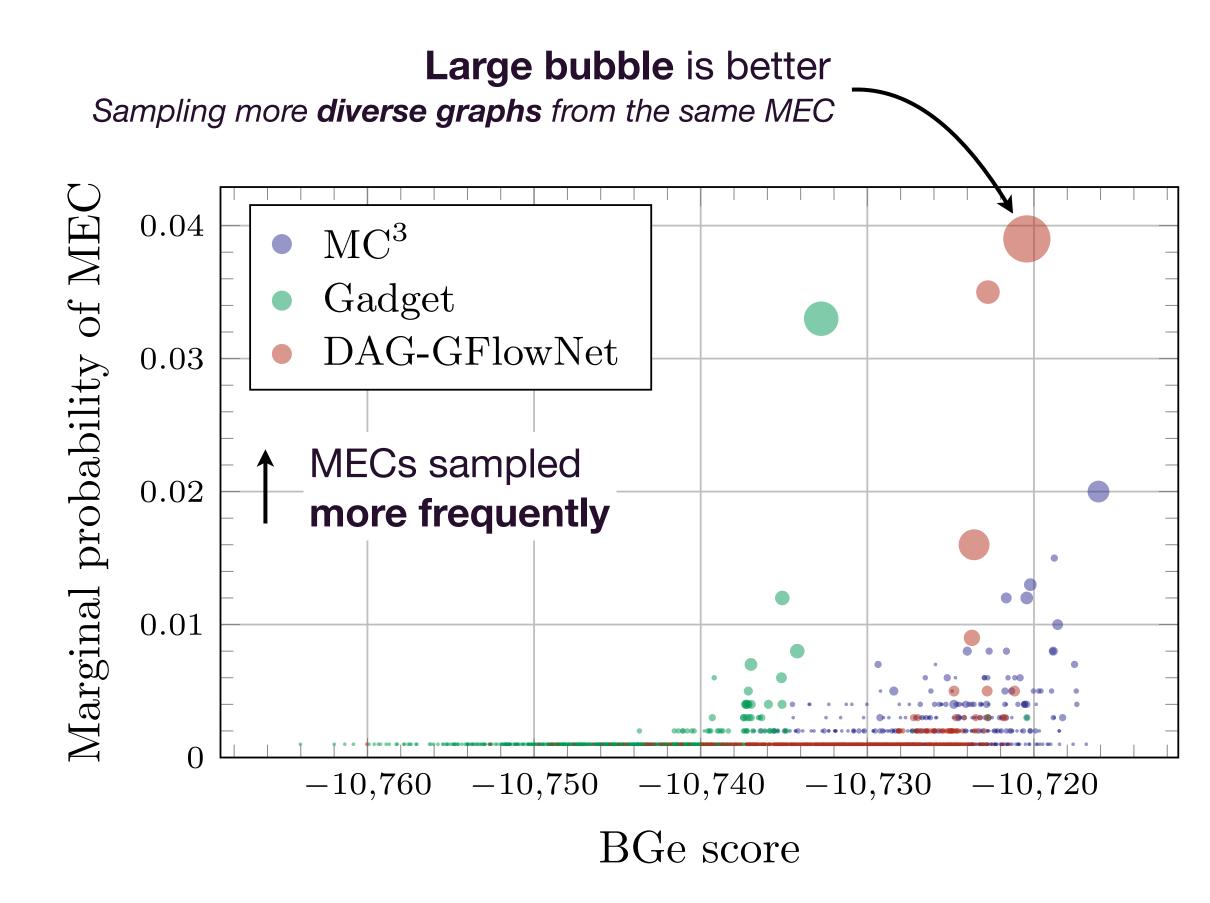
Experimental results – Simulated data





Experimental results - Flow cytometry data

- Real-world flow cytometry data, to learn protein signaling pathways.
- Data: continuous measurements of 11
 phosphoproteins. There are 853 observations.
- The ground truth graph contains 11 nodes and 17 edges.
 - The consensus graph may not represent a realistic description of the system.

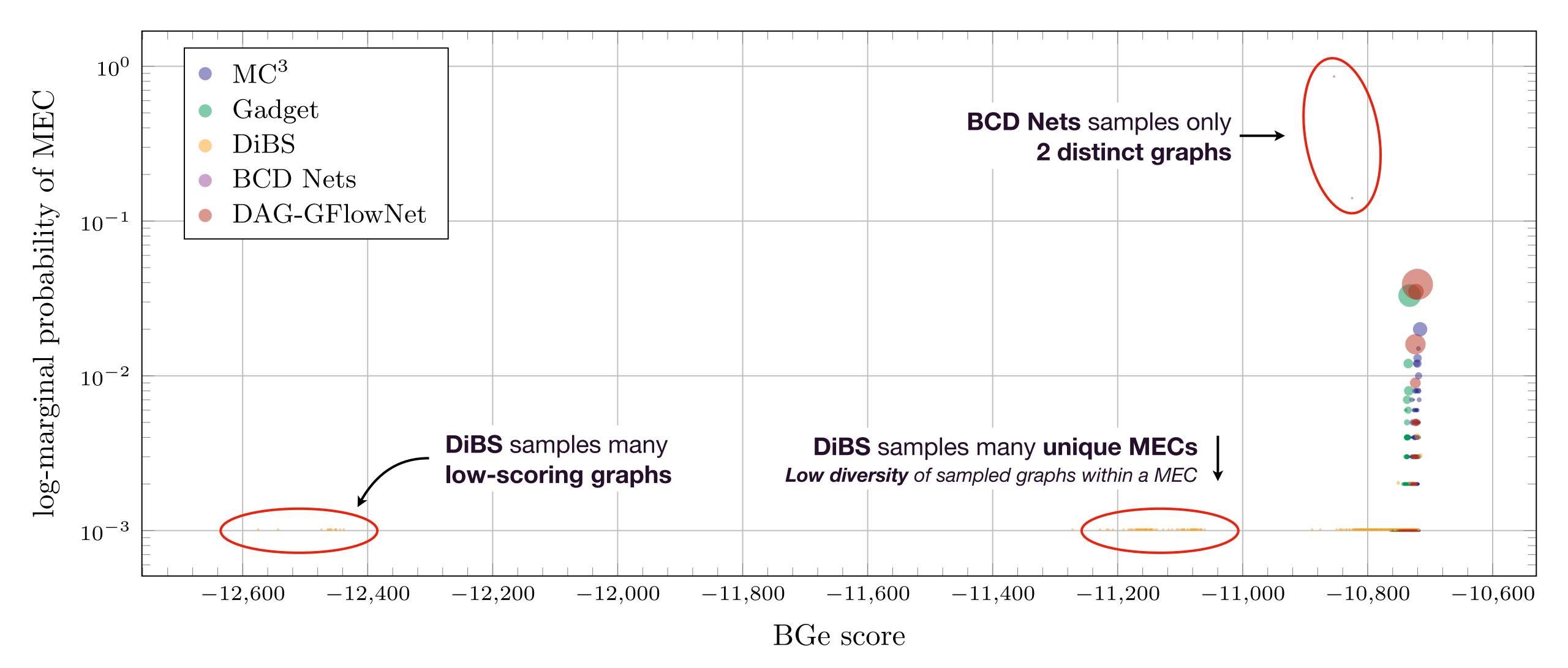


Higher score is better

Concentrated around the mode of the distribution



Experimental results – Flow cytometry data





Experimental results – Interventional data

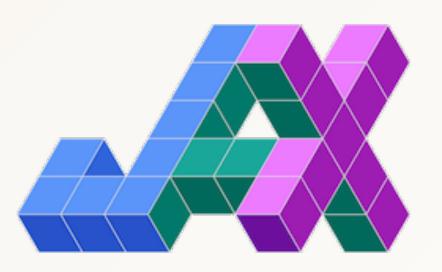
- The real world flow cytometry data also contains interventional data, based on experimentations where some phosphoproteins are inhibited.
 - We model these as **perfect interventions**, even though it may not be the case in practice.
 - We know the intervention targets.
- We can adapt the reward function (computation of the marginal likelihood) to handle a mixture of observational & interventional data.
- This is a first step toward causal discovery.

	E-# Edges	E-SHD	AUROC
Exact posterior* MC ³	-25.97 ± 0.01	25.08 ± 0.02	0.816 0.665
DAG-GFlowNet	30.66 ± 0.04	27.77 ± 0.03	0.700



Thank you





github.com/tristandeleu/jax-dag-gflownet

